Chapter 7

Quasi-conformal maps and Beltrami equation

7.1 Linear distortion

Assume that $f(x+iy) = u(x+iy) + iv(x+iy)$ be a (real) linear map from $\mathbb{C} \to \mathbb{C}$ that is orientation preserving. Let $z = x + iy$ and $w = u + iv$. The map $z \mapsto w = f(z)$ can be expressed by a matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}, \tag{7.1}$$

where T is the 2×2 matrix

$$T = Df(z) = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for some real constants a, b, c, and d. As f is orientation preserving, the determinant of the matrix T is positive, that is, $ad - bc > 0$.

The circle $|z|^2 = x^2 + y^2 = 1$ is mapped by f to an ellipse with equation $|T^{-1}w|^2 = 1$. The *distortion* of f, denoted by K_f, is defined as the eccentricity of this ellipse, that is, K_f is the ratio of the length of the major axis of the ellipse to the length of its minor axis of the ellipse. Since f is a linear map, the distortion of f is independent of the radius of the circle $|z| = 1$ we choose to define the ellipse.

A basic calculation leads to the equation

$$K_f + 1/K_f = \frac{a^2 + b^2 + c^2 + d^2}{ad - bc}.$$

for K_f in terms of a, b, c, and d. The above simple quantity and the forthcoming relations are rather complicated when viewed in real coordinates, but find simple forms in complex notations.

Any real-linear map $T : \mathbb{C} \to \mathbb{C}$ can be expressed in the form

$$w = T(z) = Az + B\overline{z}, \tag{7.2}$$

for some complex constants A and B. If T is orientation preserving, we have $\det T = |A|^2 - |B|^2 > 0$. Then, T can be also represented as

$$T(z) = A(z + \mu\bar{z}),$$

where

$$\mu = B/A, \text{ and } |\mu| < 1.$$

That is, T may be decomposed as the stretch map $S(z) = z + \mu\bar{z}$ post-composed with the multiplication by A. The multiplication consists of rotation by the angle $\arg(A)$ and magnification by $|A|$. Thus, all of the distortion caused by T is expressed in terms of μ. From μ one can find the angles of the major axis and minor axis of the image ellipse. The number μ is called the complex dilatation of T.

The maximal magnification occurs in the direction $(\arg\mu)/2$ and the magnification factor is $1+|\mu|$. The minimal magnification occurs in the orthogonal direction $(\arg\mu-\pi)/2$ and the magnification factor is $1-|\mu|$. Thus, the distortion of T, which only depends on μ, is given by the formula

$$K_T = \frac{1+|\mu|}{1-|\mu|}.$$

A basic calculation implies

$$|\mu| = \frac{K_T - 1}{K_T + 1}.$$

If T_1 and T_2 are real-linear maps from $\mathbb{C}$ to $\mathbb{C}$ one can see that

$$K_{T_2 \circ T_1} \le K_{T_2} \cdot K_{T_1}. \tag{7.3}$$

The equality in the above equation may occur when the major axis of $T_1(\partial\mathbb{D})$ is equal to the direction in which the maximal magnification of T_2 occurs and the minor axis of $T_1(\partial\mathbb{D})$ is equal to the direction at which the minimal magnification of T_2 occurs. Otherwise, one obtains strict inequality.

7.2 Dilatation quotient

Assume that $f : \Omega \to \mathbb{C}$ is an orientation preserving diffeomorphism. That is, f is homeomorphism, and both f and f^{-1} have continuous derivatives. Let $z = x + iy$ and $f(x+iy) = u(x,y) + iv(x,y)$. At $z_0 = x_0 + iy_0 \in \Omega$ and $z = x + iy$ close to zero we have

$$f(z_0 + z) = f(z_0) + \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + o(z). \tag{7.4}$$

In the above equation, the little o notation means any function $g(z)$ which satisfies $\lim_{z\to 0} g(z)/z = 0$.

We may write Equation (7.4) in the complex notation

$$f(z_0 + z) = f(z_0) + Az + B\overline{z} + o(z),$$

where A and B are complex numbers (which depend on z_0). Comparing the above two equations we may determine A and B in terms of the partial derivatives of f. That is, setting $z = 1$ and $z = i$ we obtain (respectively)

$$A + B = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad Ai - Bi = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

These imply that

$$A = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\Big), \quad B = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\Big).$$

If we introduce the notation

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \tag{7.5}$$

then the diffeomorphism f may be written in the complex notation as

$$f(z_0 + z) = f(z_0) + \frac{\partial f}{\partial z}(z_0) \cdot z + \frac{\partial f}{\partial \overline{z}}(z_0) \cdot \overline{z} + o(z).$$

In this notation, the Cauchy-Riemann condition we saw in Equation (1.4) is equivalent to

$$\frac{\partial}{\partial \overline{z}} f(z) = 0, \forall z \in \Omega, \tag{7.6}$$

and when f is holomorphic,

$$f'(z) = \frac{\partial}{\partial z} f(z).$$

Fix $\theta \in [0, 2\pi]$, and define

$$D_\theta f(z_0) = \lim_{r\to 0} \frac{f(z_0 + re^{i\theta}) - f(z)}{re^{i\theta}}.$$

This is the partial derivative of f at z_0 in the direction $e^{i\theta}$. By comparing to the distortion of real-linear maps we see that

$$\max_{\theta\in[0,2\pi]} |D_\theta f(z_))| = |A|\Big(1 + \Big|\frac{B}{A}\Big|\Big) = \Big|\frac{\partial f}{\partial z}(z_0)\Big| + \Big|\frac{\partial f}{\partial \overline{z}}(z_0)\Big|$$

and

$$\min_{\theta\in[0,2\pi]} |D_\theta f(z_))| = |A|\Big(1 - \Big|\frac{B}{A}\Big|\Big) = \Big|\frac{\partial f}{\partial z}(z_0)\Big| - \Big|\frac{\partial f}{\partial \overline{z}}(z_0)\Big|$$

The quantity μ that determines the local distortion of f at z_0 is

$$\mu = \mu_f(z_0) = \frac{\partial f/\partial \overline{z}(z_0)}{\partial f/\partial z(z_0)}.$$

Here, μ is a continuous function of z_0 defined on Ω and maps into $\mathbb{D}$. The function μ_f is called the *complex dilatation* of f. The *dilatation quotient* of f at z_0 is defined as

$$K_f(z_0) = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|} = \frac{\max_{\theta\in[0,2\pi]} |D_\theta f(z_0)|}{\min_{\theta\in[0,2\pi]} |D_\theta f(z_0)|}. \tag{7.7}$$

7.3 Absolute continuity on lines

Definition 7.1. A function $g : \mathbb{R} \to \mathbb{C}$ is called *absolutely continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every finite collection of intervals (a_1, b_1), (a_2, b_2), $\dots$, (a_n, b_n) in $\mathbb{R}$ we have

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \implies \sum_{i=1}^{n} |g(b_i) - g(a_i)| < \varepsilon.$$

A function $g : [a, b] \to \mathbb{C}$ is called absolutely continuous, if the above condition is satisfied when all the intervals lie in $[a, b]$.

For example, any C^1 function $g : [a, b] \to \mathbb{C}$ is absolutely continuous. In general, if $g : \mathbb{R} \to \mathbb{C}$ is differentiable at every $x \in \mathbb{R}$ and $|g'|$ is uniformly bounded, then g is absolutely continuous.

On the other hand, any absolutely continuous function is uniformly continuous (use with $n = 1$). But, there are uniformly continuous functions that are not absolutely continuous (for example Cantor's function).

Definition 7.2. Let $A \subset \mathbb{R}^n$, $n \geq 1$. We say that a property holds at almost every point in A if the set of points where the property does not hold forms a set of measure zero. For example, when we say that a function $f : A \to \mathbb{R}$ is continuous at almost every point in A it means that there is a set $B \subset A$ of measure zero such that for every $x \in A \setminus B$ the function f is continuous at x.

Definition 7.3. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a continuous map. We say that $f : \Omega \to \mathbb{C}$ is *absolutely continuous on lines* (ACL) if for each closed rectangle $\{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b, c \leq \operatorname{Im} z \leq d\}$ contained in Ω we have the following two properties:

(i) for almost all $y \in [c, d]$, the function $x \mapsto f(x + iy)$ is absolutely continuous on $[a, b]$,

(ii) for almost all $x \in [a, b]$, the function $y \mapsto f(x + iy)$ is absolutely continuous on $[c, d]$.

For example, if $g : \Omega \to \mathbb{C}$ is C^1, then it is ACL. If g is C^1 at all points except at a discrete set of points, it is ACL.

It is clear form the above definitions that a complex valued function is absolutely continuous iff its real and imaginary parts are absolutely continuous functions. The same statement is true for ACL property.

It follows from the standard results in real analysis that if $g : [a, b] \to \mathbb{R}$ is absolutely continuous, then it (has *bounded variation* and hence) is differentiable at almost every point. That is, at almost every $t \in [a, b]$, $g'(t)$ exists and is finite.

Proposition 7.4. *If $f : \Omega \to \mathbb{C}$ is ACL, then the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist (and are finite) at almost every $x + iy \in \Omega$.*

In particular, by the above proposition, at almost every $z \in \Omega$, the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ exist and are finite.

The proof of the above proposition may be found in any standard book on real analysis, see for example, the nice book by G. Folland [Fol99].

7.4 Quasi-conformal mappings

Definition 7.5 (Analytic quasi-conformality)**.** Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an orientation preserving homeomorphism. We say that $f : \Omega \to \mathbb{C}$ is K-quasi-conformal if we have

(i) f is absolutely continuous on lines,

(ii) for almost every $z \in \Omega$ we have $K_f(z) \leq K$.

An orientation preserving homeomorphism $f : \Omega \to \mathbb{C}$ is called quasi-conformal, if it is K-quasi-conformal for some $K \geq 1$.

Note that the condition (i) in the above definition guarantees that the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ are defined at almost every point in Ω. Hence, $\mu_f(z)$ is defined at almost every point and the condition (ii) is meaningful.

Definition 7.6. Let $f : \Omega \to \mathbb{C}$ be a quasi-conformal mapping. The quantity

$$K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$$

is called the *dilatation quotient* of f at z. The function

$$\mu_f(z) = \frac{\partial f/\partial \overline{z}}{\partial f/\partial z} \tag{7.8}$$

is called the *complex dilatation* of f. Both of these functions are defined at almost every point in Ω.

Recall that for a function $f : \Omega \to \mathbb{C}$, the *supremum norm* of f is defined as

$$\|f\|_\infty = \inf \Big\{ \sup_{z\in A} |f(z)| \mid A \subseteq \Omega, \text{ and } \Omega \setminus A \text{ has zero measure} \Big\}.$$

This is also called the *essential supremum* of f on Ω.

Note that the inequality $K_f(z) \le K$ corresponds to

$$|\mu_f(z)| \le \frac{K-1}{K+1}.$$

Thus, for a quasi-conformal map $f : \Omega \to \mathbb{C}$, we have

$$\|\mu_f\|_\infty < 1.$$

Theorem 7.7 (Pompeiu formula). *Let Ω be a simply connected domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a C^1 map which is quasi-conformal. Let γ be a piece-wise simple closed curve in Ω and B denote the bounded connected component of $\mathbb{C} \setminus \gamma$. For every $z_0 \in B$ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz - \frac{1}{2\pi i} \iint_B \frac{\partial f(z)/\partial \overline{z}}{z - z_0} dz d\overline{z}.$$

Proof. Let D be a bounded domain with a piece-wise C^1 boundary, and g be a complex valued C^1 function defined on $D \cup \partial D$. With notation $z = x + iy$ we have

$$\begin{aligned}\int_{\partial D} g(z)\, dz = \int_{\partial D} g(z)\, dx + \int_{\partial D} ig(z) dy \\ = \iint_D \Big(i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\Big)\, dxdy = 2i \iint_D \frac{\partial g}{\partial \overline{z}}\, dxdy = \iint_D \frac{\partial g}{\partial \overline{z}}\, d\overline{z}dz.\end{aligned}$$

Using $dz = dx + idy$, we have $d\overline{z}dz = (dx - idy)(dx + idy) = idxdy - idydx = 2idxdy$. This gives us the complex version of the Green's integral formula

$$\int_{\partial D} g(z)\, dz = \iint_D \frac{\partial g}{\partial \overline{z}}\, d\overline{z}dz. \tag{7.9}$$

Let z_0 be an arbitrary point in Ω and $\delta > 0$ small enough so that the closed ball $|z - z_0| \le \delta$ is contained in Ω. Define the open set

$$B_\delta = B \setminus \{z \in B : |z - z_0| \le \delta\}.$$

The function $g(z) = f(z)/(z - z_0)$ is C^1 on $B \cup \partial B$, and at every $z \in B_\delta$ we have

$$\frac{\partial}{\partial \overline{z}}\Big(\frac{f(z)}{z - z_0}\Big) = \frac{\partial f}{\partial \overline{z}} \cdot \frac{1}{z - z_0} + f(z) \cdot \frac{\partial}{\partial \overline{z}} \frac{1}{z - z_0} = \frac{\partial f}{\partial \overline{z}} \cdot \frac{1}{z - z_0}.$$

In the above equation we have used the complex version of the Cauchy-Riemann condition in Equation (7.6).

We applying the complex Green's formula to g on B_α to obtain

$$\int_{\partial B_\delta} \frac{f(z)}{z-z_0}\,dz = \iint_{B_\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz. \tag{7.10}$$

Now we want to take limits of the above equation as δ tends to 0 from above.

$$\begin{aligned}
\lim_{\delta\to 0} \int_{\partial B_\delta} \frac{f(z)}{z-z_0}\,dz &= \lim_{\delta\to 0}\Big(\int_\gamma \frac{f(z)}{z-z_0}\,dz - \int_{|z-z_0|=\delta} \frac{f(z)}{z-z_0}\,dz\Big) \\
&= \int_\gamma \frac{f(z)}{z-z_0}\,dz - \lim_{\delta\to 0}\int_{|z-z_0|=\delta} \frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\,dz \\
&= \int_\gamma \frac{f(z)}{z-z_0}\,dz - \lim_{\delta\to 0}\int_{|z-z_0|=\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz - 2\pi i f(z_0) \\
&= \int_\gamma \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0).
\end{aligned} \tag{7.11}$$

In the last line of the above equation we have used that $|f(z)-f(z_0)/(z-z_0)|$ is uniformly bounded from above.

On the other hand,

$$\iint_{B_\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz = \Big(\iint_{B} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz - \iint_{|z-z_0|\le\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz\Big)$$

and since f is C^1, and $|z-z_0|\le\delta$ is compact, there is a constant $C>0$ such that

$$\Big|\iint_{|z-z_0|\le\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz\Big| \le C\iint_{|z-z_0|\le\delta} \Big|\frac{1}{z-z_0}\Big| |d\bar{z}dz|$$

We can calculate the integral on the right hand side as in

$$\begin{aligned}
\iint_{|z-z_0|\le\delta} \Big|\frac{1}{z-z_0}\Big| |d\bar{z}dz| &= 2\iint_{|z-z_0|\le\delta} \Big|\frac{1}{z-z_0}\Big| dxdy \\
&= 2\int_0^{2\pi}\int_0^\delta \Big|\frac{1}{z-z_0}\Big| rdrd\theta = 4\pi\delta.
\end{aligned}$$

The above relations imply that

$$\lim_{\delta\to 0}\iint_{B_\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz = \iint_{B} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz. \tag{7.12}$$

Combining Equations 7.10, (7.11), and (7.12), we obtain the formula in the theorem.

□

Remark 7.8. In Theorem 7.7 it is not required to assume that f is C^1. This has an important consequence we state in Lemma 7.10. Below we give a brief argument how the statement is proved without assuming C^1 condition.

As we saw in Proposition 7.4, the ACL condition in quasi-conformality implies that the first partial derivatives of f exist and are finite at almost every point. If the first order partial derivatives are defined almost everywhere, the Jacobian of f, $\det Df$, is defined almost everywhere. Then, as f maps bounded sets to bounded set (that have bounded area), we conclude that $\det Df$ is locally in L^1. On the other hand,

$$|\partial f/\partial x|^2 \leq \max_{\theta\in[0,2\pi]} |D_\alpha f|^2 \leq (K min_{\theta\in[0,2\pi]}|D_\theta f|) \cdot max_{\theta\in[0,2\pi]}|D_\theta f| \leq K \det Df(z).$$

As $\det Df(z)$ belongs to L^1 locally, we conclude that $|\partial f/\partial x|$ belongs to L^2 locally. By a similar argument we conclude that $|\partial f/\partial y|$ also belongs to L^2. These imply that the derivatives $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ exist at almost every point and are integrable. So, the integrals in Theorem 7.7 are meaningful.

Corollary 7.9. *Let $f : \Omega \to \mathbb{C}$ be a C^1 map which is 1-quasi-conformal. Then, $f : \Omega \to \mathbb{C}$ is a conformal map.*

Proof. The condition 1-quasi-conformal implies that $\mu_f(z) = 0$ at almost every point in Ω. Hence, $\partial f/\partial \overline{z} = 0$ at almost every point. It follows from the formula in Theorem 7.7 that f satisfies the Cauchy integral formula, and therefore it is holomorphic. □

As we remarked in Remark 7.8, the C^1 condition is not required in Theorem 7.7. This stronger statement has an important consequence known as the Weyl's lemma. But the proof requires some standard real analysis that is not the prerequisite for this course!

Lemma 7.10 (Weyl's lemma)**.** *Any 1-quasi-conformal map $f : \Omega \to \mathbb{C}$ is conformal.*

Proposition 7.11. *If $f : \Omega_1 \to \Omega_2$ is K-quasi-conformal, $g : \Omega_0 \to \Omega_1$ is conformal, and $h : \Omega_2 \to \Omega_3$ is conformal, then $h \circ f \circ g : \Omega_0 \to \Omega_3$ is K-quasi-conformal.*

Proof. For the first part of the theorem we need to verify the two condition in Definition 7.5 for the map $h\circ f\circ g$. Let $A_1 \subset \Omega_1$ be the set of points where $K_f(z)$ is defined and bounded by K. As f is K-quasi-conformal, $\Omega_1 \setminus A_1$ has zero area. Define $A_0 = g^{-1}(A_1)$. It is easy to show that A_0 has zero area (use exhaustion of Ω_1 by compact sets, and use that $|g'|$ is bounded from above and below on each compact set).

Note that since g and h are holomorphic functions, by Equation (7.7), $K_g \equiv 1$ and $K_h \equiv 1$. Then, for every $w \in A_0$, by the inequality in Equation (7.3), we have

$$K_{h\circ f\circ g}(w) \leq K_g(w) \cdot K_f(g(w)) \cdot K_h(f \circ g(w)) = K_f(g(w)) \leq K.$$

This proves condition (ii) in the definition of quasi-conformality.

We need to prove that $h \circ f \circ g$ is ACL on Ω_0. Since g and h are C^1, they are ACL. In fact, for every rectangle bounded by horizontal and vertical sides, in their domain of definition, these maps are absolutely continuous on every horizontal and every vertical line. In fact, g and h are absolutely continuous on every piece-wise C^1 curves in their domain of definition. We also know that for every rectangle $R \subset \Omega_1$ bounded by horizontal and vertical sides, f is absolutely continuous on almost every horizontal and almost every every vertical line in R. With these properties, it is easy to see that $h \circ f$ is ACL. But the problem with $f \circ g$ is that g does not map horizontal lines to horizontal or vertical lines. And we do not a priori know that f is absolutely continuous on almost every analytic curves (these are images of a horizontal and vertical lines by g). As in Remark 7.8 we need to use some standard results from real analysis. That is, a homeomorphism f is ACL iff the first partial derivatives of f exist at almost every point in the domain of f and are locally in L^1. From this criterion it is easy to see that the composition of ACL homeomorphisms is ACL. (We skip the details as this requires material that are not the prerequisite for this course.) $\square$

Proposition 7.12. *If $f : \Omega_1 \to \Omega_2$ is K-quasi-conformal, $g : \Omega_0 \to \Omega_1$ is conformal, and $h : \Omega_2 \to \Omega_3$ is conformal, then for almost every $z \in \Omega_1$ and almost every $w \in \Omega_0$ we have*

$$\mu_{h \circ f}(z) = \mu_f(z), \quad \mu_{f \circ g}(w) = \Big(\frac{|g'(z)|}{g'(z)}\Big)^2 \cdot \mu_f(g(w)).$$

Proof. This is easy to see from the definition of μ in terms of the length of major and minor axis, and their direction. See Exercise 7.1 $\square$

Remark 7.13. Many theorems in complex analysis are valid, with some modifications, for quasi-conformal mappings. The Pompeiu formula is an example of such statements. In general, it is possible to show that the composition of quasi-conformal maps are quasi-conformal. If a sequence of K-quasi-conformal maps converges uniformly on compact sets to some function, then the limiting function is either constant or quasi-conformal. The class of K-quasi-conformal maps $f : \mathbb{C} \to \mathbb{C}$ normalized with $f(0) = 0$ and $f(1) = 1$ forms a normal family.

Quasi-conformal maps, in contrast to conformal maps, enjoy the flexibility that allows one to build such maps by hand. This makes them a powerful tool in complex analysis.

7.5 Beltrami equation

Given a diffeomorphism $f : \Omega \to \mathbb{C}$ with $\mu_f : \Omega \to \mathbb{D}$ one may look at f in Equation (7.8) as the solution of the differential equation

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z), \forall z \in \Omega. \tag{7.13}$$

That is, given a function $\mu : \Omega \to \mathbb{D}$, is there a diffeomorphism $f : \Omega \to \mathbb{C}$ such that the above equation holds. The above equation is called the *Beltrami equation*, and the function μ is called the *Beltrami coefficient* of f.

There is a geometric interpretation of the Beltrami equation similar to the solutions of vector fields in the plane. The function μ specifies a field of ellipses in Ω where at each $z \in \Omega$ the major axis of the ellipse has angle $(\arg \mu(z) + \pi)/2$ and size $1/(1 - |\mu(z)|)$. The minor axis of the ellipse at z has angle $\mu(z)/2$ and has size $1/(1 + |\mu(z)|)$. The solution f of the above equation is a diffeomorphism that infinitesimally maps the field of ellipses to the field of round circles.

The Beltrami equation has a long history. It was already considered by Gauss in 1820's in connection with a seemingly different problem of finding isothermal coordinates on a surface for real analytic maps. Most of the developments in the study of the Beltrami equation took place in $1950's$. These mainly focused on reducing the regularity condition required for f; see Remark 7.15.

Theorem 7.14. *[Measurable Riemann mapping theorem-continuous version] Let $\mu : \mathbb{C} \to \mathbb{D}$ be a continuous map with $\sup_{z\in\mathbb{C}} |\mu(z)| < 1$. Then, there is a quasi-conformal map $f : \mathbb{C} \to \mathbb{C}$ such that the Beltrami equation* (7.13) *holds on $\mathbb{C}$.*

Moreover, the solution f is unique if we assume that $f(0) = 0$ and $f(1) = 1$.

Remark 7.15. The condition of continuity of μ in the above theorem is not necessary. The sufficient condition is that μ is measurable and $\|\mu\|_\infty < 1$. This result is known as the measurable Riemann mapping theorem, and has many important consequences.

The relation between the regularity of the solution and the regularity of μ is not simple. For example, if μ is Hölder continuous, then the solution becomes a diffeomorphism. But, this condition is far from necessary. There are discontinuous functions μ where the solution is diffeomorphism.

7.6 An application of MRMT

In the theory of dynamical systems one wishes to understand the behavior of the sequences of points generated by consecutively applying a map at a given point. That is, if $g : X \to X$, and $x_0 \in X$, one studies the sequence $\{x_n\}$ defined as $x_{n+1} = g(x_n)$. This is called *the orbit* of x_0 under g. We shall look at the special case when $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Recall the homeomorphism π from the unit sphere $S \subset \mathbb{R}^3$ to the Riemann sphere $\hat{\mathbb{C}}$ we discussed in Section 3.1. There is a spherical metric d' on S which is defined as the Euclidean length of the shortest curve on S. We may use π and d' to define a metric on $\hat{\mathbb{C}}$ as $d(z, w) = d'(\pi^{-1}(z), \pi^{-1}(w))$.

We may naturally extend the notion of normal families for holomorphic maps of $\mathbb{C}$ we presented in Definition 5.9 to holomorphic maps of $\hat{\mathbb{C}}$. Let Ω be an open set in $\hat{\mathbb{C}}$ and $f_n : \Omega \to \hat{\mathbb{C}}$ be a sequence of maps. We say that f_n converges uniformly on E to $g : E \to \hat{\mathbb{C}}$, if for every $\varepsilon > 0$ there is $n_0 \geq 1$ such that for all $n \geq n_0$ and all $z \in E$ we have $d(f_n(z), g(z)) < \varepsilon$.

Definition 7.16. Let Ω be an open set in $\hat{\mathbb{C}}$ and $\mathcal{F}$ be a family (set) of maps $f : \Omega \to \hat{\mathbb{C}}$. We say that the family $\mathcal{F}$ is *normal*, if every sequence of maps in $\mathcal{F}$ has a sub-sequence which converges uniformly on compact subset of Ω to some $g : \Omega \to \hat{\mathbb{C}}$.

Given a holomorphic map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and any integer $n \geq 1$ we may compose the map R with itself n times to obtain a map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. We use the notation $R^{\circ n}$ to denote this n-fold composition.

Definition 7.17. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. We say that $z \in \hat{\mathbb{C}}$ is *stable* for the iterates $R^{\circ n}$, $n \geq 1$, if there is an open set $U \subset \hat{\mathbb{C}}$ containing z such that the family $\{R^{\circ n}\}_{n=0}^{\infty}$ restricted to U forms a normal family.

By the above definition, the set of stable points of a rational function forms an open subset of $\hat{\mathbb{C}}$. The set of all stable points of a rational function R is called the *Fatou* set of R, and denoted here by $\mathcal{F}(R)$. The complement of the Fatou set, $\hat{\mathbb{C}} \setminus \mathcal{F}(R)$, which is a closed subset of $\hat{\mathbb{C}}$, is called the *Julia* set of R. This is denoted by $\mathcal{J}(R)$. These are named after the pioneering works of P. Fatou and G. Julia in 1920's on properties of these sets.

Lemma 7.18. *Let $R(z) = z^d$, for some integer $d \geq 2$. Then $\mathcal{J}(R) = \partial\mathbb{D}$.*

Proof. First we show that the open disk $\mathbb{D}$ is contained in $\mathcal{F}(R)$. To see this, let E be an arbitrary compact set in $\mathbb{D}$. There is $r \in (0, 1)$ such that $E \subset B(0, r)$. Then, for every $w \in E$ we have $|R^{\circ n}(w)| = |w^{d^n}| \leq r^{d^n} \to 0$, as n tends to infinity. That is, the iterates $R^{\circ n}$ converges uniformly on E to the constant function 0. As E was an arbitrary compact set in $\mathbb{D}$, we conclude that $R^{\circ n}$ converges uniformly on compact subsets of $\mathbb{D}$ to the constant function 0.

By a similar argument, the iterates $R^{\circ n}$ converges uniformly on compact subsets of $\hat{\mathbb{C}} \setminus (\mathbb{D} \cup \partial\mathbb{D})$ to the constant function ∞.

By the above two paragraphs, $\hat{\mathbb{C}} \setminus \partial\mathbb{D}$ is contained in $\mathcal{F}(R)$. On the other hand, let $z \in \partial\mathbb{D}$ and U be an arbitrary neighborhood of z. For w in U with $|w| > 1$ we have $R^{\circ n}(w) \to \infty$ and for $w \in U$ with $|w| < 1$ we have $R^{\circ n}(w) \to 0$. Thus, there is no subsequence of $R^{\circ n}$ that converges to some continuous function on U. As U was arbitrary, we conclude that $z \notin \mathcal{F}(R)$. Then, $z \in \mathcal{J}(F)$. ☐

The above example is a very special case where the Julia set has a simple structure (is smooth). For a typical rational map the Julia set has a rather complicated structure, see some examples of Julia sets in Figure 7.6. The self-similarity of the figures is due to the invariance of $\mathcal{J}(R)$ under R we state below.

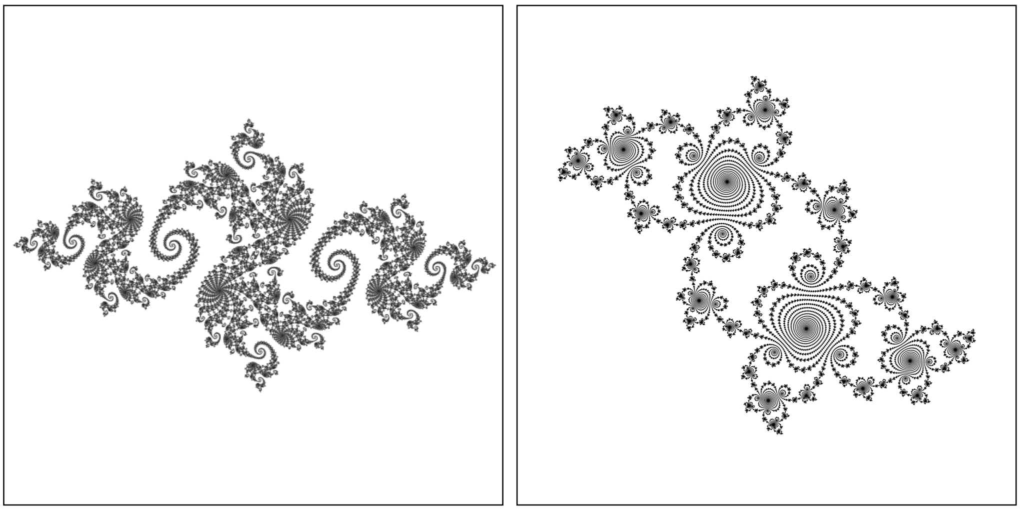

Figure 7.1: Two examples of Julia sets.

Lemma 7.19. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then, $z \in \mathcal{F}(R)$ if and only if $R(z) \in \mathcal{F}(R)$.*

Proof. See Exercise 7.5. ☐

By the above lemma $R^{-1}(\mathcal{F}(R)) = \mathcal{F}(R)$, which implies $R^{-1}(\mathcal{J}(R)) = \mathcal{J}(R)$.

Assume that the Fatou set of some $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is not empty. Let U_0 be a connected component of $\mathcal{F}(R)$. There is a sequence R^{n_k}, $k \geq 1$, that converges to some $g : U_0 \to \hat{\mathbb{C}}$. Then the function g describes the limiting behavior of the orbit $R^{\circ n_k}(z)$. But, to understand the behavior of the orbit of z one needs to know all limiting functions of convergent sub-sequences of $R^{\circ n}$.

Let U be a connected components of $\mathcal{F}(R)$. It follows from Lemma 7.19 that $R(U)$ is a components of $\mathcal{F}(R)$ which may or may not be distinct from U.

Definition 7.20. A component U of $\mathcal{F}(R)$ is called *wandering*, if $R^{\circ i}(U) \cap R^{\circ j}(U) = \emptyset$, for distinct integers i and j. A component U of $\mathcal{F}(R)$ is called *eventually periodic*, if there are positive integers $i \geq 0$ and $p \geq 1$ such that $R^{\circ(i+p)}(U) = R^{\circ i}(U)$.

By definition, if a Fatou component is not wandering, then it is eventually periodic. In 1985, D. Sullivan established the following remarkable property that settled a conjecture of Fatou from 1920's.

Theorem 7.21 (No wandering domain)**.** *Let U be a connected component of the Fatou set of a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then, U is eventually periodic.*

Remark 7.22. Theorem 7.21 is a major step towards characterizing the limiting functions of the iterates $R^{\circ n}$. When, $R^{\circ(i+p)}(U) = R^{\circ i}(U)$. The map $h = R^{\circ p}$ is a holomorphic map from $V = R^{\circ i}(U)$ to V. This allows one to study all possible limits of the iterates $R^{\circ n}$ on U. For example when V is a simply connected subset of $\hat{\mathbb{C}}$ one has Exercise 3.6.

The complete proof requires some advanced knowledge of quasi-conformal mappings. However, we present an sketch of the argument in the class, only emphasizing the use of the measurable Riemann mapping theorem.

7.7 Exercises

Exercise 7.1. Prove Proposition 7.12.

Exercise 7.2. Let Ω_1, Ω_2, and Ω_3 be open sets in $\mathbb{C}$. Assume that $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ are C^1 maps. With the notations $z \in \Omega_1$ and $w = f(z) \in \Omega_2$, prove the complex chain rules,

$$\frac{\partial(g \circ f)}{\partial z} = (\frac{\partial g}{\partial w} \circ f) \cdot \frac{\partial f}{\partial z} + (\frac{\partial g}{\partial \overline{w}} \circ f) \cdot \frac{\partial \overline{f}}{\partial z},$$

and

$$\frac{\partial(g \circ f)}{\partial \overline{z}} = (\frac{\partial g}{\partial w} \circ f) \cdot \frac{\partial f}{\partial \overline{z}} + (\frac{\partial g}{\partial \overline{w}} \circ f) \cdot \frac{\partial \overline{f}}{\partial \overline{z}}.$$

Exercise 7.3. Assume that $\mu : \mathbb{C} \to \mathbb{D}$ is a continuous map with $\sum_{z\in\mathbb{C}} |\mu(z)| < 1$, and $f, g : \mathbf{C} \to \mathbb{C}$ are diffeomorphisms with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ that satisfying the Beltrami equation. Prove that $f(z) = g(z)$ for all $z \in \mathbb{C}$. [This is a special case of the uniqueness part in Theorem 7.14.]

Exercise 7.4. We say that a function $f : [a, b] \to \mathbb{C}$ has *bounded variation*, if

$$\sup \Big\{ \sum_{i=1}^{N} |f(x_{i+1}) - f(x_i)| \;\Big|\; a = x_1 < x_2 < x_3 < \cdots < x_{N+1} = b, N \in \mathbb{N} \Big\} < \infty.$$

Prove that if $f : [a, b] \to \mathbb{C}$ is absolutely continuous, then f has bounded variation on $[a, b]$.

Exercise 7.5. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ration map. Prove that $R^{-1}(\mathcal{F}(R)) = \mathcal{F}(R)$. Then, conclude that $R^{-1}(\mathcal{J}(R)) = \mathcal{J}(R)$.

Exercise 7.6. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Assume that there is $n \in \mathbb{N}$ and $z \in \hat{\mathbb{C}}$ such that $R^{\circ n}(z) = z$. Prove that

(i) if $|(R^{\circ n})'(z)| < 1$, then z belongs to $\mathcal{F}(R)$;

(ii) if $|(R^{\circ n})'(z)| > 1$, then z belongs to $\mathcal{J}(R)$;

(iii) if $|(R^{\circ n})'(z)| = e^{2\pi i p/q}$ for some $p/q \in \mathbb{Q}$, then z belongs to $\mathcal{J}(R)$. [hint: first consider the case $n = 1$ and look at $(R^{\circ n})''(z)$ as n tends to infinity.