Chapter 5

Conformal Mappings

In the previous chapters we studied automorphisms of $\mathbb{D}$, and the geometric behavior of holomorphic maps from $\mathbb{D}$ to $\mathbb{D}$ using the Poincaré metric. A natural question is whether similar methods can be used for other domains in $\mathbb{C}$. A possible approach is the idea we used to describe $\mathrm{Aut}(\mathbb{H})$. To employ that idea for an open set $\Omega \subset \mathbb{C}$ we need a biholomorphic map $f : \mathbb{D} \to \Omega$. Then, elements of $\mathrm{Aut}(\Omega)$ obtain the form $f \circ \varphi \circ f^{-1}$, for $\varphi \in \mathrm{Aut}(\mathbb{D})$, and the Poincaré metric can be pulled back by f^{-1} to a conformal metric on Ω, etc. To carry out this idea we face the following key questions:

(i) for which domains $\Omega \subseteq \mathbb{C}$ there is a biholomorphic map from $\mathbb{D}$ to Ω;

(ii) if the answer to question (i) is positive for some Ω, when is there an explicit biholomorphic map from $\mathbb{D}$ to Ω. For instance, a biholomorphic map given by an algebraic formula, trigonometric functions, or a combination of such maps;

(iii) what if there are no elementary biholomorphic maps from $\mathbb{D}$ to Ω, but a biholomorphic map exists.

In this chapter we study the questions in parts (i) and (ii). We shall study the question in part (iii) in the next chapters.

5.1 Conformal mappings of special domains

Example 5.1. The exponential map $z \mapsto e^z = e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y)$, where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$. The exp map is biholomorphic from the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \pi\}$ to the upper half plane. It maps horizontal lines $\mathrm{Im}\, z = y_0$ to straight rays $\{z \in \mathbb{C} : \arg z = y_0\}$, and the vertical lines $\mathrm{Re}\, z = x_0$ to the arcs $\{z \in \mathbb{C} : |z| = e^{x_0}, \mathrm{Im}\, z > 0\}$.

The inverse of exp is log which is only determined up to translations by $2\pi i$. We often fix a branch of the inverse to determine which inverse of exp we are considering. For example, to map $\mathbb{H}$ to the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \pi\}$ we may work with the branch that

$0 < \operatorname{Im} \log w < 2\pi$. If we write $z = re^{i\theta}$, then

$$\log z = \log r + i\theta = \log |z| + i \arg z.$$

Also, the restriction of log provides a biholomorphic map

$$\log : \{z \in \mathbb{C} \mid |z| < 1, 0 < \operatorname{Im} z\} \to \{z \in \mathbb{C} \mid \operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi\}.$$

Example 5.2. Let n be a positive integer. The power map $p_n(z) = z^n$ is biholomorphic from the sector $S_n = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/n\}$ to the upper half plane. The inverse of this map is given by $w \mapsto w^{1/n}$. This is defined using a branch of log as $w \mapsto e^{\frac{1}{n} \cdot \log w}$, where $0 < \operatorname{Im} \log(w) < 2\pi$.

The map p_n has a rather simple behavior on S_n. To see this, we note that S_n is the union of the straight rays $R_\theta = \{re^{i\theta} : r > 0\}$, for $0 < \theta < \pi/n$. Then, p_n maps R_θ to $R_{n\theta}$. Also, we may consider S_n as the union of the arcs $C_r = \{re^{i\theta} : 0 < \theta < \pi/n\}$, for $r > 0$. Then p_n maps C_r to the arc $\{r^n e^{i\theta} : 0 < \theta < \pi\}$.

In general, for $\alpha > 0$ the power map $p_\alpha(z) = z^\alpha = e^{\alpha \log z}$ is defined and biholomorphic from the sector $\{z \in \mathbb{C} : 0 < \arg(z) < \pi/\alpha\}$ to the upper half plane. The inverse of this map is given by the formula $w \mapsto w^{1/\alpha} = e^{\frac{1}{\alpha} \log w}$, where $0 < \operatorname{Im} \log(w) < 2\pi$.

Example 5.3. Recall the biholomorphic map $G(w) = i\frac{1-w}{1+w}$ from $\mathbb{D}$ to $\mathbb{H}$ we introduced in Equation 2.1. The restriction of this map provides a biholomorphic map

$$G : \{z \in \mathbb{D} : \operatorname{Im} z > 0\} \to \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}.$$

The composition of G with the map $z \mapsto z^2$ provides a biholomorphic map from the upper half-disk to $\mathbb{H}$.

Example 5.4. Let $f(z) = z + 1/z$. For non-zero $z \in \mathbb{C}$, z and $1/z$ are mapped to the same point . On the other hand, for each non-zero $w_0 \in \mathbb{C}$, the equation $f(z) = w_0$ reduces to $z^2 - w_0 z + 1 = 0$ that has two solutions, counted with multiplicity. The two solutions are the same if and only if $z = 1/z$, which is only possible if $z = \pm 1$. This implies that f is one-to-one on the set $\{z \in \mathbb{C} : |z| > 1\}$.

When $|z| = 1$,

$$f(z) = z + 1/z = z + \overline{z} = 2 \operatorname{Re} z.$$

That is, f maps the circle $|z| = 1$ in a two-to-one fashion to the interval $[-2, +2]$. Then, the restriction of f to the set $|z| > 1$ covers $\mathbb{C} \setminus [-2, +2]$. It follows that

$$f : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C} \setminus [-2, +2].$$

is biholomorphic. By the same arguments,

$$f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus [-2, +2]$$

is also biholomorphic. Since the map $w \mapsto 1/w$ is biholomorphic from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$, the map

$$z \mapsto \frac{1}{f(z)} = \frac{z}{1+z^2} : \mathbb{D} \to \mathbb{C} \setminus \big((-\infty, -1/2] \cup [1/2, +\infty)\big)$$

is biholomorphic.

Example 5.5. The Koebe map

$$k(z) = \frac{z}{(1-z)^2} : \mathbb{D} \to \mathbb{C} \setminus (-\infty, -1/4]$$

is biholomorphic. To see this, we write

$$k(z) = \frac{1}{4}\Big(\frac{1+z}{1-z}\Big)^2 - \frac{1}{4}.$$

and observe that $(1+z)/(1-z)$ is biholomorphic from $\mathbb{D}$ to the right half-plane $\operatorname{Re} z > 0$. As we shall see in the next chapter, the Koebe function has some extreme behavior among biholomorphic maps defined on $\mathbb{D}$.

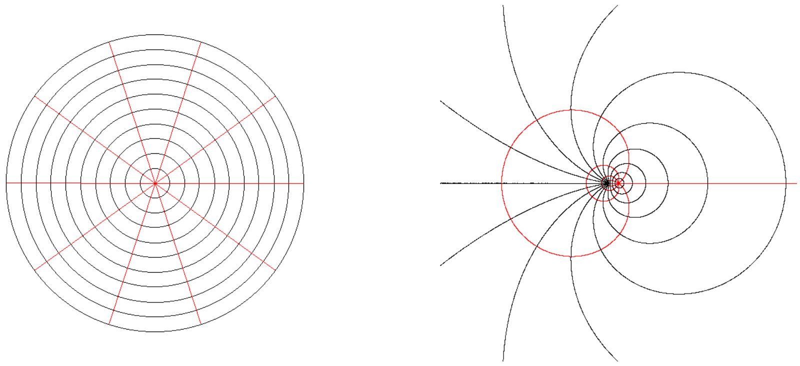

Figure 5.1: The images of the rays and circles by the Koebe function discussed in Example 5.5.

Example 5.6. Using the relations $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ we obtain a formula for the sine function $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, for real values of θ. This can be used to extend sin onto the whole complex plane, i.e.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Let $g(\zeta) = -i \log \zeta$. We have

$$\sin \circ g(\zeta) = \frac{1}{2i}\Big(\zeta - \frac{1}{\zeta}\Big) = \frac{-1}{2}\Big(i\zeta + \frac{1}{i\zeta}\Big) = \frac{-1}{2} f(i\zeta),$$

where f is the function in Example 5.4. Using Example 5.4, we obtain a biholomorphic map

$$\sin : \{z \in \mathbb{C} : -\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\} \to \mathbb{H}.$$

See Figure 5.6

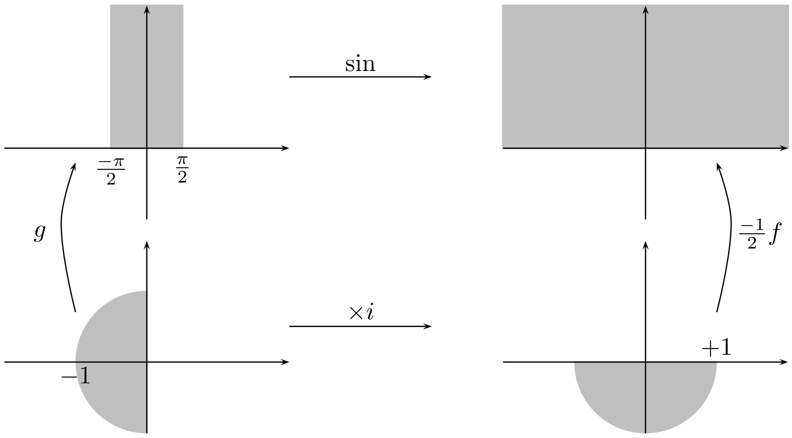

Figure 5.2: The sine function.

5.2 Normal families, Montel's theorem

Let Ω be an open set in $\mathbb{C}$. We would like to introduce a limiting process to build new holomorphic maps on Ω using limits of known holomorphic maps (just like how one build real numbers as limits of rational numbers). There are many notions of convergence of functions in analysis, but it turns out that the natural notion of limit in this setting is the "uniform convergence on compact set".

Definition 5.7. Let $f_n : \Omega \to \mathbb{C}$, for $n \geq 1$, be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. Assume that E is a subset of Ω. We say that the sequence f_n *converges uniformly on* E to some function $f : E \to \mathbb{C}$, if for every $\varepsilon > 0$ there is n_0 such that for all $n \geq n_0$ and all $z \in E$ we have $|f_n(z) - f(z)| \leq \varepsilon$.

Note that the above convergence is stronger than the *point-wise convergence* where we say that the sequence f_n converges to f on E if for every $z \in E$ and every $\varepsilon > 0$ there is

n_0 such that for all $n \geq n_0$ we have $|f_n(z) - f(z)| \leq \varepsilon$. Here ε may depend on z, but in the uniform convergence ε works for all $z \in E$. For example, the functions $f_n(z) = (1 + 1/n)z$ converge to the function $f(z) = z$ at every point $z \in \mathbb{C}$ bu the convergence is not uniform on unbounded sets $E \subset \mathbb{C}$.

Definition 5.8. Let $f_n : \Omega \to \mathbb{C}$, for $n \geq 1$, be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. We say that f_n *converges uniformly on compact sets* to $f : E \to \mathbb{C}$, if for every compact set $E \subset \Omega$ the sequence f_n converges uniformly to f on E.

It is a simple exercise to show that if $f_n : \Omega \to \mathbb{C}$, $n \geq 1$, are continuous and converge uniformly on compact sets to some $f : \Omega \to \mathbb{C}$, then $f : \Omega \to \mathbb{C}$ is also continuous. We shall prove in Theorem 5.10 that the holomorphic property also survives under convergence on compact sets.

The Cauchy's criterion also has a counter part here. The sequence f_n converges uniformly on E if and only if for every $\varepsilon > 0$ there is an $n_0 > 0$ such that for all $n, m \geq n_0$ and all $z \in E$ we have $|f_n(z) - f_m(z)| \leq \varepsilon$.

Definition 5.9. Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ be a family (set) of maps that are defined on Ω. We say that the family $\mathcal{F}$ is *normal* if every sequence of maps f_n, $n \geq 1$, in $\mathcal{F}$ has a sub-sequence that converges uniformly on every compact subset of Ω.

Note that in the above definition we do not require the limiting map to be in $\mathcal{F}$. This is often a consequence of the uniform convergence. For instance, when the maps involved in the definition are holomorphic we may use the following theorem of Weierstrass.

For example, you can show that the sequence of functions $f_n(z) = z^n$ is normal on $\mathbb{D}$. But this sequence is not normal on the ball $|z| < 2$.

Theorem 5.10. *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. Assume that the sequence f_n converges uniformly on compact sets to some function $f : \Omega \to \mathbb{C}$. Then, f is holomorphic on Ω.*

Moreover, $f_n' : \Omega \to \mathbb{C}$ converges uniformly on compact sets to $f' : \Omega \to \mathbb{C}$.

Proof. Let z_0 be an arbitrary point in Ω and choose $r > 0$ such that the disk $|z - z_0| \leq r$ is contained in Ω. Let us denote the circle $|z - z_0| = r$ by γ. Since each f_n is holomorphic on Ω, by Cauchy integral formula, for every z_1 in the disk $|z - z_0| < r$ we have

$$f_n(z_1) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(z)}{z - z_1} \, dz.$$

We wish to take limits as n tends to infinity. To that end we observe that

$$\begin{aligned}
\Big|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-z_1}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_1}\,dz\Big| &= \Big|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)-f(z)}{z-z_1}\,dz\Big| \\
&\leq \frac{1}{2\pi}\int_\gamma \Big|\frac{f_n(z)-f(z)}{z-z_1}\Big|\,|dz| \qquad (5.1)\\
&\leq \frac{r}{r-|z_1-z_0|}\max_{z\in\gamma}|f_n(z)-f(z)|.
\end{aligned}$$

The expression on the last line of the above equation tends to 0 as n tends to ∞. That is because γ is compact in Ω. Taking limits in the Cauchy integral formula for f_n, we obtain

$$f(z_1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_1}\,dz.$$

Note that the above equation holds for every point z_1 enclosed by γ. It is easy to conclude from the above formula that

$$f'(z_1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_1)^2}\,dz. \qquad (5.2)$$

In particular, $f'(z_1)$ exists and is a continuous function of z_1, as long as z_1 moves within the disk $|z-z_0|<r$. That is, $f(z)$ is holomorphic on the disk $|z-z_0|<r$. Since $z_0\in\Omega$ was arbitrary we conclude that f is holomorphic on Ω.

To prove the uniform convergence in the last part of the theorem, let E be a compact set in Ω. For every $z\in E$ there is $r_z>0$ such that the closure of the disk $B(z,2r_z)$ is contained in Ω. Thus the union of $B(z,r_z)$, for $z\in E$, provides a cover of E by open sets. Since E is compact, a finite number of such open sets covers E. Let $z_i\in E$ and $r_i>0$ be a finite collection such that the closure of each $B(z_i,2r_i)$ is contained in Ω and $E\subset\cup_i B(z_i,r_i)$.

Fix an arbitrary i and let γ_i denote the circle $|z-z_i|=2r_i$. We may repeat the inequalities in Equation 5.1 for the integral formula in Equation 5.2 to conclude that for every $z\in B(z_i,r_i)$ we have

$$\begin{aligned}
|f_n'(z)-f'(z)| &= \Big|\frac{1}{2\pi i}\int_{\gamma_i}\frac{f_n(\zeta)}{(\zeta-z)^2}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\Big| \\
&\leq \frac{1}{r_i}\max_{\zeta\in\gamma_i}|f_n(\zeta)-f(\zeta)|.
\end{aligned}$$

Since f_n converges to f uniformly on the compact set γ_i, the above inequality implies that $f_n'(z)$ converges to $f'(z)$ uniformly on the ball $B(z_i,r_i)$. As there are a finite number of such balls that covers E, and the convergence on each ball is uniform, we conclude that f_n' converges to f' uniformly on E. $\square$

When $\mathcal{F}$ is a class of one-to-one maps on Ω, we need the following theorem of A. Hurwitz to conclude that the limiting maps are also one-to-one.

Theorem 5.11. *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$ such that $f_n(z) \neq 0$ for all n and all $z \in \Omega$. If f_n converges uniformly on compact subsets of Ω to some $f : \Omega \to \mathbb{C}$, then either f is identically equal to 0 or has no 0 on Ω.*

Proof. By Theorem 5.10, f is holomorphic on Ω. Thus, by Proposition 3.12, either $f(z) \equiv 0$, or the set of solutions of $f(z) = 0$ forms a discrete subset of Ω. If the latter happens, we show that the set of solutions is empty.

Let z_0 be a solution of $f(z) = 0$. By the above paragraph, there is $r > 0$ such that the ball $B(z_0, r) \subset \Omega$ and the equation $f(z) = 0$ has no solution with $0 < |z - z_0| \leq r$. Let γ denote the circle $|z - z_0| = r$. As γ is compact in Ω, f_n converges uniformly on γ to f. Also, by Theorem 5.10, f_n' converges uniformly on γ to f'. Then, it follows that

$$\lim_{n \to +\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

For each $n \geq 1$, $f_n(z) = 0$ has no solutions in Ω. It follows that $f_n'(z)/f_n(z)$ is holomorphic on Ω. In particular, by Cauchy-Goursat Theorem (Thm 1.2), the integrals on the left hand side of the above equation are equal to 0. Then, the right hand side integral is equal to 0. However, the integral on the right hand side counts the number of points z within γ such that $f(z) = 0$. That is, $f(z) = 0$ has no solution within γ. This contradiction proves that $f(z) = 0$ has no solutions in Ω. $\square$

Corollary 5.12. *Let Ω be an open set in $\mathbb{C}$ and let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions that converge uniformly on compact sets to some function $f : \Omega \to \mathbb{C}$. If every f_n is one-to-one on Ω, then either f is a constant function, or f is one-to-one on Ω.*

Proof. Assume that there are distinct points a and b in Ω with $f(a) = f(b)$. Since each f_n is one to one on Ω, $f_n(z) - f(b)$ does not vanish on the ball $|z - a| < r$. Therefore, by Theorem 5.11, either $f(z) - f(b)$ has no zero in $|z - a| < r$ or is identically constant. As a is a zero of this function, f is a constant function. $\square$

It is often possible that a sequence of holomorphic maps has no convergent sub-sequence, or the point-wise limit exists but is not holomorphic. We are looking for criteria on a family that imply a convergent (uniformly on compact sets) sub-sequence exists.

There are some natural conditions that a family of maps must fulfill in order to be uniformly convergent on compact sets. We state these as definitions.

Definition 5.13. Let $\mathcal{F}$ be a family of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. We say that the family $\mathcal{F}$ is *uniformly bounded on compact subsets* of Ω, if for every compact set $E \subset \Omega$, there is a constant M such that for all $z \in E$ and all $f \in \mathcal{F}$ we have $|f(z)| \leq M$.

Definition 5.14. Let $\mathcal{F}$ be a family of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$, and let $K \subseteq \Omega$. We say that the family $\mathcal{F}$ is *equicontinuous* on K, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all z and z' in Ω with $|z - z'| \leq \delta$ and all $f \in \mathcal{F}$ we have $|f(z) - f(z')| \leq \varepsilon$.

By the above definition, each map in an equicontinuous family is uniformly continuous.

One can see that any normal family must satisfy the properties in Definitions 5.13 and 5.14. It turns out that the condition in Definition 5.13 is the key to the normality of a family. The following theorem is due to P. Montel.

Theorem 5.15. *Let $\mathcal{F}$ be a family of holomorphic maps defined on an open set $\Omega \subseteq \mathbb{C}$. If $\mathcal{F}$ is uniformly bounded on every compact subset of Ω, then*

(i) $\mathcal{F}$ is equicontinuous on every compact subset of Ω;

(ii) $\mathcal{F}$ is a normal family.

Before we prove the above theorem we give a basic property of compact and closed sets in the plane.

Lemma 5.16. *Let A be a compact set in $\mathbb{C}$ and B be a closed set in $\mathbb{C}$ such that $A \neq \emptyset$, $B \neq \emptyset$, and $A \cap B = \emptyset$. There is $r > 0$ such that for every $z \in A$ and $w \in B$ we have $|z - w| > r$.*

Proof. If there is no such $r > 0$, for each $n \geq 1$ there are $z_n \in A$ and $w_n \in B$ such that $|z_n - w_n| \leq 1/n$. Since A is compact, there is a sub-sequence of z_n, say z_{n_k}, that converges to some $z \in A$. The sequence w_{n_k} is bounded since $|w_{n_k}| \leq |w_{n_k} - z_{n_k}| + |z_{n_k}| \leq 1 + |z_{n_k}|$, and z_{n_k} belong to a compact set. Thus, there is a sub-sequence of w_{n_k} that converges to some w. As B is closed, $w \in B$, and since $|z_{n_k} - w_{n_k}| \leq 1/n$ we have $w = z$. This contradicts $A \cap B = \emptyset$. $\square$

The above lemma allows us to define the distance between a non-empty compact set and a non-empty closed set in $\mathbb{C}$. That is,

$$d(A, B) = \inf\{|z - w| : z \in A, w \in B\}.$$

The above set is bounded from below, and by Lemma 5.16, the infimum is strictly positive when $A \cap B = \emptyset$.

Proof of Theorem 5.15. Part (i): Let K be an arbitrary compact set in Ω. By Lemma 5.16 there is $r > 0$ such that for every $z \in K$ the ball $B(z, 3r) \subset \Omega$.

Let z and w be in K with $|z-w| \leq r$. By the choice of r, the closure of the ball $B(z, 2r)$ is contained in Ω. Let γ_z denote the boundary of the ball $B(z, 2r)$. By the Cauchy integral formula, for every $f \in \mathcal{F}$ we have

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\gamma_z} f(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\Big)\, d\zeta.$$

We note that for $\zeta \in \gamma_z$ we have

$$\Big|\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\Big| \leq \frac{|z - w|}{|\zeta - z||\zeta - w|} \leq \frac{|z - w|}{r^2}.$$

Therefore,

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} |z - w| \sup_{\zeta \in \gamma_z} \cdot |f(\zeta)|.$$

Define

$$\Gamma_K = \{a \in \Omega : d(a, K) \leq 2r\}.$$

This is a compact subset of Ω. By the hypothesis of the theorem, there is $C > 0$, depending on Γ_K, such that for all $f \in \mathcal{F}$ and every $a \in \Gamma_K$ we have $|f(a)| \leq C$. In particular, $\sup_{\zeta \in \gamma_z} |f(\zeta)| \leq C$.

By the above paragraphs, we have shown that for all $f \in \mathcal{F}$and all z and w in K with $|z-w| \leq r$ we have $|f(z)-f(w)| \leq C/r$. This implies that the family $\mathcal{F}$ is equicontinuous on K (given $\varepsilon > 0$ let $\delta = \min\{r, r\varepsilon/C\}$). As K was an arbitrary compact set in Ω, we have proved the first part of the theorem.

Part (ii): Let f_n be an arbitrary sequence in $\mathcal{F}$. There is a sequence of points $\{w_i\}$, for $i \geq 1$, that is dense in Ω. We first extract a sub-sequence of f_n that converges at each of these points w_j. The process we are going to use is known as the Cantor's diagonal process.

By the hypothesis of the theorem, $|f_n(w_1)|$, for $n \geq 1$, is uniformly bounded from above. Hence there is a sub-sequence $f_{1,1}(w_1), f_{2,1}(w_1), f_{3,1}(w_1), \dots$ of this sequence that converges to some point in $\mathbb{C}$. For the same reason, there is a sub-sequence $f_{1,2}, f_{2,2}, f_{3,2}, \dots$ of the sequence $f_{1,1}, f_{2,1}, f_{3,1}, \dots$ such that $\lim_{k\to\infty} f_{k,2}(w_2)$ exists in $\mathbb{C}$. Inductively, for $l \geq 1$, we build a sub-sequence $f_{1,l}, f_{2,l}, f_{3,l}, \dots$ of the sequence $f_{1,l-1}, f_{2,l-1}, f_{3,l-1}, \dots$ such that $\lim_{k\to\infty} f_{k,l}(w_l)$ exists in $\mathbb{C}$.

Let us define the sequence of maps $g_n = f_{n,n}$, for $n \geq 1$. This is a sub-sequence of $\{f_n\}$, and for each $j \geq 1$ the limit $\lim_{n\to\infty} g_n(w_j)$ exists and is finite. We are going to show that this sequence is uniformly convergent on compact sets of Ω. To this end we shall show that this sequence is Cauchy on compact sets. Let K be an arbitrary compact set in Ω and fix $\varepsilon > 0$.

Since $\mathcal{F}$ is equicontinuous on K, for $\varepsilon/3$ there is $\delta > 0$ such that for all $f \in \mathcal{F}$ and every z and w in K with $|z - w| < \delta$ we have $|f(z) - f(w)| \leq \varepsilon/3$. Since K is compact, there are a finite number of points $w_1, w_2, \dots, w_l$ such that $K \subset \cup_{i=1}^{l} B(w_i, \delta)$.

The sequence $\lim_{n\to\infty} g_n(w_i)$ exists for each $i = 1, 2, 3, \dots, l$. In particular each of these sequences is Cauchy. As there are a finite number of points w_i, given $\varepsilon/3$ there is $N > 0$ such that for all $m, n \geq N$ and all $i = 1, 2, 3, \dots, l$ we have

$$|g_n(w_i) - g_m(w_i)| \leq \varepsilon/3.$$

Now, for an arbitrary $w \in K$ there is $i \in \{1, 2, 3, \dots, l\}$ with $w \in B(w_i, \delta/2)$. Then for all $n, m \geq N$, we have

$$\begin{aligned}|g_n(w) - g_m(w)| &\leq |g_n(w) - g_n(w_i)| + |g_n(w_i) - g_m(w_i)| + |g_m(w_i) - g_m(w)| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

The above property implies that for each $w \in \Omega$ the sequence of points $g_n(w)$ is convergent. So, the sequence g_n converges at every point in Ω. The convergence is uniform on compact sets. □

Corollary 5.17. *Let $\mathcal{F}$ be a family of holomorphic maps $f : \Omega \to \mathbb{D}$. Then, $\mathcal{F}$ is a normal family.*

The notion of normality corresponds to the notion of pre-compactness in a metric space. To explain the correspondence we define a notion of metric on holomorphic maps such that uniform convergence on compact sets becomes equivalent to the convergence with respect to that metric.

Lemma 5.18. *Let Ω be an open set in $\mathbb{C}$. There are compact sets $E_i \subset \Omega$, for $i \in \mathbb{N}$, such that*

(i) for all $i \geq 1$, E_i is contained in the interior of E_{i+1};

(ii) for every compact set E in Ω there is $i \geq 1$ with $E \subset E_i$. In particular,

$$\Omega = \bigcup_{i=1}^{\infty} E_i.$$

A sequence of sets E_i satisfying the properties in the above lemma is called *an exhaustion of* Ω by compact sets.

Proof of Lemma 5.18. Let us first assume that Ω is bounded in the plane, that is, there is $M > 0$ such that $\Omega \subseteq B(0, M)$. For $l \geq 1$ we define the set

$$E_i = \{z \in \Omega \mid \forall w \in \partial\Omega, |w - z| \geq 1/i\}.$$

Each E_i is a bounded and closed subset of $\mathbb{C}$. Therefore, each E_i is compact. Clearly, every compact subset of Ω is contained in some E_i.

If Ω is not a bounded subset of $\mathbb{C}$, we defined the sets

$$E_i = \{z \in \Omega \mid \forall w \in \partial\Omega, |w - z| \geq 1/i\} \bigcap \{z \in \Omega \mid |z| \leq i\}.$$

One can verify that these sets satisfy the properties in the lemma. $\square$

Define the new metric d' on $\mathbb{C}$ as

$$d'(z, w) = \frac{|z - w|}{1 + |z - w|}. \tag{5.3}$$

One can see that the above function on $\mathbb{C} \times \mathbb{C}$ is a metric, see Exercise 5.5.

Let Ω be an open set in $\mathbb{C}$ and E_i, for $i \in \mathbb{N}$, be an exhaustion of Ω with compact sets. Let $C^0(\Omega)$ be the set of continuous functions on Ω with values in $\mathbb{C}$. Define the function $d'' : C^0(\Omega) \times C^0(\Omega) \to [0, \infty)$ as

$$d''(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\sup_{z \in E_i} |f(z) - g(z)|}{1 + \sup_{z \in E_i} |f(z) - g(z)|}. \tag{5.4}$$

In an exercise you will show that d'' is a metric on the class $\mathcal{F}$.

Theorem 5.19. *A class of holomorphic maps $\mathcal{F}$ defined on an open set Ω is compact with respect to d'' if and only if the family $\mathcal{F}$ is normal and the limiting functions are contained in $\mathcal{F}$.*

See Exercise 5.7 for the proof.

5.3 General form of Cauchy integral formula

Definition 5.20. Let Ω be an open set in $\mathbb{C}$, and $\gamma_1, \gamma_2 : [0,1] \to \Omega$ be continuous maps with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. We say that γ_1 is *homotopic to* γ_2 in Ω if there is a continuous map

$$F : [0,1] \times [0,1] \to \Omega,$$

such that

(i) for all $t \in [0,1]$, $F(t,0) = \gamma_1(t)$;

(ii) for all $t \in [0,1]$, $F(t,1) = \gamma_2(t)$;

(iii) for all $s \in [0,1]$, $F(0,s) = \gamma_1(0)$ and $F(1,s) = \gamma_1(1)$.

In other words, the curves γ_1 and γ_2 with the same end points are homotopic in Ω if one can continuously move one of them to the other one without moving the end points.

Definition 5.21. Let Ω be a path connected subset of $\mathbb{C}$. We say that Ω is *simply connected* if every continuous closed curve $\gamma_1 : [0,1] \to \Omega$ is homotopic to the constant curve $\gamma_2(t) \equiv \gamma(0)$, $t \in [0,1]$ in Ω.

Note that any connected open set in $\mathbb{C}$ is path connected.

Example 5.22. The open unit disk $\mathbb{D}$ is simply connected. To see this, let $\gamma : [0,1] \to \mathbb{D}$ be a closed curve. We define $F : [0,1] \times [0,1] \to \mathbb{D}$ as $F(t,s) = (1-s)\gamma(t) + s\gamma(0)$. This is clearly a continuous map satisfying the three conditions for being a homotopy from γ to the constant curve $\gamma(0)$.

The above example shows that any convex set in $\mathbb{C}$ is simply connected. But this condition is far from necessary as we see in the next example.

Example 5.23. The set $\Omega = \mathbb{C} \setminus [0, \infty)$ is simply connected. Let $\gamma[0,1] \to \Omega$ be an arbitrary continuous map. First we move the curve γ continuously to the constant curve $\gamma_1 \equiv -1$, by the homotopy $F : [0,1] \times [0,1] \to \Omega$ defined as $F(t,s) = (1-s)\gamma(t) - s$. This we move the constant curve γ_1 to the constant curve $\gamma(0)$.

Recall that a curve $\gamma : [a,b] \to \mathbb{C}$ is called a *simple closed curve* if γ is continuous, one-to-one, and $\gamma(a) = \gamma(b)$. It is a non-trivial theorem due to Jordan that every simple closed curve in $\mathbb{C}$ divides the complex plane into two connected components. That is, $\mathbb{C} \setminus \gamma$ has two connected components.

Intuitively, Ω is simply connected if for every simple closed curve γ in Ω, the bounded connected component of $\mathbb{C} \setminus \gamma$ is contained in Ω.

One can show that the set $\Omega = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ is not simply connected. In general, any open set with "holes" is not simply connected.

The above definition allows us to generalize a number of theorems you have already seen in complex analysis. We state these below without proof. The proof of these statements can be found in any standard book on complex analysis.

Theorem 5.24 (Cauchy-Goursat-theorem-general-form). *Assume that Ω is a simply connected open set in $\mathbb{C}$ and f is an analytic map defined on Ω. Let γ be a simple closed curve in Ω which is piece-wise C^1. Then,*

$$\int_\gamma f(z)\, dz = 0.$$

The inverse of the above theorem is also true and is known as the Morera Theorem. That is, if $f(z)$ is defined and continuous in a region Ω, and for all closed curves γ in Ω we have $\int_\gamma f(z)\, dz = 0$, then f is holomorphic on Ω. We shall not use this theorem in this course.

Theorem 5.25 (Cauchy Integral Formula-general version). *Assume that Ω is a simply connected open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is holomorphic. Let γ be a piece-wise C^1 simple closed curve in Ω. Then, for every z_0 in the bounded connected component of $\mathbb{C} \setminus \gamma$,*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Proposition 5.26. *Assume that Ω is a simply connected domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a holomorphic function such that for all $z \in \Omega$, $f(z) \neq 0$. Then, there is a well-defined continuous branch of $\log f(z)$ defined on Ω.*

In particular, there are well-defined continuous branches of $\sqrt[n]{f(z)}$ defined on Ω, for each $n \geq 1$.

Proof. Fix a point z_0 in Ω. By the assumptions, $f(z_0) \neq 0$. So, $\log(f(z_0))$ is defined up to an additive constant in $2\pi i\mathbb{Z}$. Let us fix a value for $\log(f(z_0))$.

By the assumption, the function $f'(z)/f(z)$ is defined and holomorphic on Ω. We define a new function $g(z)$ on Ω as follows. For z in Ω choose a continuous curve $\gamma_z : [0, 1] \to \Omega$ with $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. Then define the integral

$$g(z) = \int_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + \log(f(z_0)). \tag{5.5}$$

The above integral is independent of the choice of γ_z. To see this let $\gamma : [0,1] \to \Omega$ be another continuous curve with $\gamma(0) = z_0$ and $\gamma(1) = z$. The curve γ_z followed by the curve $\gamma(1-t)$, for $t \in [0,1]$, is a closed curve in Ω. Then, since Ω is simply connected, by Theorem 5.24,

$$\int_{\gamma_z(t)\cup\gamma(1-t)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = 0.$$

This implies that

$$\int_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta.$$

Therefore, $g(z)$ is a well-defined function on Ω.

Since $f'(\zeta)/f(\zeta)$ is continuous on Ω, $g(z)$ is differentiable on Ω with derivative

$$g'(z) = \frac{f'(z)}{f(z)}.$$

This yields

$$\frac{d}{dz}(f(z)\cdot e^{-g(z)}) = 0.$$

As Ω is connected, $f(z)e^{-g(z)}$ must be a constant. Evaluating this function at z_0 we obtain the value of the constant $f(z_0)e^{-\log(f(z_0))} = 1$. Hence, for all $z \in \Omega$ we have $f(z) = e^{g(z)}$. This finishes the proof of the first part.

In the last part of the corollary we define $\sqrt[n]{f(z)} = e^{\frac{1}{n}\log f(z)}$, where $\log f(z)$ is a continuous branch defined on Ω. $\square$

5.4 Riemann mapping theorem

There are some obvious restrictions on a subsets Ω of $\mathbb{C}$ that is biholomorphic to $\mathbb{D}$. If there is a biholomorphic map $\phi : \mathbb{D} \to \Omega$, then Ω must be connected, since the image of any connected set under a continuous map is connected. Also, Ω cannot be equal to $\mathbb{C}$, since the inverse map $\phi^{-1} : \mathbb{C} \to \mathbb{D}$ is bounded and must be constant by Liouville's theorem. Also, Ω must be simply connected, since $\mathbb{D}$ is simply connected, and homeomorphisms map homotopic curves to homotopic curves. It turns out that these three conditions on Ω are enough to guarantee the existence of a biholomorphic map from $\mathbb{D}$ to Ω.

Bernhard Riemann was the first person to state the following important theorem for domains Ω with piece-wise smooth boundaries (1851). However, the proof he presented contained a gap. The first proof of the theorem for arbitrary domains is due to William Osgood (1900), but it did not attract the attention it deserved. The proof we present here is mostly due to Carathéordory with some simplifications due to Paul Koebe.

Theorem 5.27. *Let Ω be a non-empty, connected, and simply connected subset of $\mathbb{C}$ that is different from $\mathbb{C}$. Then, there is a biholomorphic map $f : \mathbb{D} \to \Omega$.*

Proof. Fix a point $z_0 \in \Omega$. We shall prove that there is a biholomorphic map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) \in (0, \infty)$. To this end we define the class of maps

$$\mathcal{F} = \Big\{h : \Omega \to \mathbb{D} \mid h \text{ is holomorphic, one to one}, h(z_0) = 0, \text{ and } h'(z_0) \in (0, \infty)\Big\}.$$

We break the proof into several steps.

Step 1. The class $\mathcal{F}$ is not empty.

If the set Ω is bounded in $\mathbb{C}$, we may translate and re-scale the set Ω to find a map in $\mathcal{F}$. That is, there is $a \in (0, \infty)$ such that the map $f(z) = a(z - z_0)$ maps Ω into $\mathbb{D}$. Clearly f is holomorphic, one-to-one, and $f'(z_0) = a \in (0, \infty)$.

If there is $w_0 \in \mathbb{C}$ and $r > 0$ such that $B(w_0, r) \cap \Omega = \emptyset$, then $g(z) = 1/(z - w_0)$ is holomorphic and one-to-one on Ω and maps Ω to a bounded region in $\mathbb{C}$. Then, there is a linear map f as in the above paragraph such that $f \circ g$ belongs to $\mathcal{F}$.

In general, first we note that since Ω is not equal to $\mathbb{C}$ there is $w_0 \in \mathbb{C} \setminus \Omega$. Hence the function $z \mapsto z - w_0$ never vanishes on the simply connected set Ω. By Proposition 5.26, there is a continuous (and holomorphic) branch

$$f(z) = \log(z - w_0)$$

defined on Ω. As a consequence, we have $e^{f(z)} = z - w_0$. This implies that f is one-to-one on Ω. Fix a point $a \in \Omega$, and observe that for all $z \in \Omega$ we have $f(z) \neq f(a) + 2\pi i$. That is because, otherwise $z - w_0 = e^{f(z)} = e^{f(a)+2\pi i} = e^{f(a)} = a - w_0$. This implies that $z = a$, and hence $f(z) = f(a)$, which is a contradiction.

We claim that there is $r > 0$ such that $f(\Omega) \cap B(f(a) + 2\pi i, r) = \emptyset$. If this is not the case, there is a sequence of points $z_i \in \Omega$ such that $f(z_i)$ converges to $f(a) + 2\pi i$. Since the exponential map is continuous, we conclude that $z_i - w_0 = e^{f(z_i)} \to e^{f(a)+2\pi i} = e^{f(a)} = a - w_0$. This implies that $z_i \to a$, and hence $f(z_i) \to f(a)$. This is a contradiction.

The map

$$I(z) = \frac{1}{f(z) - (f(a) + 2\pi i)}$$

is holomorphic and one-to-one on Ω and $I(\Omega) \subseteq B(0, 1/r)$. As in the first paragraph, we may compose I with a linear transformation to obtain a map of the from $k(z) = a(I(z) - I(z_0))$ that is holomorphic, one-to-one, and $k(\Omega) \subset \mathbb{D}$. Finally, the map $h(z) = k(z)/k'(z_0)$ is holomorphic, one-to-one, $k(z_0) = 0$, and $k'(z_0) \in (0, \infty)$.

Step 2. There is $f \in \mathcal{F}$ such that

$$f'(z_0) = \sup\{h'(z_0) : h \in \mathcal{F}\}.$$

Let $A = \{h'(z_0) : h \in \mathcal{F}\}$. This is a subset of $(0, \infty)$, and by step 1, A is a non-empty set.

As Ω is open, there is $r > 0$ such that $B(z_0, r) \subset \Omega$. Let γ denote the circle $|z - z_0| = r$. By the Cauchy integral formula for the derivative, for every $h \in \mathcal{F}$, we have

$$|h'(z_0)| = \Big|\frac{1}{2\pi i} \int_\gamma \frac{h(\zeta)}{(\zeta - z_0)^2}\, d\zeta\Big| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} \cdot \sup_{\zeta \in \gamma} |h(\zeta)| \leq \frac{1}{r}.$$

This proves that A is bounded from above. In particular, $\sup A$ exists and is finite.

By the definition of supremum, there is a sequence of maps $f_n \in \mathcal{F}$, for $n \geq 1$, such $f_n'(0) \to \sup A$.

Every map in $\mathcal{F}$ maps Ω into the bounded set $\mathbb{D}$. In particular, the family $\mathcal{F}$ is uniformly bounded on compact sets. By Montel's theorem, $\mathcal{F}$ is a normal family. Therefore, $\{f_n\}$ has a sub-sequence converging uniformly on compact sets to some map $f : \Omega \to \mathbb{C}$.

By Theorem 5.10, f is holomorphic. In particular, $f'(z_0)$ exists, and $f'(z_0) = \sup A$ is non-zero. This implies that f is not a constant function.

Each map f_n is one-to-one. Since f is not a constant function, it follows from Corollary 5.12 that f is one-to-one. For each $z \in \Omega$, $|f_n(z)| < 1$. Hence, taking limit along the convergent sub-sequence, we conclude that $|f(z)| \leq 1$. As f is not constant, by the maximum principle, for all $z \in \Omega$, $|f(z)| < 1$, that is, f maps Ω into $\mathbb{D}$.

By the above paragraph $f \in \mathcal{F}$. This finishes the proof of Step 2.

Step 3. The map f obtained in Step 2 is onto.

We shall prove that if f is not onto, there is $h \in \mathcal{F}$ with $h'(z_0) > f'(z_0)$, contradicting the extremality of f in Step 2.

Assume that there is $a \in \mathbb{D} \setminus f(\Omega)$. Consider the automorphism of $\mathbb{D}$ that maps a to 0 and 0 to a, that is,

$$\psi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Since Ω is simply connected, the set $U = \psi_a \circ f(\Omega)$ is simply connected, and since $f(\Omega)$ does not contain a, U does not contain 0. Then, by Proposition 5.26, there is a continuous branch of the square root function defined on U, that is,

$$g(w) = e^{\frac{1}{2} \log w}, w \in U.$$

Consider the function

$$h = \psi_{g(a)} \circ g \circ \psi_a \circ f.$$

The map h is holomorphic on Ω and maps Ω into $\mathbb{D}$. The latter is because, f maps Ω into $\mathbb{D}$, all of the other functions in the composition map $\mathbb{D}$ into $\mathbb{D}$.

As each of the maps f, ψ_a, $\psi_{g(a)}$ and g are one-to-one, h is also one-to-one. We also have $h(z_0) = 0$.

Let $p_2(z) = z^2$ denoted the square function. Define the map $I : \mathbb{D} \to \mathbb{D}$ as

$$I = \psi_a^{-1} \circ p_2 \circ \psi_{g(a)}^{-1}.$$

The function I maps 0 to 0, since $g(a)$ is the square root of a.

We have

$$I \circ h = \big(\psi_a^{-1} \circ p_2 \circ \psi_{g(a)}^{-1}\big) \circ \big(\psi_{g(a)} \circ g \circ \psi_a \circ f\big) = f.$$

In particular, we have

$$I'(0) \cdot h'(z_0) = f'(z_0).$$

On the other hand, by the Schwarz lemma $|I'(0)| < 1$, unless I is a rotation of the circle. However, since, $p_2 : \mathbb{D} \to \mathbb{D}$ is not one-to-one, $I : \mathbb{D} \to \mathbb{D}$ is not one-to-one. In particular, I may not be a rotation of the circle, and we have $|I'(0)| < 1$. This implies that $h'(z_0) > f'(z_0)$. This finishes the proof of Step 3.

All together we have proved that $f : \Omega \to \mathbb{D}$ is holomorphic, one-to-one, and onto. Thus, f is biholomorphic. $\square$

5.5 Exercises

Exercise 5.1. Prove that the function $z \mapsto z + 1/z$ maps the circle $|z| = r > 1$ to the ellipse

$$\frac{x^2}{(r + 1/r)^2} + \frac{y^2}{(r - 1/r)^2} = 1.$$

Exercise 5.2. Show that

$$\tan z = \frac{1}{i}\Big(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\Big) = i\big(\frac{1 - e^{2iz}}{1 + e^{2iz}}\Big)$$

Write tan as a composition of the map G in Equation (2.1) and the map $z \mapsto e^{2iz}$. Conclude that

$$\tan : \{z \in \mathbb{C} : -\pi/4 < \operatorname{Re} z < \pi/4, \operatorname{Im} z > 0\} \to \mathbb{H}$$

is biholomorphic.

Exercise 5.3. Let $\Omega = \mathbb{D} \setminus (1/2, 1)$. Find a biholomorphic map from Ω to $\mathbb{D}$ as a composition of some elementary maps.

Exercise 5.4. Prove that if a family of holomorphic maps defined on the same domain Ω is normal then the family is uniformly bounded on compact sets.

Exercise 5.5. Prove that the function d' defined in Equation (5.3) defines a metric on $\mathbb{C}$.

Exercise 5.6. Prove that the function d'' in Equation (5.4) defines a metric on the space $C^{)}(\Omega)$.

Exercise 5.7. Let Ω, $\mathcal{F}$, and d'' be as in Equation 5.4. Prove that a sequence of functions $f_n \in \mathcal{F}$, $n \geq 1$, converges to some function $f : \Omega \to \mathbb{C}$ uniformly on compact sets if and only if $d''(f_n, f) \to 0$. In particular, the statement of Theorem 5.19 is independent of the choice of the exhaustion E_i in the definition of the metric d''.

Exercise 5.8. Let $\mathrm{Hol}(\Omega)$ denote the space of all holomorphic maps from Ω to $\mathbb{C}$. Define the map $D : \mathrm{Hol}(\Omega) \to \mathrm{Hol}(\Omega)$ as $D(f) = f'$, that is, $D(f)(z) = f'(z)$. Prove that D is continuous from $\mathrm{Hol}(\Omega)$ to $\mathrm{Hol}(\Omega)$ with respect to the metric d''.

Exercise 5.9. Let $\mathcal{F}$ denote the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the from

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots,$$

such that for all $n \geq 2$ we have $|a_n| \leq n$. Prove that the family $\mathcal{F}$ is normal.

Exercise 5.10. Let $f_n : \Omega \to \mathbb{C}$, $n \geq 1$, be a sequence of holomorphic functions that is uniformly bounded on compact sets. Assume that for every $z \in \Omega$ the sequence $f_n(z)$ converges in $\mathbb{C}$. Prove that the sequence f_n converges uniformly on compact sets.