Chapter 4

Conformal geometry on the disk

4.1 Poincare metric

Let X be a set. Recall that a *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that for all x, y, and z in X we have

(i) $d(x, y) = d(y, x)$,

(ii) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

A metric on a space X allows us to talk about distances on X. The most familiar example is probably the *Euclidean distance* on $\mathbb{R}$ given by the absolute value. That is, $d(x, y) = |x-y|$, for x and y in $\mathbb{R}$. This notion of distance respects the underlying operation of addition that is described by the relation $d(x, y) = d(x + c, y + c)$, for all $c \in \mathbb{R}$. That is, d is *invariant* under translations.

Another example of a metric on $\mathbb{R}$ is given by the function

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

The above notion of metric is rather general for our purpose. There is a more restrictive notion of metric that is suitable in the world of complex analysis. Before we define that, recall that a set $A \subset \mathbb{C}$ is called a discrete set, if for every $z \in A$ there is an open set U in $\mathbb{C}$ with $A \cap U = \{z\}$.

Definition 4.1. Let Ω be a domain in $\mathbb{C}$. A *conformal metric* on Ω is a continuously differentiable (C^1) function

$$\rho : \Omega \to [0, \infty),$$

where $\rho(z) \neq 0$ except on a discrete subset of Ω. If $z \in \Omega$ and $\xi \in \mathbb{C}$ is a vector, we define the *length of ξ at z with respect to the metric ρ* as

$$\| \xi \|_{\rho,z} = \rho(z) \cdot |\xi|.$$

Here, $|\xi|$ denotes the Euclidean norm of ξ, i.e. $\sqrt{\xi\bar{\xi}}$.

Remark 4.2. In contrast to what we learn in a calculus course that a vector has only direction and magnitude, in the above notion of the metric, a vector has direction, magnitude, and position. That is the length of a vector also depends on its position.

Definition 4.3. Assume that $\gamma : [a, b] \to \Omega$ is a C^1 curve. The *length of* γ *with respect to the metric* ρ is defined as

$$\ell_\rho(\gamma) = \int_a^b \left\| \frac{\partial \gamma(t)}{\partial t} \right\|_{\rho, \gamma(t)} dt = \int_a^b \rho(\gamma(t)) \cdot \left| \frac{\partial \gamma(t)}{\partial t} \right| dt.$$

The length of a piece-wise C^1 curve is defined as the sum of the lengths of its C^1 parts.

As in the definition of the integration along a curve in complex analysis, the above notion of length is independent of the parameterization of the curve.

It is convenient to think of the tangent vector to $\gamma'(t)$ at $\gamma(t)$ as a vector based at $\gamma(t)$. See Figure 4.1

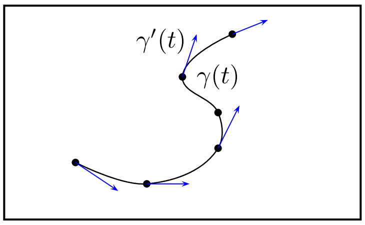

Figure 4.1: The tangent vectors to a C^1 curve γ in the calculation of the length of γ with respect to a conformal metric.

For every $c \in \mathbb{C}$ with $|c| = 1$ we have

$$\| c \cdot \xi \|_{\rho,z} = \| \xi \|_{\rho,z} .$$

By the above relation, the length of a vector ξ at some $z \in \Omega$ is independent of its direction. This feature makes conformal metrics natural in complex analysis, as we shall see in this section.

In the classical literature in analysis, sometimes you find the notation

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)\, |dz|,$$

for the length of γ with respect to the metric ρ. This is consistent with the definition of integration along curves you learn in complex analysis.

Definition 4.4. A set A in $\mathbb{C}$ is called *path connected* if for every two points z and w in A there is a continuous map $\gamma : [0,1] \to A$ with $\gamma(0) = z$ and $\gamma(1) = w$.

It follows that any path connected subset of $\mathbb{C}$ is connected, but there are connected subsets of $\mathbb{C}$ that are not path connected.

Definition 4.5. Let ρ be a conformal metric defined on an open and path connected set $\Omega \subseteq \mathbb{C}$. Given z and w in Ω let $\Gamma_{z,w}$ denote the set of all piece-wise C^1 curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = z$ and $\gamma(1) = w$. Define

$$d_\rho(z, w) = \inf\{\ell_\rho(\gamma) : \gamma \in \Gamma_{z,w}\}.$$

It follows (Exercise 4.1) that d_ρ defines a metric on Ω, that is, $d_\rho : \Omega \times \Omega \to [0, \infty)$ satisfies the required properties listed at the beginning of this chapter.

Remark 4.6. One should not confuse the notion of $d_\rho(z, w)$ with the notion of the length of the vector $w - z$ at z with respect to ρ. In general, these are different values and not related.

Example 4.7. When $\rho(z) \equiv 1$ on Ω, the length of a piece-wise C^1 curve γ with respect to ρ, $\ell_\rho(\gamma)$, becomes the Euclidean length of γ (which we learn in calculus). When $\Omega = \mathbb{C}$, d_ρ becomes the Euclidean distance. In general, when Ω is a convex set, that is, the line segment connecting any two points in Ω lies in Ω, then d_ρ is the restriction of the Euclidean metric to Ω. But in general, there may not be a curve of shortest length between two points in Ω. See Figure 4.2.

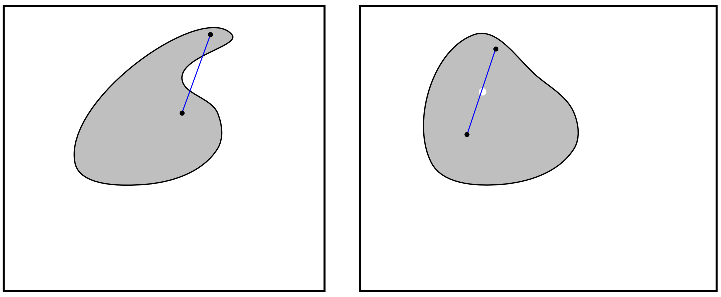

Figure 4.2: Examples of non-convex domains; one with a point omitted, and the other with a special shape.

Definition 4.8. The Poincaré metric on $\mathbb{D}$ is defined as

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

The Poincaré metric has been used to gain deep understanding into the complex analysis on the disk and beyond. We shall study this metric in details.

For any vector $\xi \in \mathbb{C}$ we have

$$\begin{aligned}
\| \xi \|_{\rho,0} &= \rho(0) \cdot |\xi| = |\xi|, \\
\| \xi \|_{\rho,(1/2+0i)} &= \rho(1/2 + 0i) \cdot |\xi| = \frac{4}{3} \cdot |\xi|, \\
\| \xi \|_{\rho,(0+i/2)} &= \rho(0 + i/2) \cdot |\xi| = \frac{4}{3} \cdot |\xi|, \\
\| \xi \|_{\rho,(0.99+0i)} &= \rho(0.99 + 0i) \cdot |\xi| = (50.251256 \dots) \cdot |\xi|.
\end{aligned}$$

The metric ρ has a rotational symmetry about 0, i.e. $\rho(c \cdot z) = \rho(z)$ for all $c \in \mathbb{C}$ with $|c| = 1$. Also, $\rho(z) \to \infty$ as $|z|$ tends to $+1$ from below. There are many conformal metrics on $\mathbb{D}$ with rotational symmetry and diverging to $+\infty$ near the boundary, but the speed of divergence in the Poincaré metric is chosen to guarantee some significant properties.

Example 4.9. Let us calculate the length of the curve $[0, 1 - \varepsilon]$ with respect to the Pioncaré metric ρ on $\mathbb{D}$. Define $\gamma(t) = t + 0i$, for $t \in [0, 1 - \varepsilon]$. Then,

$$\ell_\rho(\gamma) = \int_0^{1-\varepsilon} \rho(\gamma(t)) \cdot |\gamma'(t)| \, dt = \int_0^{1-\varepsilon} \frac{1}{1-t^2} \, dt = \frac{1}{2} \log\Big(\frac{1+t}{1-t}\Big)\Big|_{t=0}^{t=1-\varepsilon} = \frac{1}{2} \log\Big(\frac{2-\varepsilon}{\varepsilon}\Big).$$

We note that the above quantity tends to $+\infty$ as ε tends to 0. This means that the point $+1$ is at distance ∞ from the point 0 along the curve γ, with respect to the Poincaré metric on $\mathbb{D}$.

Proposition 4.10. *Let ρ be the Poincaré metric on $\mathbb{D}$. We have*

$$d_\rho(0, 1-\varepsilon) = \frac{1}{2} \log\Big(\frac{2-\varepsilon}{\varepsilon}\Big).$$

Proof. Let $\eta : [a, b] \to \mathbb{D}$ be a C^1 curve with $\eta(a) = 0$ and $\eta(b) = 1 - \varepsilon + 0i$. In coordinates, let $\eta(t) = \eta_1(t) + i\eta_2(t)$, for $t \in [a, b]$. Both η_1 and η_2 are C^1 and moreover, for all $t \in [a, b]$,

$$|\eta'(t)| = |\eta_1'(t) + i\eta_2'(t)| \geq |\eta_1'(t)|.$$

Also, since $|\eta(t)| \geq |\eta_1(t)|$, for all $t \in [a, b]$ we have

$$\rho(\eta(t)) \geq \rho(\eta_1(t)).$$

Note that $\eta_1 : [a, b] \to (-1, 1)$ is a C^1 curve with $\eta_1(a) = 0$ and $\eta_1(b) = 1 - \varepsilon$. However, η_1 may not a monotone function of $t \in [a, b]$. Its image may cover some parts of $[0, 1 - \varepsilon]$ several times. If necessary, we may throw away parts of this curve and keep a piece-wise

C^1 and monotone part of η_1 that maps a subset of $[a,b]$ to $[0,1-\varepsilon]$. Let $A \subseteq [a,b]$ be that set. Using the above inequalities,

$$\begin{aligned}\ell_\rho(\eta) &= \int_a^b \rho(\eta(t)) \cdot |\eta'(t)|\,dt \\ &\geq \int_a^b \rho(\eta_1(t)) \cdot |\eta_1'(t)|\,dt \\ &\geq \int_A \rho(\eta(t)) \cdot |\eta'(t)|\,dt \\ &= \frac{1}{2}\log(\frac{2-\varepsilon}{\varepsilon}).\end{aligned}$$

□

Figure 4.3 shows the graph of the function $r \mapsto d_\rho(0,r)$, on $(0,1)$. Note how on a large interval $(0,r)$ (with r close to 1) the distance of the points from 0 is less than 5.

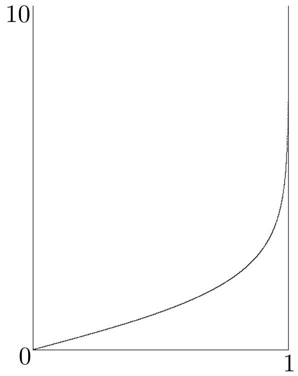

Figure 4.3: The graph of the function $r \mapsto d_\rho(0,r)$, for $r \in (0,1)$.

4.2 Isometries

Definition 4.11. Assume that Ω_1 and Ω_2 are open sets in $\mathbb{C}$ and

$$f : \Omega_1 \to \Omega_2$$

is a holomorphic map. Let ρ_2 be a conformal metric on Ω_2. The *pull-back* of ρ_2 by f is defined as

$$(f^*\rho_2)(z) = \rho_2(f(z)) \cdot |f'(z)|.$$

It is clear that if ρ_2 is C^1 then $f^*\rho_2$ is also C^1. On the other hand, $f^*\rho_2(z) = 0$ if and only if either $\rho(f(z)) = 0$ or $f'(z) = 0$. Since, f' is a holomorphic function on Ω_1, the set of points where it becomes 0 is a discrete set. These imply that the pull-back of a conformal metric under a holomorphic map is a conformal metric. Indeed, this is the reason for the name conformal metric. These are metrics that behave well under holomorphic transformations.

By the above definition, if ξ is a vector in $\mathbb{C}$ and $z \in \Omega_1$, then

$$\| \xi \|_{f^*\rho_2,z} = \rho_2(f(z)) \cdot |f'(z)| \cdot |\xi| = \rho_2(f(z)) \cdot |f'(z) \cdot \xi|.$$

Let us denote the metric $f^*\rho_2$ on Ω_1 by ρ_1. If γ_1 is a C^1 curve in Ω_1, and $\gamma_2 = f \circ \gamma_1$, then it follows that $\ell_{\rho_1}(\gamma_1) = \ell_{\rho_2}(\gamma_2)$.

$$\ell_{\rho_2}(\gamma_2) = \int_a^b \rho_2(\gamma_2(t)) \cdot |\gamma_2'(t)|\, dt = \int_a^b \rho_2(f(\gamma_1(t))) \cdot |(f \circ \gamma_1)'(t)|\, dt =$$
$$\int_a^b \rho_2(f(\gamma_1(t))) \cdot |f'(\gamma_1(t))| \cdot |\gamma_1'(t)|\, dt = \int_a^b \rho_1(\gamma_1(t))|\gamma_1'(t)|\, dt = \ell_{\rho_1}(\gamma_1)$$

Definition 4.12. Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ and $f : \Omega_1 \to \Omega_2$ be an onto holomorphic map. Let ρ_i be a conformal metric on Ω_i, for $i = 1, 2$. Then, f is called an *isometry* from (Ω_1, ρ_1) to (Ω_2, ρ_2) if for all $z \in \Omega_1$ we have

$$f^*\rho_2(z) = \rho_1(z).$$

Proposition 4.13. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ with conformal metrics ρ_1 and ρ_2, respectively. Assume that f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2). Then for every C^1 curve $\gamma : [0, 1] \to \Omega_1$ we have*

$$\ell_{\rho_1}(\gamma) = \ell_{\rho_2}(f \circ \gamma).$$

The curve $f \circ \gamma$ is often called the *push-forward* of the curve γ by f, and is denoted by $f_*\gamma$. That is,

$$f_*\gamma(t) = f \circ \gamma(t), \text{for } t \in [0, 1].$$

Proof. By Definition 4.3 we have

$$\ell_{\rho_1}(\gamma) = \int_0^1 \| \gamma'(t) \|_{\rho_1,\gamma(t)}\ dt, \quad \ell_{\rho_2}(f \circ \gamma) = \int_0^1 \| (f \circ \gamma)'(t) \|_{\rho_2, f\circ\gamma(t)}\ dt.$$

To prove that the above integrals give the same value, it is enough to show that the integrands are equal on $[0,1]$. That is,

$$\begin{aligned}\| (f\circ\gamma)'(t) \|_{\rho_2, f\circ\gamma(t)} &= \rho_2(f\circ\gamma(t))\cdot|(f\circ\gamma)'(t)| \\ &= \rho_2(f\circ\gamma(t))\cdot|f'(\gamma(t))\cdot\gamma'(t)| \\ &= \rho_2(f\circ\gamma(t))\cdot|f'(\gamma(t))|\cdot|\gamma'(t)| \\ &= (f^*\rho_2)(\gamma(t))\cdot|\gamma'(t)| \\ &= \| \gamma'(t) \|_{\rho_1,\gamma(t)}\ .\end{aligned}$$

□

Note that if f is an isometry from (Ω_1,ρ_1) to (Ω_2,ρ_2) we may not conclude that for every z and w in Ω_1 we have

$$d_{\rho_1}(z,w) = d_{\rho_2}(f(z), f(w)). \tag{4.1}$$

That is because not every curve in Ω_2 from $f(z)$ to $f(w)$ is obtained from push-forward of a curve in Ω_1 from z to w. We illustrate this by an example.

Example 4.14. Let

$$\Omega_1 = \{z\in\mathbb{C} : e^{-1} < |z| < e\}, \qquad \Omega_2 = \{z\in\mathbb{C} : e^{-2} < |z| < e^2\}$$

and define

$$f:\Omega_1\to\Omega_2, \quad f(z) = z^2.$$

Consider the conformal metrics

$$\rho_1(z) = \frac{\pi}{2}\cdot\frac{1}{|z|\cdot\cos(\frac{\pi\log|z|}{2})},$$

$$\rho_2(z) = \frac{\pi}{4}\cdot\frac{1}{|z|\cdot\cos(\frac{\pi\log|z|}{4})}.$$

We have

$$\begin{aligned}f^*\rho_2(z) &= \rho_2(f(z))\cdot|f'(z)| \\ &= \frac{\pi}{4}\cdot\frac{1}{|z|^2\cdot\cos(\frac{\pi\log|z|^2}{4})}\cdot 2|z| \\ &= \frac{\pi}{2}\cdot\frac{1}{|z|\cdot\cos(\frac{\pi\log|z|}{2})} = \rho_1(z).\end{aligned}$$

That means that f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2).

Note that f is not one-to-one, for example, $f(+1) = f(-1) = +1$. If γ is a curve in Ω_1 connecting $+1$ to -1, $f \circ \gamma$ is a curve in Ω_2 that connects $+1$ to itself and wraps around in Ω_2 at least once. But, there is a constant curve with zero length from $+1$ to $+1$ in Ω_2. The constant curve is not the image of any continuous curve from $+1$ to -1 in Ω_1. This is the only issue that prevents us from having Equation (4.1). As you will show in Exercise 4.5, if $f : \Omega_1 \to \Omega_2$ is one-to-one, then Equation (4.1) holds for all z and w in Ω_1.

Let Ω_1, Ω_2, and Ω_3 be open sets in $\mathbb{C}$ with conformal metrics ρ_1, ρ_2, and ρ_3, respectively. Assume that f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2), and g is an isometry from (Ω_2, ρ_2) to (Ω_3, ρ_3). You can show that $g \circ f$ is an isometry from (Ω_1, ρ_1) to (Ω_3, ρ_3).

Theorem 4.15. *Every automorphism of $\mathbb{D}$ is an isometry from $(\mathbb{D}, \rho)$ to $(\mathbb{D}, \rho)$, where ρ is the Poincaré metric on $\mathbb{D}$.*

Proof. By Theorem 2.5 every automorphism of $\mathbb{D}$ is of the form

$$z \mapsto e^{i\theta} \cdot \frac{z-a}{1-\overline{a}z},$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. We shall prove the theorem in two steps.

First assume that $h(z) = e^{i\theta} \cdot z$. We have

$$(h^*\rho)(z) = \rho(h(z)) \cdot |h'(z)| = \frac{1}{1-|h(z)|^2} \cdot |e^{i\theta}| = \frac{1}{1-|z|^2} = \rho(z).$$

Thus, h is an isometry of $(\mathbb{D}, \rho)$.

Now assume that $h(z) = (z-a)/(1-\overline{a}z)$. we have

$$\begin{aligned}
(h^*\rho)(z) &= \rho(h(z)) \cdot |h'(z)| \\
&= \frac{1}{1-\left|\frac{z-a}{1-\overline{a}z}\right|^2} \cdot \frac{1-|a|^2}{|1-\overline{a}z|^2} \\
&= \frac{1-|a|^2}{|1-\overline{a}z|^2 - |z-a|^2} \\
&= \frac{1-|a|^2}{1-|z|^2-|a|^2+|a|^2|z|^2} \\
&= \rho(z).
\end{aligned}$$

That is, h is an isometry of $(\mathbb{D}, \rho)$. Since the composition of isometries is an isometry, see Exercise 4.4, we conclude that any member of $\mathrm{Aut}(\mathbb{D})$ is an isometry of $(\mathbb{D}, \rho)$. $\square$

As a corollary of the above theorem, and Proposition 4.10, we are able to calculate the Poincaré distant between any two points on $\mathbb{D}$.

Proposition 4.16. *Let p and q be two points in $\mathbb{D}$ equipped with Poincaré metric ρ. We have*
$$d_\rho(p,q)=\frac{1}{2}\log\left(\frac{1+\left|\frac{p-q}{1-\overline{p}q}\right|}{1-\left|\frac{p-q}{1-\overline{p}q}\right|}\right).$$

Proof. When $p=0$ and q is a positive real number the formula in the proposition reduces to the one in Proposition 4.10. Now, define
$$\varphi_1(z)=\frac{z-p}{1-\overline{p}z},$$
and
$$\varphi_2(z)=\frac{|\varphi_1(q)|}{\varphi_1(q)}\cdot z.$$
Note that both of φ_1 and φ_2 belong to $\mathrm{Aut}(\mathbb{D})$. Then, by Theorem 4.15 and Exercise 4.5, we must have
$$d_\rho(p,q)=d_\rho(\varphi_1(p),\varphi_1(q))=d_\rho(0,\varphi_1(q))=d_\rho(\varphi_2(0),\varphi_2(\varphi_1(q)))=d_\rho(0,|\varphi_1(q)|).$$
Using Proposition 4.10 with $1-\varepsilon=|\varphi_1(q)|$ we obtain
$$d_\rho(0,|\varphi_1(q)|)=\frac{1}{2}\log\left(\frac{1+|\varphi_1(q)|}{1-|\varphi_1(q)|}\right).$$
This finishes the proof of the proposition. $\square$

The proof of the above proposition also provides us with the shortest curve connecting the two points p and q. We state this in a separate proposition.

Proposition 4.17. *Let p and q be two distinct points in $\mathbb{D}$. The shortest curve with respect to ρ connecting p to q is given by the formula*
$$\gamma_{p,q}(t)=\frac{t\frac{q-p}{1-q\overline{p}}+p}{1+t\overline{p}\frac{q-p}{1-q\overline{p}}},\quad 0\le t\le 1.$$

Proof. In Proposition 4.10 and its preceding example, the shortest curve connecting 0 to a point z on $(0,1)\subset\mathbb{D}$ is given by the interval $[0,z]$. As the rotation $z\mapsto e^{i\theta}\cdot z$, for each fixed $\theta\in\mathbb{R}$, is an isometry of $(\mathbb{D},\rho)$, we conclude that the shortest curve connecting 0 to a given point $z\in\mathbb{D}$ is the curve $t\mapsto t\cdot z$, $0\le t\le 1$.

Consider the automorphism
$$\varphi_1(z)=\frac{z-p}{1-\overline{p}z}.$$
We have $\varphi_1(p)=0$ and $\varphi_1(q)\in\mathbb{D}$. The inverse of this map is given by the formula
$$\varphi_1^{-1}(z)=\frac{z+p}{1+\overline{p}z}.$$

By the above paragraph, the shortest curve connecting 0 to $\varphi_1(q)$ is given by the formula $\theta(t) = t \cdot \varphi_1(q)$. Since φ_1^{-1} is an isometry of the pair $(\mathbb{D}, \rho)$, the image of this curve under φ^{-1} is the shortest curve connecting p to q. The formula for this curve is

$$t \mapsto \varphi_1^{-1}(t \cdot \varphi_1(q)) = \frac{t \cdot \varphi_1^{-1}(q) + p}{1 + \overline{p}t \cdot \varphi^{-1}(q)}.$$

This finishes the proof of the proposition. $\square$

Definition 4.18. Let Ω be an open set in $\mathbb{C}$ and ρ be a conformal metric on Ω. A continuous curve $\gamma : [a, b] \to \Omega$ is called *geodesic* if for every $t \in [a, b]$ there is $\varepsilon_t > 0$ such that for all x and y in $[a, b] \cap [t - \varepsilon_t, t + \varepsilon_t]$ we have

$$d_\rho(\gamma(x), \gamma(y)) = \ell_\rho(\gamma : [x, y] \to \Omega).$$

In other words, the curve γ is locally the shortest curve connecting points together. For example, straight lines on $\mathbb{C}$ are geodesics with respect to the conformal metric $\rho \equiv 1$. The curves $\gamma_{p,q}$ in the above proposition provide examples of geodesics with respect to the Poincaré metric on $\mathbb{D}$.

There is an intuitive way to visualize the curve $\gamma_{p,q}$. To present this, we need to recall a basic property of holomorphic maps.

Definition 4.19. A holomorphic map $f : \Omega \to \mathbb{C}$ is called *conformal* at $z \in \Omega$ if $f'(z) \neq 0$. A holomorphic map $f : \Omega \to \mathbb{C}$ is called *conformal*, if it is conformal at every point in Ω.

If $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is one-to-one, it follows that f is conformal at every point in U; see Exercise 3.3. In particular, biholomorphic maps are conformal at every point in their domain of definition. However, note that a map that is conformal at every point in its domain of definition is not necessarily one-to-one from its domain to its range. For example, the map $z \mapsto z^2$ is conformal on the set

$$\{w \in \mathbb{C} \mid \arg(w) \in (-3\pi/4, 3\pi/4), |w| \in (1, 2)\}$$

but is not one-to-one on this set.

Recall from complex analysis that conformal maps preserve angles. We state this below for future reference.

Proposition 4.20. *Let U and V be two open subsets of $\mathbb{C}$ and $f : U \to V$ be a holomorphic map that is conformal at some $z \in U$. Then f preserves angles at z.*

The curve $t \mapsto t \cdot \varphi_1(q)$ is part of a straight line segment in $\mathbb{D}$. By Exercise 3.5, the image of any line segment in $\mathbb{D}$ under φ_1^{-1} is either a line segment or part of a circle. The image may be a line segment only if the three points p, q, and 0 lie on a straight line segment, and other wise the curve is part of a circle. Moreover, since the line segment passing through 0 is orthogonal to the boundary of $\mathbb{D}$, and conformal maps preserve angles, the circle passing through p and q is orthogonal to the circle $|z| = 1$. See Figure 4.4.

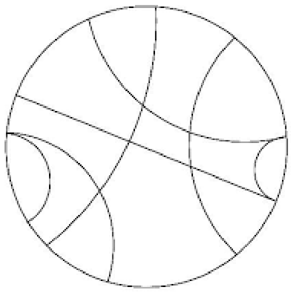

Figure 4.4: Some examples of geodesics with respect to the Poincaré metric ρ on $\mathbb{D}$.

4.3 Hyperbolic contractions

Theorem 4.21 (Schwarz-Pick Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and ρ denote the Poincaré metric on $\mathbb{D}$. Then, f is distance decreasing with respect to ρ, that is, for every $z \in \mathbb{D}$ we have*

$$f^*\rho(z) \leq \rho(z).$$

In particular, if $\gamma : [0, 1] \to \mathbb{D}$ is a C^1 curve then

$$\ell_\rho(f_*\gamma) \leq \ell_\rho(\gamma).$$

Therefore, if z and w belong to $\mathbb{D}$, then

$$d_\rho(f(z), f(w)) \leq d_\rho(z, w).$$

Proof. Recall that

$$f^*\rho(z) = \rho(f(z)) \cdot |f'(z)| = \frac{1}{1 - |f(z)|^2} \cdot |f'(z)|,$$

and

$$\rho(z) = \frac{1}{1-|z|^2}.$$

Hence, the inequality in the theorem reduces to the Schwarz-Pick lemma we saw earlier in Exercise 2.2.

The latter parts of the theorem follow directly from the definitions. $\square$

Theorem 4.22 (Farkas-Ritt)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and assume that $f(\mathbb{D})$ has a compact closure in $\mathbb{D}$, that is, every sequence in $f(\mathbb{D})$ has a sub-sequence converging to some point in $\mathbb{D}$. Then,*

i) there is a unique point $p \in \mathbb{D}$ such that $f(p) = p$;

ii) for every w_0 in $\mathbb{D}$ the sequence of points w_i defined as $w_{i+1} = f(w_i)$, for $i \geq 0$, converges to p in the Euclidean metric.

Proof. Define

$$A = \{f(z) : z \in \mathbb{D}\}.$$

By the hypothesis, the closure of A is contained in $\mathbb{D}$. This implies that there is $\delta > 0$ such that for every $z \in \mathbb{C}$ with $|z| \geq 1$ and every $w \in A$ we have $|w - z| > \delta$.

Fix an arbitrary $z_0 \in \mathbb{D}$. Define the map

$$g(z) = f(z) + \frac{\delta}{2}(f(z) - f(z_0)), \quad \forall z \in \mathbb{D}.$$

The map g is holomorphic on $\mathbb{D}$, and maps $\mathbb{D}$ into $\mathbb{D}$ since

$$|g(z)| \leq |f(z)| + \frac{\delta}{2}|f(z) - f(z_0)| < (1 - \delta) + \frac{\delta}{2} \cdot 2 = 1.$$

We have $g(z_0) = f(z_0)$ and $g'(z_0) = (1 + \delta/2)f'(z_0)$. By Theorem 4.21, g is non-expanding the Poincaré metric at z_0, that is,

$$g^*\rho(z_0) \leq \rho(z_0).$$

Writing the definition of g^*, this yields

$$(1 + \delta/2) \cdot |f'(z_0)| \cdot \rho(f(z_0)) \leq \rho(z_0).$$

As $z_0 \in \mathbb{D}$ was arbitrary, we conclude that the above inequality holds for all $z_0 \in \mathbb{D}$. In particular, if γ is any C^1 curve in $\mathbb{D}$, then

$$(1 + \delta/2) \cdot \ell_\rho(f \circ \gamma) \leq \ell_\rho(\gamma).$$

This implies that for arbitrary points z and w in $\mathbb{D}$ we have

$$d_\rho(f(z), f(w)) \le (1+\delta/2)^{-1} \cdot d_\rho(z, w).$$

Fix an arbitrary w_0 in $\mathbb{D}$ and define the sequence of points $w_{i+1} = f(w_i)$, for $i \ge 0$. Inductively using the above inequality we conclude that for every $i \ge 2$ we have

$$d_\rho(w_{i+1}, w_i) \le (1+\delta/2)^{-1} \cdot d_\rho(w_i, w_{i-1}) \le \cdots \le (1+\delta/2)^{-i} \cdot d_\rho(w_1, w_0).$$

Since $\sum_{i=0}^{\infty}(1+\delta/2)^{-i}$ is finite, the sequence w_i is Cauchy with respect to d_ρ. The space $\mathbb{D}$ with respect to d_ρ is a complete metric space, see Exercise 4.3. This means that any Cauchy sequence (w.r.t d_ρ) in $\mathbb{D}$ converges (w.r.t d_ρ) to some point in $\mathbb{D}$. By Exercise 4.2, the sequence w_i converges with respect to the Euclidean metric on $\mathbb{D}$. Let p denote the limit of this sequence. Taking limit from the relation $w_{i+1} = f(w_i)$ as i tends to $+\infty$, we conclude that $f(p) = p$.

If there is q in $\mathbb{D}$ with $f(q) = q$, by the above inequalities,

$$d_\rho(p, q) = d_\rho(f(p), f(q)) \le (1+\delta/2)^{-1} d_\rho(p, q).$$

As $\delta > 0$, this is only possible if $p = q$. This shows the uniqueness of p. So far we have completed the proof of Part i).

By the above arguments, $d_\rho(w_i, p) \le (1+\delta/2)^{-i} d_\rho(w_0, p)$. Hence, w_i converges to p with respect to d_ρ. It follows that w_i converges to p with respect to the Euclidean metric, see Exercise 4.2. □

4.4 Exercises

Exercise 4.1. Show that $d_\rho : \Omega \times \Omega \to \mathbb{R}$ defined in Definition 4.5 is a metric on Ω.

Exercise 4.2. Let z_i, $i \ge 1$, be an infinite sequence in $\mathbb{D}$, and ρ be the Poincaré metric on $\mathbb{D}$. Show that z_i converges to some point z in $\mathbb{D}$ with respect to d_ρ iff it converges to $z \in \mathbb{D}$ with respect to the Euclidean metric.

Exercise 4.3. Show that the disk $\mathbb{D}$ equipped with the Poincaré metric ρ is a complete metric space. That is, every Cauchy sequence in $\mathbb{D}$ with respect to d_ρ converges to some point in $\mathbb{D}$ with respect to the distance d_ρ.

Exercise 4.4. Let ρ be the Poincaré metric on $\mathbb{D}$. For $z \in \mathbb{D}$ and $r > 0$, the circle of radius r about z with respect to the metric d_ρ is defined as

$$\{w \in \mathbb{D} : d_\rho(z, w) = r\}.$$

Show that for every $z \in \mathbb{D}$ and $r > 0$, the circle of radius r about z is an Euclidean circle. Find the center of this circle.

Exercise 4.5. Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ with conformal metrics ρ_1 and ρ_2, respectively. Assume that $f : \Omega_1 \to \Omega_2$ is a one-to-one holomorphic map that is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2). Prove that for all z and w in Ω_1 we have

$$d_{\rho_1}(z, w) = d_{\rho_2}(f(z), f(w)).$$

Exercise 4.6. Recall the biholomorphic map $F : \mathbb{H} \to \mathbb{D}$ given in Equation (2.1), and let ρ be the Poincaré metric on ρ. Show that for all $w \in \mathbb{H}$ we have

$$(F^* \rho)(w) = \frac{1}{2|\operatorname{Im} w|}.$$