Chapter 4

Dynamical Systems

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1. Introduction

Dynamical system is the branch of mathematics that studies the time evolution of a system. The evolution is given by a law as for example a recursion relation, a (partial) differential equation, an integral equation or even a random mechanism. The theory of dynamical systems takes a global and more qualitative view point trying to work out properties which are genuine and have features independent of the details of the considered dynamics like the long time behaviour, its sensitivity w.r.t. when the law is modified and how to module complicated system with highly complex dynamics by simpler dynamics which have nevertheless the same complex dynamical structure.

Mathematically, a dynamical systems often consists of a phase space (or state space) X representing all possible states of the system, and a map from the phase space to itself that represents the evolution law of the system, that is, if the systems at time s is in the state x then the function $\Phi_{s,t}(x)$ gives the state the system has evolved to at time t. Examples include the mathematical models that describe the swinging of a clock pendulum, the temperature at each point on earth on 1st of January each year as well complicated systems as the earth climate system as a whole. Other examples are fluid flows, numerical algorithm, stationary processes.

A dynamical system may be based on a *discrete* or *continuous* time. In the former case, the system is often describe as $f : X \to X$, where the state at time $n+1$ is obtained from applying f to the state at time n. In the latter case, the system may be described e.g. by a differential equation, where the state at time $t > s$ is obtained from the state at time s by running along the flow of the equation for the time $t-s$. These systems are *deterministic*, that is, only one future state follows from the current one, but one may consider stochastic systems as well, that is, states are randomly selected from a collection of maps f_t. In general, the law may depend explicitly on the time itself.

Given a system $f : X \to X$ the *orbit* (path for continuous time) of x, is defined as the sequence of states x, $f(x)$, $f \circ f(x)$, $f \circ f \circ f \circ f(x)$, $\dots,$. One main goal in dynamical system is to describe the behavior of individual orbits (or trajectories, respectively). While it is possible to know the orbits for some (rare) simple dynamical systems, most dynamical systems are too complicated to be understood in terms of individual orbits, however one can give an effective stochastic description.

In Section 2 and Section 3 two explicit examples are studied in detail qualitatively and explicitly, which are diametral in their properties. In Section 2 the "rotations of the circle" where if we understand one orbit then we understand any other orbits and in Section 3 the *doubling map*, an example of a *chaotic* system where nearby points move apart at an exponential rate (and come back together). Although X is low dimensional, that is one-dimensional, the doubling map shows surprisingly complex behaviour like sensitive dependence on initial condition, the existence of a space filling orbit and that the periodic orbits are dense. The rest of the section is dedicated to study the counter-intuitive properties in more details. Such type of complicated systems are typical in the sense that randomly chosen f will show such a complicated behaviour. In Section 4 it is shown that chaotic systems can be completely described by simple model where the dynamic is just a shift in an infinite word. A surprising consequence of this description is that chaotic systems form large classes which are qualitatively similar. With qualitative similar we mean that the full dynamical picture of two dynamical systems differ only by a coordinate transformation. More importantly, they are structurally stable, in the sense that a small change of the dynamics will lead to a qualitatively similar system. The latter is not true for the rotation on the circle for example, which the paradigmatic example of an integrable system. In Section 5 the so-called topological entropy is introduce which measure the complexity of a dynamic. Qualitative similar dynamics have the same complexity.

The chaotic nature of a system restricts our ability to make deterministic predictions for large times into the future, like weather forecast. However, probabilistic predictions on the contrast will get easier the further we look into the future, as for example prediction of climate. Some basic techniques of probabilistic description of dynamical systems are given in Section **??** and how it can be used to predict the behaviour in the far future of dynamical systems. In Section **??** we demonstrate that expanding dynamical systems fulfil all the properties required in Section **??** and even much stronger regularity results for the probabilistic description. It is shown that time average can be effectively described by a suitable unique space average and that the influence of the starting point decays exponentially, cf. Subsection **??**. The space average is described by a probability on the set of all possible states of the system, this probability one would call the "statistics of the dynamical system". The fluctuations of the temporal average can also be related to the aforementioned probability on states, see Subsection **??**. Another aspect is that the aforementioned probability on states is not only stable under the changes of dynamics but even differentiable, see Subsection **??**. Finally, the techniques are applied to derive the convergence and consistency of the simple stochastic numerical algorithm. Section 6 is based on,[??] and.[?] Section 7 follows mainly and.[?]

2. Homeomorphisms of the circle

2.1. *Rigid rotations*

It is convenient to define the circle as a subset of the complex plane as
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$
For $\alpha \in [0, 2\pi)$, the rotation of angle α is defined as
$$R_\alpha : S^1 \to S^1, \quad R_\alpha(z) = e^{i\alpha} \cdot z, \text{ for all } z \in S^1.$$
That is, the point $e^{i\theta}$ on S^1 is mapped to the point $e^{i\alpha} \cdot e^{i\theta} = e^{i(\alpha+\theta)}$ on S^1.

However, sometimes it is convenient to use an alternative notation for the rotation of the circle. We may identify S^1 with the quotient space $\mathbb{R}/\mathbb{Z}$, which is the same as the interval $[0, 1]$ with 0 and 1 identified. The identification is given by the explicit map $x \in [0, 1] \mapsto e^{2\pi i x} \in S^1$. Then, for $a \in [0, 1)$, the rotation of angle $2\pi a$ becomes
$$T_a : [0, 1) \to [0, 1), \quad T_a(x) = x + a \pmod 1$$
I.e.
$$T_a(x) = \begin{cases} x + a, & \text{if } 0 \le x + a < 1, \\ x + a - 1, & \text{if } 1 \le x + a < 2. \end{cases}$$

Exercise 4.1. *Show that the above definition provides a well-defined homeomorphism of the circle* $\mathbb{R}/\mathbb{Z}$.

Recall that the orbit of a point $z \in S^1$ under the rotation R_α is defined as the sequence
$$z, R_\alpha(z), R_\alpha \circ R_\alpha(z), R_\alpha \circ R_\alpha \circ R_\alpha(z), \dots.$$
To simplify the notations, we use the expression $f^{\circ n}$ to denote the map obtained from composing f with itself n times. For example, $f^{\circ 1} = f$, $f^{\circ 2} = f \circ f$, $f^{\circ 3} = f \circ f \circ f$, etc. Following the standard conventions, $f^{\circ 0}$ denotes the identity map.

Due to the basic algebraic form of the rigid rotations, we are able to obtain a simple formula for the orbits. However, this is very exceptional in the study of dynamical systems. Let us first consider the case that $\alpha = 2\pi \cdot \frac{p}{q}$, where p/q is a rational number. We assume that $q \neq 0$, $p \in \mathbb{Z} - \{0\}$, and p/q is in the reduced form, that is, p and q are relatively prime. Then, the orbit of z under R_α becomes
$$\begin{aligned} &z, e^{2\pi i \frac{p}{q}} \cdot z, e^{2\pi i \frac{2p}{q}} \cdot z, \dots, e^{2\pi i \frac{qp}{q}} \cdot z, \dots \\ &\qquad = z, e^{2\pi i \frac{p}{q}} \cdot z, e^{2\pi i \frac{2p}{q}} \cdot z, \dots, e^{2\pi i \frac{(q-1)p}{q}} \cdot z, \\ &\qquad\qquad z, e^{2\pi i \frac{p}{q}} \cdot z, e^{2\pi i \frac{2p}{q}} \cdot z, \dots, e^{2\pi i \frac{(q-1)p}{q}} \cdot z, z, \dots. \end{aligned}$$
This is a periodic sequence of q points on the circle.

In contrast, when α is irrational the situation is very different. Before we discuss that, we recall some basic definitions. A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all x_1, x_2, and x_3 in X we have

(i) $d(x_1,x_2) = d(x_2,x_1)$;
(ii) $d(x_1,x_2) \geq 0$, where the equality occurs only if $x_1 = x_2$;
(iii) $d(x_1,x_2) \leq d(x_1,x_3) + d(x_3,x_2)$.

The Euclidean metric defined as $d(x,y) = |x-y|$ on $\mathbb{R}$ (or on any Euclidean space $\mathbb{R}^n$) is a prominent example of a metric. On S^1 (and any n-dimensional sphere) we may defined the function $d(x,y)$ as the length of the shortest arc on S^1 connecting x to y.

The notion of metric allows one to talk about convergence of sequences on X. We say that a sequence x_n, $n \geq 1$, on X *converges* to some point $x \in X$ with respect to some metric d defined on X if the sequence of real numbers $d(x_n,x)$ tends to 0 as n tends to infinity. Let X be a set equipped with the metric d. An orbit x_n, $n \geq 1$, is said to be *dense* on X if for every x in X there is a sub-sequence of the sequence x_n that converges to x. This is equivalent to saying that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $d(x_n,x) \leq \epsilon$.

Proposition 4.1. *If a is an irrational number, then for each $z \in S^1$, the orbit $\{R_{2\pi a}^{\circ n}(z) : n \in \mathbb{Z}\}$ is infinite and dense on S^1.*

Proof. Let z be an arbitrary point on S^1, and let $\alpha = 2\pi a$. If $R_\alpha^{\circ m}(z) = R_\alpha^{\circ n}(z)$ for some integers m and n, we must have $e^{(m-n)\alpha i} \cdot z = z$. As $z \neq 0$, and $(m-n)\alpha$ cannot be an integer multiple of 2π, we must have $m = n$. In other words, the orbit of z is an infinite sequence.

Fix an arbitrary $w \in S^1$ and an $\epsilon > 0$. We aim to find $n \in \mathbb{Z}$ with $d(R_\alpha^{\circ n}(z), w) \leq \epsilon$.

Choose $n_0 > 0$ with $(2\pi)/n_0 < \epsilon$. Consider the $n_0 + 1$ points on the circle $R_\alpha^i(z)$, for $0 \leq i \leq n_0$. There must be integers $0 \leq l < k \leq n_0$ such that $d\big(R_\alpha^{\circ k}(z), R_\alpha^{\circ l}(z)\big) < 2\pi/n_0$, where d is the arc length metric on S^1. Since R_α is an isometry, that is, it preserves distances, we must have $d\big(z, R_\alpha^{\circ(k-l)}(z)\big) < \epsilon$.

As $R_\alpha^{\circ(k-l)}$ is an isometry, by the above paragraph the sequence z, $R_\alpha^{\circ(k-l)}(z)$, $R_\alpha^{\circ 2(k-l)}(z)$, $R_\alpha^{\circ 3(k-l)}(z)$ consists of points on the circle that are at most ϵ apart. In particular, there is $j \in \mathbb{N}$ such that $d(R_\alpha^{\circ j(k-l)}(z), w) < \epsilon$. □

Recall that a metric space (X,d) is *compact* if any sequence in X has a subsequence converging to some point in X. For example, the interval $(0,1]$ is not compact since the sequence $1/n$, $n \geq 1$, does not converge to some point in $(0,1]$. On the other hand each interval $[a,b]$ (with respect to the Euclidean metric), the circle S^1 with respect to the arc length, and the two dimensional sphere as a subset of $\mathbb{R}^3$ are compact spaces.

Definition 4.1. *Let X be a compact metric space and $T : X \to X$ be a continuous map. We say that $T : X \to X$ is topologically transitive if there exists $x \in X$ such that the orbit $\{T^{\circ n}(x) : n \in \mathbb{Z}\}$ is dense in X. We say that $T : X \to X$ is minimal if for every $x \in X$, the orbit $\{T^{\circ n}(z), n \in \mathbb{Z}\}$ is dense in X.*

A topologically transitive dynamical system cannot be decomposed into two disjoint sets with nonempty interiors which do not interact under the transformation.

Exercise 4.2. *Give an example of a metric space X and a continuous map $T: X \to X$ that is topologically transitive, but not minimal.*

2.2. *Distribution of orbits*

We now look at the problem of quantifying the time of visiting an interval. If α is irrational then the proportion of the orbit $z, R_\alpha(z), R_\alpha^{\circ 2}(z), \ldots$ which lies inside a given arc becomes the length of the arc divided by 2π. This is made precise in the next theorem.

Theorem 4.1. *If a is irrational and $\phi : [0,1] \to \mathbb{R}$ is a continuous function with $\phi(0) = \phi(1)$, then for any $x \in [0,1)$,*

$$\lim_{n\to\infty} \Big(\frac{1}{n} \sum_{k=0}^{n-1} \phi\big(T_a^{\circ k}(x)\big)\Big) = \int_{[0,1]} \phi(y)\, dy.$$

Proof. Let us first consider the functions

$$e_m(x) = e^{2\pi i m x} = cos(2\pi m x) + isin(2\pi m x), m \in \mathbb{Z}.$$

We have $e_m(T_a^{\circ k}(x)) = e^{2\pi i m(x+ka)} = e^{2\pi i m k a} e_m(x)$. Thus, for $m \neq 0$,

$$\begin{aligned}\left| \frac{1}{n} \sum_{k=0}^{n-1} e_m\big(T_a^{\circ k}(x)\big)\right| &= \frac{1}{n} \cdot |e^{2\pi i m x}| \cdot \left| \sum_{k=0}^{n-1} e^{2\pi i m k a} \right| \\ &= \frac{1}{n} \cdot 1 \cdot \left| \frac{1 - e^{2\pi i n m a}}{1 - e^{2\pi i m a}} \right| \le \frac{1}{n} \cdot \frac{2}{|1 - e^{2\pi i m a}|} \to 0\end{aligned}$$

as n tends to infinity. Thus, if $\phi(x) = \sum_{m=-N}^{N} a_m e_m(x)$, with $a_{-N}, a_{-N+1}, \ldots, a_N \in \mathbb{C}$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi\big(T_a^{\circ k}(x)\big) = a_0 = \int \phi(y)\, dy.$$

Since trigonometric polynomials are dense in the space of all periodic continuous functions, we obtain the result in the theorem. □

Exercise 4.3. *By an example show that the continuity assumption in Theorem 4.1 is necessary.*

As an application of the above theorem, we look at the distribution of the first digits of 2^n, $n \ge 1$.

Proposition 4.2. *Fix $p \in \{1, 2, \ldots, 9\}$. The frequency of those n for which the first digit of 2^n is equal to p, that is,*

$$\lim_{N\to\infty} \frac{\big\{1 \le n \le N : \textit{first digit of } 2^n \textit{ is equal to } p\big\}}{N} = \log_{10}\big(1 + \frac{1}{p}\big).$$

Proof. The first digit of 2^n is equal to p if and only if for some $k \geq 1$,

$$p \times 10^k \leq 2^n < (p+1)10^k.$$

This is equivalent to $\log_{10} p + k \leq n \log_{10} 2 < \log_{10}(p+1) + k$, which is also equivalent to

$$n \log_{10} 2 \pmod 1 \in [\log_{10} p, \log_{10}(p+1)).$$

Let us define

$$\chi(x) = \begin{cases} 1, & \text{if } x \in [\log_{10} p, \log_{10}(p+1)] \\ 0, & \text{otherwise.} \end{cases}$$

Let $a = \log_{10} 2 \in \mathbb{R} \setminus \mathbb{Q}$, and $x = \log_{10} 2$. We claim that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi\big(T_a^{\circ n}(x)\big) = \int_{[0,1]} \chi(y)\, dy = \log_{10}\big(1 + \frac{1}{p}\big). \tag{1}$$

If χ was continuous, this would directly follows from the theorem. However, we need a bit more work. Given $\delta > 0$, there are continuous functions $\chi_1 \leq \chi \leq \chi_2$ defined on $[0,1]$ such that $\int_{[0,1]} |\chi_1 - \chi_2|\, dy < \delta$. Then,

$$\begin{aligned} \int \chi_2\, dy = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_2\big(T_a^{\circ n}(x)\big) &\geq \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=0}^{N-1} \chi\big(T_a^{\circ n}(x)\big) \\ &\geq \underline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=0}^{N-1} \chi\big(T_a^{\circ n}(x)\big) \geq \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_1\big(T_a^{\circ n}(x)\big) = \int \chi_1\, dy. \end{aligned}$$

The notation $\overline{\lim}$ denotes the lim sup (supremum limit) of a given sequence. When a sequence is convergent, $\overline{\lim}$ gives the same limit, but when there are many convergent sub-sequences, it gives the maximum of all the limits of all convergent sub-sequences. The advantage is that $\overline{\lim}$ always exists, although it may be infinite.

Since $\delta > 0$ was arbitrary, the above inequalities imply that the limit in Eq. (1) exists. On the other hand, as $\delta \to 0$, $\int \chi_2\, dy \to \int \chi\, dy$ and $\int \chi_1\, dy \to \int \chi\, dy$. Hence, the limit must be equal to $\int \chi(y)\, dy$.

It is clear that Eq. (1) implies the equality in the theorem. □

2.3. *Homeomorphisms of the circle*

Consider the natural projection

$$\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}, \quad \pi(x) = x \pmod 1.$$

For all $i \in \mathbb{Z}$ and $x \in \mathbb{R}$, $\pi(x+i) = \pi(x)$.

Proposition 4.3. *Let $f : S^1 \to S^1$ be a homeomorphism of the circle. Then there exists a homeomorphism $F : \mathbb{R} \to \mathbb{R}$, called a lift of f, such that $f \circ \pi = \pi \circ F$ on $\mathbb{R}$.*

Moreover, F is unique up to adding an integer. That is, if F and G are lifts of f then there is $n \in \mathbb{Z}$ such that for all $x \in \mathbb{R}$, $F(x) = G(x) + n$.

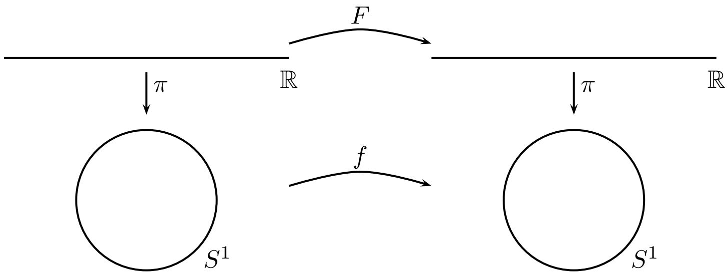


Fig. 1. Illustration of the maps in Proposition 4.3.

Proof. Given x and $y = f(x) \in S^1$, choose x', and $y' \in \mathbb{R}$ with $\pi(x') = x$, $\pi(y') = y$. Define $F(x') = y'$. Now one can use the functional equation $f \circ \pi = \pi \circ F$ to extend F to a continuous map from $\mathbb{R}$ to $\mathbb{R}$.

If G is another lift, we must have $G(x') = y' + n'$, for some integer n'. This implies that for all $x \in \mathbb{R}$, $G(x) = F(x) + n'$. $\square$

Exercise 4.4. *Show that any lift F of a circle homeomorphism under π satisfies $F(x+i) = F(x)+i$, for all i in $\mathbb{Z}$ and $x \in \mathbb{R}$. On the other hand, prove that any homeomorphism $F : \mathbb{R} \to \mathbb{R}$ that satisfies $F(x+c) = F(x)+c$, for some positive constant c, induces a homeomorphism of the circle.*

Example 4.1. *Let $f : S^1 \to S^1$ be a rotation by $\alpha = 2\pi a$. The lifts of f are given by the formulas*

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x) = x + a + n,$$

where $n \in \mathbb{Z}$.

Exercise 4.5. *Let $f_\epsilon : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined as $f_\epsilon(x) = x + \epsilon sin(2\pi x) \pmod 1$, for $|\epsilon| < \frac{1}{2\pi}$. Then, the lifts of f_ϵ are defined by*

$$F_\epsilon(x) = x + \epsilon sin(2\pi x) + n,$$

for $n \in \mathbb{Z}$. Show that if f_ϵ is a homeomorphism, we must have $|\epsilon| < \frac{1}{2\pi}$.

Exercise 4.6. *Is $F(x) = x + \frac{1}{2} sin(x)$ the lift of a circle homeomorphism? How about $F(x) = x + \frac{1}{4x} sin(2\pi x)$?*

Remark 4.1. *We always assume that $f : S^1 \to S^1$ is orientation preserving, that is, the graph of f is strictly increasing.*

It is possible to assign a rotation number to a homeomorphism of the circle that records the "combinatorial rotation" of the map on the circle. Note that individual points may be rotated by different values.

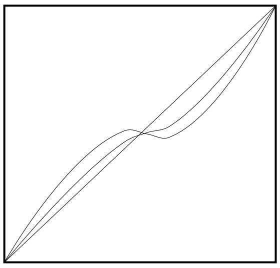


Fig. 2. The graph of the function f_ϵ for three different values of ϵ.

Proposition 4.4. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism, and let $F : \mathbb{R} \to \mathbb{R}$ be a lift of F. Then, for each $x \in \mathbb{R}$, the limit*

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \pmod 1$$

exists. Moreover, the limit is independent of $x \in \mathbb{R}$ and the choice of the lift F.

Proof. We present the proof of the above proposition in several steps.

Step 1. If the limit exists, it is independent of the choice of the lift.

If G is another lift of f, by Proposition 4.3, there is $k \in \mathbb{Z}$ such that $G(x) = F(x)+k$. By Ex. 4.4, for all $x \in \mathbb{R}$ and all $i \in \mathbb{Z}$ we have $G(x+i) = G(x)+i$. Hence,

$$G^{\circ 2}(x) = G(G(x)) = G(F(x)+k) = G(F(x))+k = F(F(x))+k+k = F^{\circ 2}(x)+2k.$$

In general, one can see that for all $n \in \mathbb{N}$, $G^{\circ n}(x) = F^{\circ n}(x)+nk$. Therefore,

$$\lim_{n\to\infty} \frac{G^{\circ n}(x)}{n} = \lim_{n\to\infty} \frac{F^{\circ n}(x)+nk}{n} = \lim_{n\to\infty} \frac{F^{\circ n}(x)}{n} + k.$$

Therefore, if the limit exists, we obtain the same values modulo 1.

Step 2. The limit is independent of the choice of $x \in \mathbb{R}$.

Let $y \in \mathbb{R}$ be another choice that satisfies $|x-y| < 1$. Note that by the definition of the lift, for each x and y in $\mathbb{R}$ with $|x-y| < 1$ we have $|F(x)-F(y)| < 1$. Repeating this property inductively, we conclude that for all $n \geq 1$, $|F^{\circ n}(x) - F^{\circ n}(y)| < 1$. Hence, $|F^{\circ n}(x)/n - F^{\circ n}(y)/n| < \frac{1}{n} \to 0$, as $n \to \infty$.

By the above paragraph, when $|x-y| < 1$, $\lim_{n\to\infty} F^{\circ n}(x)/n$ is the same as $\lim_{n\to\infty} F^{\circ n}(y)/n$, provided they exist. For arbitrary x and y in $\mathbb{R}$, there is a finite sequence of points $x = t_0 < t_1 < t_2, ..., t_n = y$ with all $|t_{i+1} - t_i| < 1$. Then, provided the limits exist, we must have

$$\lim_{n\to\infty} \frac{F^{\circ n}(x)}{n} = \lim_{n\to\infty} \frac{F^{\circ n}(t_1)}{n} = ... = \lim_{n\to\infty} \frac{F^{\circ n}(y)}{n}.$$

Step 3. The limit exists.

For each $n \geq 1$, there is an integer k_n with $k_n \leq F^{\circ n}(0) < k_n + 1$. Then,

$$\left| \frac{F^{\circ n}(0)}{n} - \frac{k_n}{n} \right| \leq \frac{1}{n}.$$

Since each iterate $F^{\circ n}$ is a monotone map, $t_1 \leq t_2$ implies $F^{\circ n}(t_1) \leq F^{\circ n}(t_2)$. Thus,

$$2k_n \leq k_n + F^{\circ n}(0) \leq F^{\circ n}(k_n) \leq F^{\circ n}(F^{\circ n}(0)) = F^{\circ 2n}(0),$$

and

$$F^{\circ 2n}(0) = F^{\circ n}(F^{\circ n}(0)) \leq F^{\circ n}(k_n + 1) = k_n + 1 + F^{\circ n}(0) \leq 2(k_n + 1).$$

In general, repeating the above argument several times one concludes that for $m \geq 1$,

$$mk_n \leq F^{\circ(nm)}(0) \leq m(k_n + 1).$$

Thus,

$$\left| \frac{F^{\circ(nm)}(0)}{nm} - \frac{k_n}{n} \right| \leq \frac{1}{n},$$

and so

$$\begin{aligned} &\left| \frac{F^{\circ m}(0)}{m} - \frac{F^{\circ n}(0)}{n} \right| \\ &\leq \left| \frac{F^{\circ m}(0)}{m} - \frac{k_m}{m} \right| + \left| \frac{k_m}{m} - \frac{F^{\circ(nm)}(0)}{nm} \right| + \left| \frac{F^{\circ(nm)}(0)}{nm} - \frac{k_n}{n} \right| + \left| \frac{k_n}{n} - \frac{F^{\circ n}(0)}{n} \right| \\ &\leq \frac{1}{m} + \frac{1}{m} + \frac{1}{n} + \frac{1}{n}. \end{aligned}$$

In particular, $F^{\circ n}(0)/n$ forms a Cauchy sequence, and hence it converges. □

Exercise 4.7. *Let $f : S^1 \to S^1$ be a homeomorphism of S^1. Show that $\rho(f^{\circ m}) = m\rho(f) \mod 1$, where $\rho(f)$ denotes the rotation number of f.*

Exercise 4.8. *Let f and g be orientation preserving homeomorphisms of S^1. Prove that $\rho(f) = \rho(g^{-1}fg)$, where ρ denotes the rotation number.*

The notion of rotation number defined in Proposition 4.4 is quite informative, as illustrated in the next two lemmas.

Lemma 4.1. *If a homeomorphism $f : S^1 \to S^1$ has a periodic point $f^{\circ N}(z) = z \in S^1$, then, $\rho(f)$ is a rational number.*

Proof. Let F be a lift of f and choose x with $\pi(x) = z$. By the definition of the lift, we have $\pi \circ F^{\circ N}(x) = f^{\circ N} \circ \pi(x) = f^{\circ N}(z) = z$. Thus, there is $l \in \mathbb{Z}$ such that $F^{\circ N}(x) = x + l$.

For each $n \geq 1$ there are $k \geq 0$ and r with $0 \leq r \leq N - 1$ such that $n = kN + r$. Then,

$$\begin{aligned} \lim_{n\to\infty} \frac{F^{\circ n}(x)}{n} &= \lim_{n\to\infty} \frac{F^{\circ(kN+r)}(x)}{n} = \lim_{n\to\infty} \frac{F^{\circ r}\big(F^{\circ(kN)}(x)\big)}{n} \\ &= \lim_{n\to\infty} \frac{F^{\circ r}\big(x + kl\big)}{n} = \lim_{n\to\infty} \frac{F^{\circ r}(x) + kl}{n} = \lim_{n\to\infty} \frac{kl}{kN + r} = \frac{l}{N}. \end{aligned}$$

□

Exercise 4.9. *Let $F(x) = x + c + b\sin(2\pi x)$. Show that if $|2\pi b| < 1$ then this is an orientation preserving homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. If $|c| < |b|$ show that $\rho(f) = 0$ for the induced map $f : S^1 \to S^1$.*

Lemma 4.2. *If $f : S^1 \to S^1$ is a homeomorphism of the circle with a rational rotation number, then f has a periodic point.*

Proof. Let F be a lift of f with $\lim_{n\to\infty} \frac{F^{\circ n}(x)}{n} \pmod 1 = \frac{p}{q} \in \mathbb{Q}$. Note that $F^{\circ q}$ is a lift of $f^{\circ q}$, and we have

$$\lim_{n\to\infty} \frac{(F^{\circ q})^{\circ n}(x)}{n} = \lim_{n\to\infty} \frac{F^{\circ qn}(x)}{n} = q \lim_{n\to\infty} \frac{F^{\circ qn}(x)}{qn} = q\frac{p}{q} = p = 0 \mod 1.$$

Thus, $\rho(f^{\circ q}) = 0$. The map $G = F - p$ is also a lift of $f^{\circ q}$ and we have

$$\lim_{n\to\infty} \frac{G^{\circ n}(x)}{n} = 0. \tag{2}$$

We claim that $G : \mathbb{R} \to \mathbb{R}$ must have a fixed point. Assuming this for a moment, the fixed point projects to a fixed point for $f^{\circ q}$, which must be a periodic point for f.

If in the contrary G has no fixed point, then either (i) for all $y \in \mathbb{R}$ we have $G(y) > y$, or (ii) for all $y \in \mathbb{R}$ we have $G(y) < y$. If (i) occurs, since $G(y) - y$ is continuous on the closed interval $[0,1]$, and strictly positive, there is $\delta > 0$ such that $G(y) - y \geq \delta$. As $G : \mathbb{R} \to \mathbb{R}$ is a lift, for all $x \in \mathbb{R}$ and all $i \in \mathbb{Z}$, we have $G(x+i) = G(x) + i$. These imply that for all $y \in \mathbb{R}$, we have $G(y) - y \geq \delta$. Repeating this inequality inductively, we have $G^{\circ n}(0) \geq 0 + n\delta = n\delta$. Thus,

$$\lim_{n\to\infty} \frac{G^{\circ n}(0)}{n} \geq \frac{n\delta}{n} = \delta.$$

This contradicts Eq. (2).

The proof in case (ii) is similar to the above one where one shows that $\lim_{n\to\infty} G^{\circ n}(0)/n \leq -\delta$. ☐

The following is a classical result on the homeomorphisms of the circle. See [**?**, Thm 11.2.7] for a proof.

Theorem 4.2 (Poincaré)**.** *Assume that $f : S^1 \to S^1$ is a homeomorphism that is minimal and $\rho(f)$ is irrational. Then there is a homeomorphism $\phi : S^1 \to S^1$ such that, $R_{2\pi\rho(f)} \circ \phi = \phi \circ f$.*

The statement of the above theorem may be illustrated by the commutative diagram

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\phi"] & S^1 \arrow[d, "\phi"] \\ S^1 \arrow[r, "R_{2\pi\rho(f)}"] & S^1 \end{tikzcd}$$

The homeomorphism ϕ in the above proposition is called *topological conjugacy.* It motivates the following definition.

Definition 4.2. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps. We say that f is topologically conjugate to g if there is a homeomorphism $\phi : X \to Y$ such that $g \circ \phi = \phi \circ f$ holds on X. The conjugacy is called C^1, or smooth, or analytic, if we further require that ϕ is C^1, or C^∞, or analytic, respectively.*

Exercise 4.10. *Let $Q_c(x) = x^2 + c$. Prove that if $c < \frac{1}{4}$ there is a unique $\mu > 1$ such that Q_c is topologically conjugate to $f_\mu(x) = \mu x(1-x)$ via a map of the form $h(x) = \alpha x + \beta$.*

When two dynamical systems are topologically conjugate, the two systems behave the same in terms of topological properties. For example, if some sub-sequence $f^{\circ n_k}(x)$ converges to some point $x' \in X$ the corresponding sub-sequence $g^{\circ n_k}(\phi(x))$ converges to $\phi(x')$. However, $f^{\circ n_k}(x)$ may converge to x' exponentially fast, but the latter convergence may be very slow. In general higher regularity of the conjugacy is required to have similar fine properties for the two systems.

Exercise 4.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map, and $x \in \mathbb{R}$ be a periodic point of f of minimal period q. That is, q is the smallest positive integer with $f^{\circ q}(x) = x$. The quantity $(f^{\circ q})'(x)$ is called the multiplier of f at x. Show that all points in the orbit of x have the same multipliers, i.e. the notion of multiplier is well-defined for a periodic orbit.*

Definition 4.3. *We say that a continuous function $w : [0,1] \to \mathbb{R}$ has bounded variation if*

$$\sup\Big\{\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)| : 0 \le x_1 < x_2 < ... < x_n = 1\Big\} < \infty.$$

Exercise 4.12. *For $n = 1, 2$, define the function $w_n : [0,1] \to \mathbb{R}$ as*

$$\begin{cases} 0, & \text{if } x = 0, \\ x^n sin(\frac{1}{x}), & \text{if } x \neq 0. \end{cases}$$

Show that w_1 is not a function of bounded variation, but w_2 is a function of bounded variation.

Theorem 4.3 (Denjoy)**.** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle with irrational rotation $\rho(f) = \rho$. Moreover, assume that $f : S^1 \to S^1$ is continuously differentiable and that $w(x) = \log|f'(x)|$ has bounded variation. Then $f : S^1 \to S^1$ is minimal.*

Exercise 4.13. *Assume $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be smooth maps that are conjugate by a C^1 map ϕ. Prove that the map ϕ preserve the multipliers of periodic points. That is, if x is a periodic point of f, then x and $\phi(x)$ have the same multipliers. By giving an example, show that if the conjugacy is not smooth but only a homeomorphism, the multipliers are not necessarily preserved.*
[hint: build topologically conjugate maps that have fixed points with distinct multipliers.]

We do not give a proof of the above theorem in these notes, see for instance [**?**, thm 12.1.1], but instead we present an example that shows the necessity of the assumption. This construction is known as "surgery", and is widely used in constructions of examples in dynamics and other areas of mathematics.

Example 4.2 (Denjoy's Example)**.** *For each irrational ρ, there is a C^1 diffeomorphism $f : S^1 \to S^1$ with rotation number $\rho(f) = \rho$, which is not minimal.*

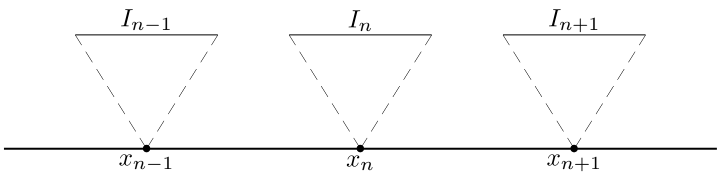
Let us introduce the positive numbers

$$l_n = \frac{1}{(|n|+3)^2}, \quad n \in \mathbb{Z}.$$

We have

$$\sum_{n\in\mathbb{Z}} l_n \le 2\sum_{n=3}^{\infty} \frac{1}{n^2} \le 2\int_2^{\infty} \frac{1}{x^2}\,dx = 1.$$

Fix $x \in [0,1)$, and note that since ρ is irrational, the points in the orbit $x_n = T_\rho^{\circ n}(x)$, $n \in \mathbb{Z}$, are distinct. For each $n \in \mathbb{Z}$, we remove the point x_n from the segment $[0,1)$ and replace it by a closed interval I_n of length l_n. After repeating this process for all points, we end up with an interval of length $1 + \sum_{n\in\mathbb{Z}} l_n$.



On the complement of the intervals $\cup_n I_n$, we define f as the map induced from the rotation T_ρ. On the intervals I_n we want to arrange the map so that $f(I_n) = I_{n+1}$, for each $n \in \mathbb{Z}$. It is enough to specify f' in the intervals I_n so that f' is equal to 0 at the end of the intervals. Let $I_n = [a_n, a_n + l_n]$ and set

$$f'(x) = \begin{cases} 1, & x \notin \bigcup_{n\in\mathbb{Z}} I_n \\ 1 + c_n - \frac{c_n}{l_n}|2(x-a_n) - l_n|, & x \in I_n, \text{ for some } n \in \mathbb{Z} \end{cases}$$

where $c_n = 2\big(\frac{l_{n+1}}{l_n} - 1\big)$. We have chosen c_n such that

$$\int_{I_n} f'(x) = \int_{a_n}^{a_n+l_n} \big(1 + c_n - \frac{c_n}{l_n}|2(x-a_n) - l_n|\big)\,dx = l_n + c_n l_n - \frac{c_n}{l_n}\frac{l_n^2}{2} = l_{n+1}.$$

Exercise 4.14. *Show that the map f introduced above is not transitive. [hint: look at the orbit of x when $x \in \cup_n I_n$ and when $x \in S^1 \setminus \cup_n I_n$.]*

3. Expanding Maps of the Circle

In this section we consider a different class of dynamical systems on the unit circle S^1.

Definition 4.4 (Expanding). *A C^1 map $f : S^1 \to S^1$ is called expanding if for all $x \in S^1$, $|f'(x)| > 1$.*

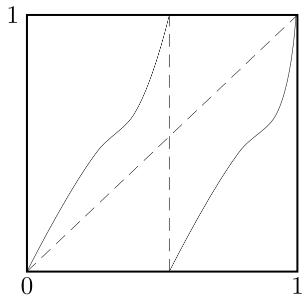


Fig. 3. The graph of an expanding degree two map of the circle.

An expanding map $f : S^1 \to S^1$ cannot be a homeomorphism. Also, since f' is continuous and S^1 is compact, there is $\beta > 1$ such that for all $x \in S^1$, $|f'(x)| \geq \beta$.

Example 4.3. *Let $m \geq 2$ be an integer, and define $f : [0,1) \to [0,1)$ as $f(x) = mx$* (mod 1). *If we regard the circle as $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, then f can be written as $f(z) = z^m$. Each of these maps is expanding.*

Definition 4.5. *The degree of an expanding map $f : S^1 \to S^1$, denoted by* $\deg(f)$*, is defined as the number of points in the set $f^{-1}(x)$, for $x \in S^1$. One can see that the notion of degree is independent of the choice of x.*

Lemma 4.3. *If f and $g : S^1 \to S^1$ are expanding maps, then we have $deg(f \circ g) = deg(f) \cdot deg(g)$. In particular, $deg(f^{\circ n}) = (deg(f))^n$, for $n \geq 1$.*

Proof. Since for each $y \in f^{-1}(x)$ the set $g^{-1}(y)$ has $deg(g)$ elements, $(f \circ g)^{-1}(x)$ has $deg(g)deg(f)$ elements. □

Proposition 4.5. *If $f : S^1 \to S^1$ is an expanding map with* $\deg(f) = d \geq 2$*, the number of periodic points of period n is $(d^n - 1)$.*

Proof. First assume $n = 1$, the number of fixed points of f is equal to the number of points on the intersection of the diagonal with the graph of f, which is $d - 1$. For arbitrary $n \geq 2$, we consider $f^{\circ n}$ with $\deg(f^{\circ n}) = d^n$. Note that the number of periodic points of period n is equal to the number of the fixed points of $f^{\circ n}$, that is, $d^n - 1$. □

Proposition 4.6. *Let X be a compact metric space, $f : X \to X$ be continuous. The following are equivalent*

(i) f is topologically transitive;
(ii) for all non-empty and open sets U and V in X, there is $n \in \mathbb{N}$, with $f^{-n}(V) \bigcap U \neq \emptyset$.

Proof. First we prove that (i) implies (ii). Let $\{f^{\circ n}(x)\}_{n=1}^{\infty}$ be a dense orbit in X. Choose integers $m > n$ with $f^{\circ n}(x) \in U$ and $f^{\circ m}(x) \in V$. Then,

$$f^{\circ n}(x) \in U \cap f^{-(m-n)}(V).$$

Thus, the intersection is non-empty.

Now we prove that (ii) implies (i). Let $Y = \{y_i\}_{i=1}^{\infty}$ be a countable dense set in X. Any compact metric space has a countable dense set. (For instance, when $X = S^1$, one can take all the points with rational angles.) Let U_i denoted the ball of radius $1/i$ about y_i. We aim to find a point $x \in X$ whose orbit visits every U_i.

Choose $N_1 \geq 0$ such that $f^{-N_1}(U_2) \cap U_1 \neq \emptyset$. Then choose an open disk V_1 of radius less than $1/2$ such that

$$V_1 \subseteq \overline{V}_1 \subseteq U_1 \bigcap f^{-N_1}(U_2).$$

Above, $\overline{V}_1$ denotes the closed disk obtained from adding the boundary of V_1 to V_1. Then choose N_2 such that $f^{-N_2}(U_3) \cap V_1 \neq \emptyset$. Choose an open disk V_2 of radius less than $1/2^2$ such that

$$V_2 \subseteq \overline{V}_2 \subseteq V_1 \cap f^{-N_2}(U_3).$$

Inductively repeating the above process, we obtain disks $V_1 \supseteq V_2 \supseteq V_3 \supseteq ...$ with radius $V_n \leq \frac{1}{2^n}$ and

$$\overline{V}_{n+1} \subseteq V_n \cap f^{-N_{n+1}}(U_{n+2}).$$

Now we define x as the unique point in the intersection $\bigcap_{n=1}^{\infty} \overline{V}_n$. It easily follows that $f^{\circ N_{n-1}}(x) \in U_n$, for $n \geq 1$. This implies that $\{f^{\circ n}(x)\}_{n=1}^{\infty}$ is dense in X. □

Exercise 4.15. *Let $f : X \to X$ be a continuous map of a compact metric space. A point $p \in X$ is called topologically recurrent if for any open set V containing p, there exists $n \geq 1$ with $f^{\circ n}(p) \in V$. Clearly every periodic point is recurrent.*

(i) Give an example of a map $f : X \to X$ with a non-periodic recurrent point.
(ii) Give an example of a map $f : X \to X$ with a non-periodic recurrent point p whose orbit is not dense in X. [hint: look at the map in Example 4.2.]

Definition 4.6. *Let X be a compact metric space, and $f : X \to X$ be continuous. We say that f is topologically mixing if for any two non-empty open sets U and V in X, there exists $N \geq 0$ such that for all $n \geq N$, $U \cap f^{-n}(V) \neq \emptyset$.*

By Proposition 4.6, any mixing transformation is topologically transitive. But, the notion of topological mixing is stronger than the notion of topological transitivity.

Exercise 4.16. *Show that an irrational rotation $R_\alpha : S^1 \to S^1$ is transitive, but is not topologically mixing.*

Exercise 4.17. *Let X be a compact metric space with more than one point and $f : X \to X$ be an isometry. Show that f cannot be topologically mixing.*

Exercise 4.18. *Let X be a compact metric space with at least three distinct points and let $f : X \to X$ be an isometry.*

(i) Show that f is not mixing.
(ii) What if X has only two points?

Proposition 4.7. *An expanding map $f : S^1 \to S^1$ is mixing.*

Proof. Because f is expanding, there is $\beta > 1$ such that for all $z \in S^1$, $|f'(z)| \geq \beta$. Let $d = \deg(f) \geq 2$. There is a lift of f to a homeomorphisms $F : \mathbb{R} \to \mathbb{R}$ that satisfies $\pi \circ F(x) = f \circ \pi(x)$ and $F(x+1) = F(x) + d$, for all $x \in \mathbb{R}$. The proof of this is similar to the one for Proposition 4.3.

It follows that $\forall x \in \mathbb{R}$, $|F'(x)| \geq \beta$. For an open set U in $\mathbb{R}$, choose an interval $(a,b) \subseteq U$. Since F is C^1 and one-to-one,

$$|F(b) - F(a)| = \int_a^b F'(t)\,dt \geq \beta(b-a).$$

That is, F increases the length of intervals by a factor of β. Similarly, $F^{\circ n}$ increases the length of intervals by a factor of at least β^n. Choose N large enough so that $\beta^N > \frac{1}{b-a}$. Then for $n \geq N$, the length of $F^{\circ n}(a,b)$ is at least 1.

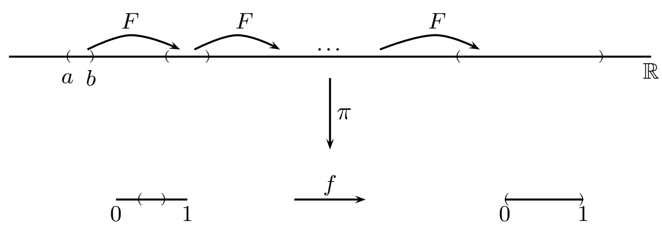


Fig. 4. The iterates of the maps F and f on an open set.

On the other hand, the relation $\pi \circ F = f \circ \pi$ implies that for all $n \geq N$, $\pi \circ F^{\circ N}(a,b) = S^1$, and hence $f^{\circ n}(U) \supseteq f^{\circ n}(a,b) = S^1$. In particular, for all open sets V, and all $n \geq N$, $f^{\circ N}(U) \cap V \neq \emptyset$, which implies,

$$U \cap f^{-n}(V) \neq \emptyset.$$

□

As a corollary of the above proof, any expanding map of the circle is topologically transitive.

Definition 4.7 (Choatic)**.** *A continuous map $f : X \to X$ of a compact metric space is called chaotic if,*

(i) f is topologically transitive; and
(ii) the set of periodic points of f is dense in X.

The notion of chaotic behavior is invariant under topological conjugacy. That is, if two maps are topologically conjugate and one of them is chaotic, the other one is also chaotic.

Example 4.4. *Consider the linear expanding map $f : S^1 \to S^1$, defined as $f(x) = mx \pmod 1$, $m \geq 2$. The periodic points of f take the form $x = \frac{j}{m^n-1}$, $0 \leq j < m^n - 1$. That is because*

$$f^{\circ n}(x) = m^n\big(\frac{j}{m^n-1}\big) = j\big(\frac{m^n-1}{m^n-1}\big) + \frac{j}{m^n-1} = x \pmod 1.$$

Such points form a dense subset of $[0, 1)$. In Theorem 4.4 we shall show that any C^1 expanding map of the circle is chaotic.

Exercise 4.19. *Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ defined as $A(x) = 2x$, observe that A induces a map $f : T^n \to T^n$, where $T^n = S^1 \times S^1 \times ... \times S^1$ (n times) is the n-dimensional torus.*

(i) Prove that the periodic points of f are dense in T^n.
(ii) Prove that eventual fixed points, i.e. the points $x \in T^n$ with $f\big(f^{\circ m}(x)\big) = f^{\circ m}(x)$, for some m, are dense in T^n.
(iii) Prove that $f : T^n \to T^n$ is chaotic.

Definition 4.8. *A continuous map $f : X \to X$ on a compact metric space is said to have sensitive dependence on initial conditions if there is $\delta > 0$ such that for all $x \in X$ and all $\epsilon > 0$, there are $y \in X$ and a positive integer $n \geq 0$ with $d(x, y) < \epsilon$ and $d(f^{\circ n}(x), f^{\circ n}(y)) \geq \delta$.*

Proposition 4.8. *Expanding circle maps have sensitive dependence on initial condition.*

Proof. By the expansion property, continuity of f', and the compactness of S^1, there is $\beta > 0$ such that $|f'(z)| \geq \beta$, for all $z \in S^1$. For the same reason, there is $\alpha > 0$ such that $|f'(z)| \leq \alpha$, for all $z \in S^1$. Note that $\alpha \geq \deg(f)$.

For x and y in S^1 with $d(x, y) < 1/2$, let $I_{x,y}$ denote the arc of smallest length connecting x to y. By the above paragraph, if $d(x, y) < 1/(2\alpha)$, then f is monotone on $I_{x,y}$ and $d(f(x), f(y)) < 1/2$. In particular, $d(f(x), f(y)) \geq \beta d(x, y)$. We claim that $\delta = 1/(2\alpha)$ satisfies the definition of sensitive dependence on initial condition.

Let x and $\epsilon > 0$ be given. If $\epsilon > \delta$ we choose $y \in S^1$ with $d(x, y) = \delta$ and $n = 0$. If $\epsilon < \delta$, we may choose any $y \in S^1$ with $0 < d(x, y) < \epsilon$. By the expansion of f and the above paragraph, there is an integer n such that $d(f^{\circ n}(x), f^{\circ n}(y)) \geq \delta$. $\square$

Proposition 4.9. *A chaotic map $f : X \to X$ of a compact metric space X is either a single periodic orbit, or has sensitive dependence on initial conditions.*

Proof. Since the set of periodic points of f are dense, if X is not a single periodic set, we must have two distinct sets

$$A = \{x, f(x), f^{\circ 2}(x), ..., f^{\circ n-1}(x) = x\}, B = \{y, f(y), f^{\circ 2}(y), ..., f^{\circ m-1}(y) = y\}.$$

Let

$$\delta = \frac{1}{8} \min \Big\{ d(f^{\circ i}(x), f^{\circ j}(y)) : 0 \leq i \leq n-1, 0 \leq j \leq (m-1) \Big\} > 0.$$

We aim to show that f satisfies the sensitive dependence on initial condition with respect to the constant δ. Fix an arbitrary $z \in X$, and $\epsilon > 0$. We may assume that $\epsilon < \delta$, otherwise, we may make ϵ smaller so that this condition holds.

We must have one of the following:

(i) $d(z, A) = \min\{d(z, w) : w \in A\} \geq 4\delta$;
(ii) $d(z, B) = \min\{d(z, w) : w \in B\} \geq 4\delta$.

We write the proof when (i) occurs. For (ii), one only needs to replace A with B in the following argument. Since periodic points are dense in X, there is a periodic point p with $d(p, z) \leq \epsilon$. Let N be the smallest positive integer with $p = f^{\circ N}(p)$.

Define

$$V = \{w \in X : d(f^{\circ i}(w), f^{\circ i}(x)) < \delta, \text{ for } 0 \leq i \leq N\}.$$

Since f has a dense orbit, there is $a \in X$ with $d(z, a) < \epsilon$ and $f^{\circ k}(a) \in V$. There exists an integer k' with $0 \leq k' \leq N - 1$ and $k + k' = jN$, for some $j \in \mathbb{N}$. Now $f^{\circ jN}(p) = p$, and

$$\begin{aligned} d(f^{\circ jN}(p), f^{\circ jN}(a)) &= d(p, f^{\circ jN}(a)) \\ &\geq d(z, f^{\circ jN}(a)) - d(p, z) \\ &\geq d(z, f^{\circ k'}(x)\big) - d(f^{\circ k'}(x), f^{\circ jN}(a)) - d(p, z) \\ &\geq 4\delta - \delta - \delta = 2\delta. \end{aligned}$$

However, a and p belong to $B(z, \epsilon)$ and $d(f^{\circ jN}(p), f^{\circ jN}(a)) \geq 2\delta$. Therefore, by the triangle inequality, at least one of $d(f^{\circ jN}(p), f^{\circ jN}(z))$ and $d(f^{\circ jN}(a), f^{\circ jN}(z))$ must be bigger than δ. This finishes the proof of the proposition. □

4. Symbolic dynamics

In this section we introduce an approach to build a symbolic model for a dynamical system. We shall focus on two examples, but the method is far reaching.

4.1. *Coding expanding maps of the circle*

For an integer $n \geq 2$, define the set

$$\Sigma_n = \{(w_0, w_1, w_2, \dots) \mid \forall i \geq 0, w_i \in \{1, 2, \dots, n\}\}.$$

We define a metric on Σ_n as

$$d((w_i)_{i=0}^{\infty}, (w_i')_{i=0}^{\infty}) = \sum_{i=0}^{\infty} \frac{|w_i - w_i'|}{2^i}.$$

The shift map $\sigma : \Sigma_n \to \Sigma_n$ is defined as

$$\sigma(w_0, w_1, w_2, \dots) = (w_1, w_2, w_3, \dots).$$

Let $f : S^1 \to S^1$ be an expanding map of degree 2, where S^1 denotes the unit circle. There is a unique fixed point $p \in S^1$. Let $q \neq p$ be the other pre-image of p, i.e. $f(q) = p$. Let Δ_1 and Δ_2 denote the closed arcs on S^1 bounded by p and q, so that $S^1 = \Delta_1 \bigcup \Delta_2$.

Given $x \in S^1$, we want to associate a $w = (w_i)_{i=0}^{\infty} \in \Sigma_2$ such that,

$$f^{\circ n}(x) \in \Delta_{w_n}, \forall n \geq 0.$$

However, if $f^{\circ n}(x) \in \Delta_1 \cap \Delta_2 = \{p, q\}$ then there are ambiguities. In this case, we can finish the sequence $w_n, w_{n+1}, ... =$ with either, $1,1,1,1,...$ or $2,2,2,2,...$ if $f^{\circ n}(x) = p$, and either $2,1,1,1,...$ or $1,2,2,2,...$ if $f^{\circ n}(x) = q$. To illustrate the situation we look at a familiar example.

Example 4.5. *Let $T : S^1 \to S^1$ be defined as $T(x) = 2x \pmod 1$. Then $p = 0$ and $q = 1/2$, $\Delta_1 = [0, 1/2]$, $\Delta_2 = [1/2, 1]$. Here, the sequence $w = (w_n)_{n=0}^{\infty}$ associated to x corresponds to a dyadic expansion*

$$x = \sum_{n=0}^{\infty} \frac{w_n - 1}{2^{n+1}}.$$

The coding is similar to the decimal expansion, with similar ambiguities.

Definition 4.9. *Let X and Y be metric spaces, and $f : X \to X$ and $g : Y \to Y$. We say that g is a factor of f if there is a continuous and surjective map $\pi : X \to Y$ such that $g \circ \pi = \pi \circ f$ on X.*

Proposition 4.10. *If $f : S^1 \to S^1$ is an expanding map of degree 2, then f is a factor of $\sigma : \Sigma_2 \to \Sigma_2$.*

Proof. We define the map $\pi : \Sigma_2 \to S^1$ as follows. For $w_0, w_1, \dots, w_{n-1} \in \{1, 2\}$, define

$$\Delta_{w_0, \dots, w_{n-1}} = \Delta_{w_0} \bigcap f^{-1}(\Delta_{w_1}) \bigcap f^{-2}(\Delta_{w_2}) \bigcap \cdots \bigcap f^{-(n-1)}(\Delta_{w_{n-1}}).$$

Given $w = (w_n)_{n=0}^{\infty}$ we obtain a nest of closed intervals

$$\Delta_{w_0} \supset \Delta_{w_0 w_1} \supset \Delta_{w_0 w_1 w_2} \supset \dots.$$

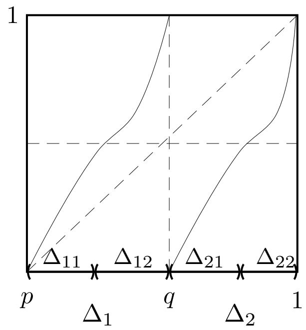


Fig. 5. The first two generations of the partitions.

That is because, $f^{\circ n}: \Delta_{w_0\dots w_{n-1}} \to S^1$ is monotone and

$$2\pi = \int_{\Delta_{w_0 w_1 \dots w_{n-1}}} \left|(f^{\circ n})'(x)\right| |dx| \geq \beta^n \cdot \text{length}(\Delta_{w_0 w_1 \dots w_{n-1}})$$

where β is the expansion constant of f. This implies that length$(\Delta_{w_0 w_1 \dots w_{n-1}})$ tends to 0 as n tends to infinity. In particular, the nest $\bigcap_{n=1}^{\infty} \Delta_{w_0 w_1 \dots w_{n-1}}$ shrinks to a single point, which we defined it as $\pi(w)$.

The map π is surjective. That is because, for $x \in S^1$ define $w = (w_n)_{n=0}^{\infty}$ according to $f^{\circ n}(x) \in \Delta_{w_n}$. This gives us $\pi(w) = x$.

The map π is continuous. That is because, if $w = (w_i)_{i=0}^{\infty}$ is close to $w' = (w'_i)_{i=0}^{\infty}$, then there is a large N such that $w_n = w'_n$ for all $n \geq N$. Then $\pi(w)$ and $\pi(w')$ belong to $\Delta_{w_0 \dots w_{N-1}}$ and we have

$$\left|\pi(w) - \pi(w')\right| \leq \text{length}(\Delta_{w_0 \dots w_{N-1}}) \leq \frac{2\pi}{\beta^N}.$$

Finally, the relation $f \circ \pi = \pi \circ \sigma$ follows immediately from the definition. That is,

$$\pi(\sigma(w)) = \bigcap_{n=1}^{\infty} \Delta_{w_1 \dots w_n} = f\Big(\bigcap_{n=0}^{\infty} \Delta_{w_0 \dots w_{n-1}}\Big) = f(\pi(w)).$$

□

Exercise 4.20. *If distinct points w and w' in Σ_2 satisfy $\pi(w) = \pi(w') = x$. Then, there is $n \geq 0$ such that $f^{\circ n}(x) = p$.*

Exercise 4.21. *Show that $\pi : \Sigma_2 \to S^1$ cannot be a homeomorphism. [Hint: Σ_2 is a union of two disjoint and closed sets, but S^1 may not be decomposed as a union of two non-empty, disjoint, and closed sets.*

Corollary 4.1. *Let $f : S^1 \to S^1$ be an expanding map of the unit circle. We have,*

(i) the periodic points of f are dense in S^1,
(ii) the map $f : S^1 \to S^1$ is topologically mixing.

Proof. Let $d = \deg(f) \geq 2$. Part (i) follows from the corresponding statement for $\sigma : \Sigma_d \to \Sigma_d$. Since, if $\sigma^{\circ p}(w) = w$ then $\pi(\sigma^{\circ p}(w)) = f^{\circ p}(\pi(w)) = \pi(w)$. That is, the image of a σ- periodic point is an f-periodic point. As σ-periodic points are dense in Σ_d, and π is continuous and surjective, the result follows.

For part (ii) we already proved that every expanding map of the circle is mixing (and hence is transitive). Here we give an alternative proof.

For nonempty and open sets U and V in S^1 we can choose $w_0, w_1, ..., w_{m-1}$ and $w'_0, w'_1, ..., w'_{m-1}$ in Σ_d such that

$$\Delta_{w_0 w_1 ... . w_{m-1}} \subseteq U, \quad \Delta_{w'_0 w'_1, ... w'_{m-1}} \subseteq V.$$

Let $[a_0, a_1, \ldots, a_k] = \{(w_i)_{i=0}^{\infty} \in \Sigma_d \mid \forall i = 0, 1, \ldots, k, w_i = a_i\}$. Then, for all $n \geq m$,

$$[w_0, w_1, ..., w_{m-1}] \bigcap \sigma^{-n}[w'_0, w'_1, ..., w'_{m-1}] \neq \emptyset.$$

For w in the above intersection,

$$x = \pi(w) \in \Delta_{w_0 \ldots w_{m-1}} \bigcap f^{-n} \Delta_{w'_0 \cdots w'_{m-1}}.$$

This finishes the proof of Part (ii). □

We now come to the main classification result for expanding maps $f : S^1 \to S^1$ of degree 2.

Theorem 4.4. *If $f : S^1 \to S^1$ and $g : S^1 \to S^1$ are two expanding maps of degree 2, then f and g are topologically conjugate. That is, there exists a homeomorphism $\pi : S^1 \to S^1$ such that $f \circ \pi = \pi \circ g$.*

By the above theorem, every expanding map of the circle of degree 2 is topologically conjugate to the linear one in Example 4.4.

Proof. Consider the conjugacies $\pi_f : \Sigma_2 \to S^1$ and $\pi_g : \Sigma_2 \to S^1$ associated to the two expanding maps f and g in Proposition 4.10. We claim that $\pi(x) = \pi_g(\pi_f^{-1}(x))$ induces a well-defined map from S^1 to S^1. That is because,

(i) if $\pi_f^{-1}(x)$ is a single point then $\pi(x)$ is well defined;
(ii) if $\pi_f^{-1}(x)$ is two points, then the sequences end with infinitely many 1's or 2's. But, then $\pi_g \circ \big(\pi_f^{-1}(x)\big)$ is again a single point.

It easily follows that the map $x \mapsto \pi_g\big(\pi_f^{-1}(x)\big)$ is one-to-one and onto.

The map π is continuous, since

$$\pi : \bigcap_{k=0}^{n-1} f^{-k} \Delta_{w_k} \to \bigcap_{k=0}^{n-1} g^{-k} \Delta_{w_k}.$$

Finally,

$$g(\pi(x)) = g(\pi_g(\pi_f^{-1}(x))) = \pi_g(\sigma(\pi_f^{-1}(x))) = \pi_g(\pi_f^{-1}(f(x))) = \pi(f(x)).$$

□

Exercise 4.22. *Show that in the above theorem, even if $f : S^1 \to S^1$ and $g : S^1 \to S^1$ are real analytic, π may not even be C^1. [Hint, give examples of f and g whose fixed points have different multipliers.]*

Theorem 4.4 extends in an obvious manner to expanding maps of degree n.

4.2. *Coding horseshoe maps*

For an integer $n \geq 2$, define the set

$$\Sigma'_n = \{(\dots, w_{-2}, w_{-1}, w_0, w_1, w_2, \dots) \mid \forall i \in \mathbb{Z}, w_i \in \{1, 2, \dots, n\}\}.$$

We define a metric on Σ'_n as

$$d((w_i)_{i\in\mathbb{Z}}, (w'_i)_{i\in\mathbb{Z}}) = \sum_{i\in\mathbb{Z}} \frac{|w_i - w'_i|}{2^{|i|}}.$$

The shift map $\sigma : \Sigma'_n \to \Sigma'_n$ is defined as $\sigma(\dots, w_{-2}, w_{-1}, w_0, w_1, w_2, \dots) = w' \in \Sigma'_n$, where the entry in the i-th coordinate of w is w_{i+1}. Note that $\sigma : \Sigma'_n \to \Sigma'_n$ is continuous, one-to-one, and onto.

Definition 4.10 (Linear Horseshoes)**.** *Let $\Delta = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be a rectangle. Assume $f : \Delta \to f(\Delta) \subset \mathbb{R}^2$ is a diffeomorphism onto its image such that*

(i) $\Delta \cap f(\Delta)$ is a disjoint union of two (horizontal) sub-rectangles Δ_1 and Δ_2 with heights $\leq 1/2$;
(ii) the restriction of f to the components of $\Delta \cap f^{-1}(\Delta)$ are linear maps.

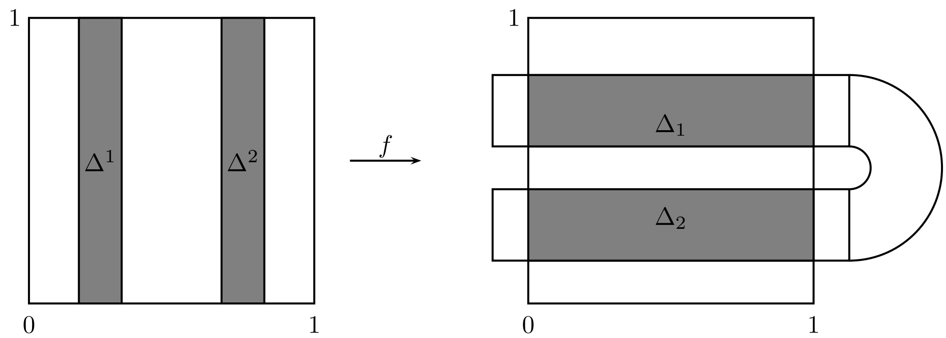


Fig. 6. A schematic presentation of the horseshoe map. The gray rectangles are mapped to the gray rectangles by some linear maps.

One can write $\Delta \cap f^{-1}(\Delta) = \Delta^1 \cup \Delta^2$, where Δ^1 and Δ^2 are (vertical) sub-rectangles of width $\leq \frac{1}{2}$.

The set of points in Δ that can be iterated infinitely many times forward and backward under f is

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^{-n}(\Delta).$$

Proposition 4.11. *The map $f:\Lambda\to\Lambda$ is topologically conjugate to $\sigma:\Sigma_2'\to\Sigma_2'$.*

Proof. Observe that $\Delta\cap f(\Delta)\cap f^{\circ 2}(\Delta)$ consists of four thin rectangles denoted by

$$\Delta_{i,j} = \Delta_i \cap f(\Delta_j), \text{ for } i,j\in\{1,2\}.$$

Continuing inductively, for each $n\geq 1$, the intersection

$$\bigcap_{i=0}^{n-1} f^{\circ i}(\Delta) = \bigcap_{i=0}^{n-1} f^{\circ i}(\Delta_1\cup\Delta_2)$$

consists of 2^n thin and disjoint horizontal rectangles. For $w_0,w_1,...,w_{n-1}\in\{1,2\}$, let

$$\Delta_{w_0 w_1,\dots,w_{n-1}} = \bigcap_{i=0}^{n-1} f^{\circ i}(\Delta_{w_i}).$$

On the other hand, for each $n\geq 1$, $\bigcap_{i=0}^{n-1} f^{-i}(\Delta)$ consists of 2^n thin and disjoint vertical rectangles. For every finite sequence $w_{-(n-1)}, \dots, w_{-1}, w_0$ in $\{1,2\}$, we let $\Delta^{w_0 w_1,\dots,w_{n-1}} = \bigcap_{i=0}^{n-1} f^{-i}(\Delta^{w_{-i}})$.

For $w=(w_n)_{n\in\mathbb{Z}}$, we define $\pi(w)$ as

$$\pi(w) = \Big(\bigcap_{n=0}^{\infty} \Delta_{w_0 w_1\dots w_{n-1}}\Big)\bigcap\Big(\bigcap_{n=0}^{\infty} \Delta^{w_0 w_{-1}\dots w_{-n+1}}\Big).$$

The image of a cylinder

$$\{(w_i')_{i\in\mathbb{Z}} \mid w_i' = w_i, \forall i \text{ with } -(n-1)\leq i\leq n-1\}$$

under π is a square $\Delta_{w_0\dots w_{n-1}}\bigcap\Delta^{w_0\dots w_{-(n-1)}}$ of size bounded by $1/2^{n-1}\times 1/2^{n-1}$. It follows that

(i) π is continuous;
(ii) π is invertible (and a homeromorphism);
(iii) Λ is a cantor set (that is, Λ is compact, totally disconnected, and every point in Λ is a limit of a sequence of points in Λ);
(iv) $\pi\circ\sigma = f\circ\pi$.

□

By Proposition 4.11, $f:\Lambda\to\Lambda$ inherits some dynamical features of $\sigma:\Sigma_2'\to\Sigma_2'$.

Exercise 4.23. *Let $f:\Delta\to\mathbb{R}^2$ be a linear horseshoe map. We have,*

(i) The periodic points of $f:\Lambda\to\Lambda$ are dense in Λ;
(ii) The number of periodic points of $f:\Lambda\to\Lambda$ is 2^n;
(iii) $f:\Lambda\to\Lambda$ is topologically mixing.

5. Topological Entropy

We have seen qualitative indications of chaos: transitivity, density of periodic orbits, sensitive dependence on initial conditions. Now, we would like to quantify the complexity of f, to obtain a finer invariant under topological conjugacy.

Let X be a compact set equipped with a metric d, and $f : X \to X$ be a continuous map. For each $n \geq 1$, we define a metric

$$d_n(x,y) = \max\{d(f^{\circ i}(x), f^{\circ i}(y)) : 0 \leq i \leq n-1\}. \tag{3}$$

Then, define

$$B(x,n,\epsilon) = \{y \in X : d_n(x,y) < \epsilon\}.$$

A finite set $E \subseteq X$ is called an (n,ϵ)-dense set if $X \subseteq \bigcup_{x\in E} B(x,n,\epsilon)$. This is also called (n,ϵ)-spanning set. Note that since X is compact and f is continuous, there is always an (n,ϵ)-dense set with a finite number of elements.

Let $S(n,\epsilon)$ be the minimum cardinality of all (n,ϵ)-dense sets. In other words, this is the list of information needed to keep track of all orbits up to time n and ϵ-error. One can ask how fast does the sequence $S(n,\epsilon)$ grow as n tends to infinity. It turns out that it is suitable to look at the exponential growth rate

$$h(f,\epsilon) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log S(n,\epsilon).$$

By definition, if $\epsilon < \epsilon'$, then $S(n,\epsilon) \geq S(n,\epsilon')$, and hence,

$$h(f,\epsilon) \geq h(f,\epsilon').$$

This implies that as ϵ tends to 0 from the right-hand side, the sequence $h(f,\epsilon)$ increases. Recall that any increasing sequence has a limit (potentially infinite). Thus, we define

$$h(f) = \lim_{\epsilon\to 0} h(f,\epsilon) \geq 0.$$

The above quantity is called the *topological entropy* of f. This notion is to some extent independent of the choice of the metric on X. Two metrics d and d' on a space X are called equivalent, if the convergence with respect to any of these metrics implies the convergence with respect to the other one.

Lemma 4.4. *Let d and d' be two equivalent metrics on X that make it a compact space, and let $f : X \to X$ be a continuous map. The topological entropy of f with respect to the metrics d and d' are the same.*

Proof. Consider the identity map $I : (X,d) \to (X,d')$. By the equivalence of the metrics d and d', I is a homeomorphisms (it is one-to-one, onto, continuous, and its inverse is also continuous). Moreover, since X is compact, I is indeed uniformly continuous. This implies that, given $\epsilon > 0$, there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(x,y) < \epsilon$. In particular, $d_n(x,y) < \delta$ implies that $d'_n(x,y) < \epsilon$. Therefore, any (n,δ)-dense set with respect to d is also (n,ϵ)-dense set with respect to d'. Hence,

$$S_d(n,\delta) \geq S_{d'}(n,\epsilon), \quad \forall n \geq 1.$$

Taking limits as n tends to infinity, we obtain $h_d(f,\delta) \geq h_{d'}(f,\epsilon)$. Therefore,

$$h_{d'}(f) = \lim_{\epsilon \to 0} h_{d'}(f,\epsilon) \leq \lim_{\delta \to 0} h_d(f,\delta) = h_d(f).$$

Repeating the above argument for the map $I : (X,d') \to (X,d)$, we also obtain $h_d(f) \leq h_{d'}(f)$. Therefore, the two quantities must be equal. □

Corollary 4.2. *Topologically conjugate maps have the same topological entropy.*

Proof. Let $\pi : X \to Y$ be a conjugacy between $f : X \to X$ and $g : Y \to Y$, i.e. $\pi \circ f = g \circ \pi$. If d_X is a metric on X, then define d_Y on Y by

$$d_Y(y,y') = d_X\big(\pi^{-1}(y), \pi^{-1}(y')\big).$$

Thus $\pi : (X,d_X) \to (Y,d_Y)$ is an isometry. This implies that $h_{d_X}(f) = h_{d_Y}(g)$, and then by Lemma 4.4, $h(f) = h(g)$. □

Example 4.6. *Consider the expanding map $f : S^1 \to S^1$, $f(x) = dx \pmod 1$, $d \geq 2$. Observe that for any $n \geq 1$, a d_n-ball $B(x,n,\epsilon)$ has diameter $(2\epsilon)/d^n$. Thus, we need at least $d^n/(2\epsilon)$ balls to cover $[0,1)$, and $d^n/(2\epsilon)+1$ balls is enough to cover this set. That is,*

$$S(n,\epsilon) \leq \left(\frac{d^n}{2\epsilon}\right) + 1, \text{ and } S(n,\epsilon) \geq \left(\frac{d^n}{2\epsilon}\right).$$

In particular,

$$h(T) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log(S(n,\epsilon)) = \log d.$$

By Theorem 4.4, any expanding map $f : S^1 \to S^1$ of degree $d \geq 2$ is topologically conjugate to the linear expanding map of degree d. Hence, such maps have the same topological entropy $\log d$.

Proposition 4.12. *If $f : X \to X$ is an isometry, then $h(f) = 0$.*

Proof. By definition, for all $n \geq 1$, $d_n(x,y) = d(x,y)$. In particular, $S(n,\epsilon)$ is independent of n, and thus, $h(f) = 0$. □

The rigid rotations $f : S^1 \to S^1$ are isometries, so $h(f) = 0$. Then, by Theorem 4.3, certain homeomorphisms of S^1 are conjugate to a rotation, and must have zero entropy.

Exercise 4.24. *Prove that the topological entropy of any C^1 (continuously differentiable) map of $S^1 \times S^1$ (torus) is finite. [hint: consider the maximum size of its derivatives.]*

Due to the definition of $S(n,\epsilon)$, we often obtain an upper bound on this quantity. That is because, an example of (n,ϵ)-dense set provides an upper bound for $S(n,\epsilon)$. This leads to an upper bound on $h(f)$. Below we give an alternative definition of the topological entropy that is conveniently used to give a lower bound on $h(f)$. The combination of the two methods is often used to calculate $h(f)$.

Given a compact metric space (X,d) and a continuous map $f: X \to X$, consider the metrics d_n defined in Eq. (3). Let $N(n,\epsilon)$ be the maximal number of points in X whose pairwise d_n distances are at least $\epsilon > 0$. A set of such points is called an (n,ϵ)-separated set.

Lemma 4.5. *We have,*

(i) $N(n,\epsilon) \geq S(n,\epsilon)$;
(ii) $S(n,\epsilon) \geq N(n,2\epsilon)$.

Proof. **(i).** Let E_n be an (n,ϵ)-separated set with $N(n,\epsilon)$ elements. Then, E_n must also be an (n,ϵ)-spanning set, since otherwise, we could enlarge the separating set by adding a point not already covered. Thus $N(n,\epsilon) \geq S(n,\epsilon)$.

(ii). Let E_n be an arbitrary $(n,2\epsilon)$-separated set, and F_n be an arbitrary (n,ϵ)-dense set. We define a map $\phi_n : E_n \to F_n$ as follows. By the definition of (n,ϵ)-dense set, the set $\cup_{x\in F_n} B(x,n,\epsilon)$ covers X. Then, for any $x \in E_n$, there is $\phi_n(x) \in F_n$ such that $d_n(x,\phi_n(x)) < \epsilon$. The map ϕ is well-defined and one-to-one. That is because, if $\phi(x) = \phi(y)$, then

$$d_n(x,y) \leq d_n(x,\phi_n(x)) + d_n(\phi_n(y),y) < \epsilon + \epsilon = 2\epsilon.$$

However, since E_n is $(n,2\epsilon)$-separated, we must have $x = y$.

The injectivity of $\phi_n : E_n \to F_n$ implies that the number of elements in E_n is bigger than the number of elements in F_n. Since E_n and F_n where arbitrary, we must have $S(n,\epsilon) \geq N(n,2\epsilon)$. This finishes the proof of the proposition by taking limits as $\epsilon \to 0$. □

Proposition 4.13.

$$h(f) = \lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N(n,\epsilon).$$

Proof. By Part (i) of Lemma 4.5,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log N(n,\epsilon) \geq \overline{\lim_{n\to\infty}} \frac{1}{n} \log S(n,\epsilon) = h(f,\epsilon).$$

By Part (ii) of Lemma 4.5,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log N(n,2\epsilon) \leq \overline{\lim_{n\to\infty}} \frac{1}{n} \log S(n,\epsilon) = h(f,\epsilon).$$

Letting $\epsilon \to 0$ we obtain the desired formula in the proposition. □

Proposition 4.14. *Let $g: Y \to Y$ be a factor of $f: X \to X$, that is, there is a continuous surjective map $\pi: X \to Y$ with $g \circ \pi = \pi \circ f$. Then $h(g) \leq h(f)$.*

Proof. Let d^X and d^Y denote the metrics on X and Y respectively. For $\epsilon > 0$, choose $\delta > 0$ such that if $d^X(x_1, x_2) < \delta$ then $d^Y(\pi(x_1), \pi(x_2)) < \epsilon$. Thus, a δ-ball with respect to d_n^X, $B(x, n, \delta)$, is mapped under π into $B(\pi(x), n, \epsilon)$. In particular,

$$S_{d^X}(n, \delta) \geq S_{d^Y}(n, \epsilon).$$

This implies the inequality in the proposition. □

Exercise 4.25. *Let $f : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ be defined as,*

$$f(x, y, z) = (x, x + y, y + z) \pmod 1.$$

Find $h_{top}(f)$.

Exercise 4.26. *Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ and for each $\lambda \in [0, 1]$ define $f_\lambda : D \to D$ as $f_\lambda(z) = \lambda z^2$.*

(i) Show that $h_{top}(f_\lambda) \geq \log 2$, when $\lambda = 1$.
(ii) Show that $h_{top}(f_\lambda) = 0$, when $0 \leq \lambda < 1$.

Therefore, topological entropy does not depend continuously on the map.