Dynamical Systems

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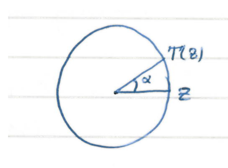
1 *Introduction*

Q: What is a dynamical system?
It is "something" that "evolves" with time!

- It may be a solution to a differential equation, for example, $w''(x) + cw(x) = 0$.

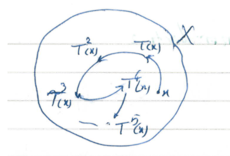
- More generally we consider a map, $T : X \to X$ on a set X.
  - For example, $T(x) =$ solution to the above differential equation with initial condition x at time t.
  - Rotation on a unit circle. Let $S^1 := \{z \in \mathbb{C}; |z| = 1\} = \{e^{i\theta}; 0 \le \theta \le 2\pi\}$
  fix some $\alpha \in (0, 2\pi)$ and define $T : S^1 \to S^1$ as $T : z \to e^{i\alpha}z$ or $T(e^{i\theta}) = e^{i(\theta+\alpha)}$



- The doubling map on the unit circle $T : S^1 \to S^1$. $T(z) = z^2$, or equivalently $e^{i\theta} \to e^{2i\theta}$. This 'doubles' the angle. There are higher dimensional analogues of these.

Q: What can we say about a dynamical systems?
- The orbits: We would like to understand what happens to points as we iteratively apply the map $T: X \to X$:

$$T^0 = \mathrm{Id}; \quad T^2 = T \circ T : X \to X; \quad \ldots \quad T^n = \underbrace{T \circ T \circ \cdots \circ T}_{\text{n times}} : X \to X.$$



How can we describe the behaviour of individual orbits?
$x, T(x), T^2(x), \ldots.,$ e.g. $x \to \frac{x}{2}$

How can we describe the global behaviour of the dynamical system?
"attractors", their "topology" and "geometry"?

Examples:
- Rotations of the circle: deterministic, if we understand one orbit then we understand any other orbits.
- The doubling map: example of a "chaotic" system where nearby points move apart at an exponential rate (and come back together).

Broadly speaking, there are two aims:

- To understand general features of some dynamical systems (simple ones) "qualitatively" and "quantitatively".
- To understand complicated systems using simple models.

Dynamical systems is one of the widest and oldest branches of mathematics. It breaks down into many branches:

- Smooth dynamical systems: $T: M \to M$, M is manifold and T is C^∞.
- Holomorphic dynamical systems: $T: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ or $T: \mathbb{C} \to \mathbb{C}$, T is a holomorphic map.
- Ergodic theory: $T: P \to P$, P is a probability space, there is no metric, but can say how large sets are.
- Group actions and number theory: $G: M \to M$, G is a group.
- There are interactions with many other areas of mathematics.

- There are many applications outside of mathematics.

The plan is to cover a little of each area:

- circle homeomorphisms

- expanding maps

- horseshoe maps, toral automorphisms, and other examples of hyperbolic maps

- structural stability, shadowing, closing, Markov partitions, symbolic dynamics

- conjugacy and topological entropy

Recommended books:

- R. Devancy: *An introduction to chaotic dynamical systems*
- B. Hasselblat and A. Katok, *A first course in dynamics*
- B. Hasselblat and A. Katok, *Introduction to the modern theory of dynamical systems*

2 *Rigid rotations on the unit circle and equidistribution*

Recall that we denoted the unit circle,

$$S^1 := \{z \in \mathbb{C}; |z| = 1\}$$

Given $\alpha \in [0,1]$, let $a = 2\pi\alpha$, and then the map $T : S^1 \to S^1$ given by the formula

$$T(z) = e^{ia}z$$

is a rotation by the angle a.

Sometimes it is convenient to use an alternative notation for this map. We may identify S^1 with the set $\mathbb{R}/\mathbb{Z}$. This is the same as the interval $[0,1]$ with 0 and 1 identified.

In this notation we can write,

$$T : [0,1) \to [0,1)$$

$$T : X \to X + \alpha (mod 1)$$

I.e.

$$T(x) = \begin{cases} x+\alpha, & 0 \le x+\alpha < 1 \\ x+\alpha-1, & 1 \le x+\alpha < 2 \end{cases}$$

Orbits:

First consider the case that $\alpha = \frac{p}{q}$ is rational. We assume that $q \neq 0$, $p \in \mathbb{Z}^*$, and p and q are relatively prime ($(p,q) = 1$). Then it is clear that the orbit,

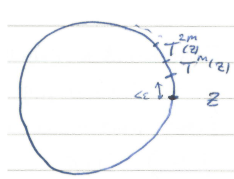
$$\bigcup_{n\ge 1} T^n(z) = \{z, T(z), \ldots,\} = \{z, ze^{2\pi i/q}, ze^{2\pi i 2/q}, \ldots, ze^{2\pi i \frac{(q-1)}{q}}\}$$

is finite.

In contrast, when α is irrational the situation is very different.

Proposition 1. *If α is irrational then for each $z \in S^1$, the orbit $\{T^n(z); n \in \mathbb{Z}\}$ is infinite and dense on S^1.*

Proof. Fix any choice of $x \in S^1$, and $\varepsilon > 0$.
By the pigeonhole principle there exists $0 \le l < k \le \frac{1}{\varepsilon} + 1$ such that $d(T^k(z), T^l(z)) < \varepsilon$ where d is the natural metric on S^1 and so $d(z, T^{k-l}(z)) < \varepsilon$, since applying T^{-l} preserves distances.
Denote $m = k - l$, clearly the orbit $\{T^n(z); n \in \mathbb{Z}\} \supseteq \{z, T^m(z), T^{2m}(z), \ldots,\}$ and this latter set is ε-dense.



In particular, for $x \in S^1$, and $\varepsilon > 0$, we have that,

$$\left\{T^n(z) : n \in \mathbb{Z}\right\} \bigcap B(x, \varepsilon) \neq 0$$

Since x and ε are arbitrary, the orbit is dense.

□

More generally, let $T : X \to X$ be a homeomorphism of a compact metric space X.

Definition 1. *We say that $T : X \to X$ is topologically transitive if there exists $x \in X$, whose orbit $\left\{T^n(z); n \in \mathbb{Z}\right\}$ is dense in X.*

Stronger still is the following.

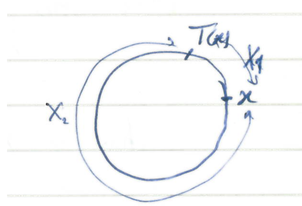
Definition 2. *We say that $T : X \to X$ is minimal if for every $x \in X$, the orbit $\left\{T^n(z), n \in \mathbb{Z}\right\}$ is dense in X.*

Remark 1. *For homeomorphisms we could also consider the "one sided" and "two sided" transitivity.*

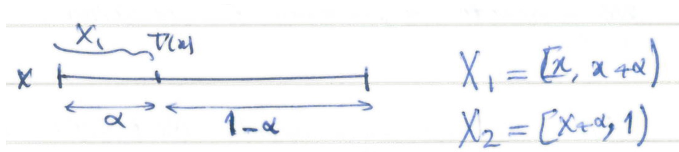
There are quantitative versions of the above notions. For example, one may ask for an irrational rotation of the circle, how long might one have to wait for the orbit of a point to enter a given interval?

Renormalizations and continued fractions:

Consider the orbit $x = 0, T(x), T^2(x), \ldots$ where $T := R_\alpha : x \to x + \alpha (mod 1)$. Here, $2\pi\alpha$ is the length of the arc from x to $T(x)$.



We can partition S^1 as $X_1 \bigcup X_2$ where X_1 is the arc from x to $T(x)$ and X_2 is the remaining part. We can also represent this using the intervals as $X_1 = [0, \alpha)$ and $X_2 = [\alpha, 1 - \alpha)$.



Assume that we can only "see" points when they land in the arc X_1. Furthermore, assume that the first time the orbit of x returns to X_1 is at the point $T^s(x)$, $(s \geq 1)$. This is called the "first return of x to X_1.

Let us denote $T' = T^s$. Continuing the trajectory: $T^{s+1}(x), T^{s+2}(x), T^{s+3}(x), \ldots$

We might see $T^{2s}(x) = T'\big(T'(x)\big) = \big(T'\big)^2(x) \in X_1$. So this is the second return time It could also happen that $T^{2s-1}(x) \in X_1$, so the second return time is not always $2s$. Observe that

$$d\big(x, T'(x)\big) = d\big(T'(x), T'\big(T'(x)\big)\big) = \big(\frac{1}{\alpha} - \{\frac{1}{\alpha}\}\big)\alpha + (\alpha - 1) = \alpha\big(1 - \{\frac{1}{\alpha}\}\big),$$

where $\{y\} = y(mod 1)$. So, $s = \frac{1}{\alpha} - \{\frac{1}{\alpha}\} + 1$.

We can sub-partition $X_1 = X_1' \bigcup X_2'$ such that X_1' is an interval of length $\alpha\{\frac{1}{\alpha}\}$ and X_2' has length $\alpha(1 - \{\frac{1}{\alpha}\})$, and define,

$$\begin{cases} T'\big|_{X_1'} = T^s \\ T'\big|_{X_2'} = T^{s-1} \end{cases}$$

as the first return times to X_1.
Thus, "up to renormalization lengths" $T'\big|_{X_1}$ = rotation by $1 - \{\frac{1}{\alpha}\}$ on X_1.

Basic Algorithm:

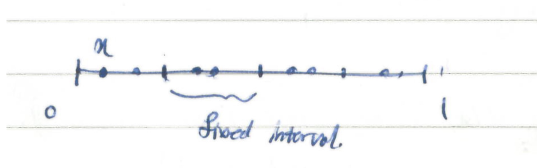
- Restrict T^s to the interval X_1'(= rotation by $1 - \{\frac{1}{\alpha}\}$);
- replace T by T^s (set $s = s_1$) and S by X_1';
- repeat the operation.

We generate a sequence of "rotations" $T, T^{s_1}, T^{s_1 s_2}, T^{s_1 s_2 s_3}, \ldots$ on decreasing intervals $X_1 \supset X_1' \supset X_1'' \ldots$. More over, α has "continued fraction"

$$\alpha = \cfrac{1}{s_1 + \cfrac{1}{s_2 + \cfrac{1}{s_3 + \ldots}}}$$

Distribution of orbits

We now go back to the problem of quantifying the time of visiting to an interval. The distribution of the orbit is "uniform" on average. If α is irrational then the proportion of the orbit $x, T(x), T^2(x), \ldots$ for any x which lies inside a given interval comes to the length of the interval. This is made precise in the next theorem.



Theorem 1. *If α is irrational and $\phi : S^1 \to \mathbb{C}$ is continuous then for any $x \in S^1$,*

$$\lim_{n\to\infty}\Big(\frac{1}{n}\sum_{k=0}^{n-1}\phi\big(R_\alpha^k(x)\big)\Big) = \int_{S^1}\phi\big(y\big)dy$$

Proof. Let us consider instead,

$$\phi(x) = e_m(x) := e^{2\pi imx} = cos(2\pi mx) + isin(2\pi mx)$$

Then $e_m(R_\alpha x) = e^{2\pi im(x+\alpha)} = e^{2\pi im\alpha}e_m(x)$. Thus for $\phi(x) = e_m(x)$ $(m \neq 0)$,

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=0}^{n-1}e_m\big(R_\alpha^k(x)\big)\right| &= \frac{1}{n}\left|\sum_{k=0}^{n-1}e^{2\pi imk\alpha}\right||e^{2\pi imx}| \\ &= \frac{1}{n}\left|\frac{1-e^{2\pi inm\alpha}}{1-e^{2\pi im\alpha}}\right| \\ &\leq \frac{2}{|1-e^{2\pi im\alpha}|}\frac{1}{n} \to 0\end{aligned}$$

as $n \to \infty$.

Thus, if $\phi(x) = \sum_{m=-N}^{N} a_m e_m(x)$, with $a_{-N}, a_{-N+1}, ..., a_N \in \mathbb{C}$, then

$$lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi\big(R_\alpha^k x\big) = a_0 = \int\phi\big(y\big)dy$$

This result extends to the uniform closure of all such trigonometric polynomials, i.e. all continuous functions.

□

More generally,

Definition 3. *Let X be a compact metric space and let $T : X \to X$ be a homeomorphism. We say T is uniquely ergodic if for every continuous function $\phi : X \to \mathbb{R}$ the sequence*

$$\{\frac{1}{n}\sum_{k=0}^{n-1}\phi\big(T^k(x)\big)\}_{n=0}^{\infty}$$

converges to a constant c independent of $x \in X$.

Application:

As an application of the above theorem we look at the distribution of first digits of 2^n, $n \geq 1$.

Proposition 2. *Fix $p \in \{0,1,2,...,9\}$. The frequency of those n for which the first digit of 2^n is equal to p, that is,*

$$\lim_{N\to\infty}\frac{\{1 \leq n \leq N; \textit{first digit of } 2^n \textit{ is equal to } p\}}{N} = \log_{10}\big(1+\frac{1}{p}\big).$$

For example, when $p=7$ the frequency is $\log_{10}(8/7)$.

Proof. 2^n has first digits p iff for some k:

$$p \times 10^k \leq 2^n < (p+1)10^k$$

Equivalently, $\log_{10} p + k \leq n \log_{10} 2 < \log_{10}(p+1) + k$, i.e.

$$n \log_{10} 2 (mod 1) \in [\log_{10} p, \log_{10}(p+1)).$$

Define,

$$\chi(x) = \begin{cases} 1, & x \in [\log_{10} p, \log_{10}(p+1)] \\ 0, & otherwise \end{cases}$$

Let $\alpha = \log_{10} 2 \in \mathbb{R} \setminus \mathbb{Q}$,

Claim 1. *We claim claim that*

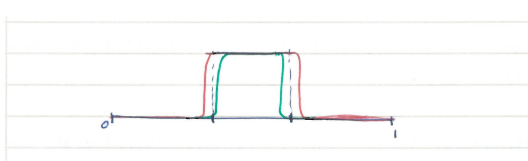
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{B} \chi\left(R_\alpha^n x\right) = \int_{[0,1]} \chi dy.$$

If χ was continuous, this would follow directly from the theorem. However, it suffices to approximate χ from above and below by continuous functions, $\chi_1 \leq \chi \leq \chi_2$ with $\int |\chi_1 - \chi_2| dy < \delta$, say.

Thus,

$$\begin{aligned} \int \chi_2 dy &= \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \chi_2\left(R_\alpha^n x\right) \\ &\geq \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \chi\left(R_\alpha^n x\right) \\ &\geq \underline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \chi\left(R_\alpha^n x\right) \\ &\geq \underline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \chi_1\left(R_\alpha^n x\right) \\ &= \int \chi_1 dy \end{aligned}$$

Since, $|\int \chi dy - \int \chi_1 dy| < \delta$ and $|\int \chi dy - \int \chi_2 dy| < \delta'$, and $\delta > 0$ can be arbitrarily small, the result follows. $\square$

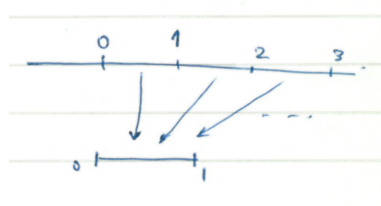


Homeomorphisms of the unit circle

Consider the natural projection,

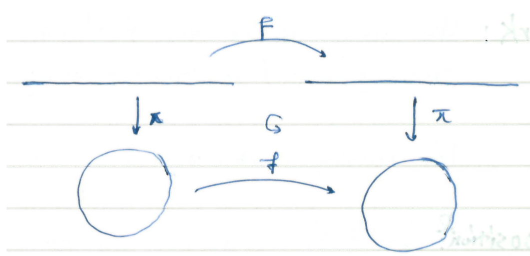
$$\pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$$
$$\pi(x) = x(mod 1)$$

It follows that for all $i \in \mathbb{Z}$, $\pi(x+i) = \pi(x)$.



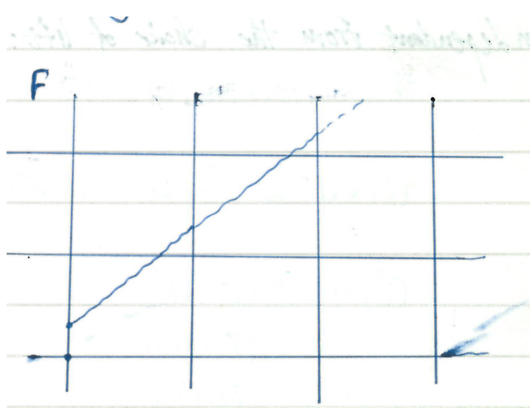
Proposition 3. *Let $f : S^1 \to S^1$ be a homeomorphism of the circle. Then there exists a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ (called a lift of f) such that,*

$$f \circ \pi = \pi \circ F$$



Moreover, F is unique up to adding an integer; if F is a lift, so is $F+n$, for $n \in \mathbb{Z}$.

Proof. Given x and $y = f(x) \in S^1$, choose x', and $y' \in \mathbb{R}$ with $\pi(x') = x$, $\pi(y') = y$. Define $F(x') = y'$. One can use the functional equation to extend this map onto $\mathbb{R}$.
If G is another lift then, we must have $G(x') = y' + n$. This implies that $G(t) = F(t) + n$ for all $t \in \mathbb{R}$.



Note that $F(x+i) = F(x) + i$, $i \in \mathbb{Z}$, $x \in \mathbb{R}$.

□

Example 1. *Let $f : S^1 \to S^1$ be a rotation by α. The lifts of f are given by the formulas*

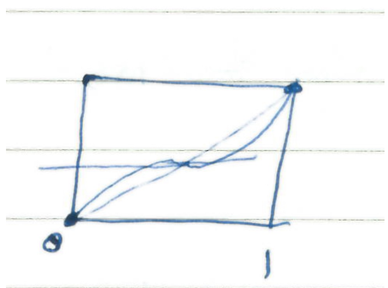
$$\begin{cases} F : \mathbb{R} \to \mathbb{R} \\ F(x) = x + \alpha + n \end{cases}$$

where $n \in \mathbb{Z}$.

Example 2. *Let $f : S^1 \to S^1$ be defined by $f(x) = x + \varepsilon sin(2\pi x)$ for $|\varepsilon| < \frac{1}{2\pi}$. Then, the lifts are defined by*

$$\begin{cases} F : \mathbb{R} \to \mathbb{R} \\ F(x) = x + \varepsilon sin(2\pi x) + n \end{cases}$$

for any $n \in \mathbb{Z}$. If f is a homeomorphism then we must have $|\varepsilon| < \frac{1}{2\pi}$.



Remark 2. *We always assume that $f : S^1 \to S^1$ is orientation preserving, that is, the graph of f is strictly increasing.*

The rotation number of $f : S^1 \to S^1$ is defined in the following proposition.

Proposition 4. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with a lift $F : \mathbb{R} \to \mathbb{R}$. Then the limit,*

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \pmod 1$$

for $x \in \mathbb{R}$, exists, and is independent of $x \in \mathbb{R}$ and the choice of the lift F.

We present the proof of the above proposition in several steps.

Independence from the choice of the lift:

If G is another lift of f, then $G(x) = F(x) + N$, for some $N \in \mathbb{Z}$. Since, G is a lift, then for all $x \in \mathbb{R}$ and all $i \in \mathbb{Z}$ we have

$$G(x+i) = G(x) + i.$$

Hence,

$$\begin{aligned} G^2(x) &= G(G(x)) \\ &= G(F(x)+N) \\ &= G(F(x))+N \\ &= F(F(x))+N+N \\ &= F^2(x)+2N \end{aligned}$$

and in general $G^n(x) = F^n(x) + nN$. Therefore,

$$\overline{\lim_{n\to\infty}} \frac{G^n(x)}{n} = \overline{\lim_{n\to\infty}} \frac{F^n(x)+nN}{n} = \overline{\lim_{n\to\infty}} \frac{F^n(x)}{n} + N.$$

So, if the limit exists, we obtain the same values modulo 1.

Independent from the choice of $x \in \mathbb{R}$:

Let $y \in \mathbb{R}$ be another choice.

Let us first assume that $|x-y| < 1$. By the definition of the lift $|x-y| < 1 \implies |F(x)-F(y)| < 1$ and therefore,

$$\begin{aligned} |x-y| < 1 &\implies \left|F^2(x)-F^2(y)\right| < 1 \\ \ldots &\implies \ldots \\ \ldots &\implies |F^n(x)-F^n(y)| < 1 \end{aligned}$$

Hence, $\left|\frac{F^n(x)}{n} - \frac{F^n(y)}{n}\right| < \frac{1}{n} \to 0$, as $n \to \infty$.

This implies that $\lim_{n\to\infty} \frac{F^n(x)}{n}$ is the same as $\lim_{n\to\infty} \frac{F^n(y)}{n}$ as $n \to \infty$, provided they exist.

In general, for x and y in $\mathbb{R}$ there is a finite sequence of points

$$x = t_0 < t_1 < t_2, \ldots, t_n = y$$

with $|t_{i+1} - t_i| < 1$, for $i = 0, \ldots, n-1$. By the above argument, if the limit exists, we must have

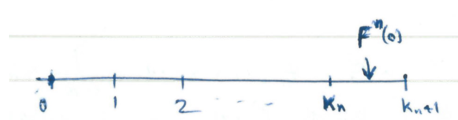
$$\lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{F^n(t_1)}{n} = \ldots = \lim_{n\to\infty} \frac{F^n(y)}{n}.$$

The limit in Proposition 4 exists:

For each $n \geq 1$, choose $k_n \in \mathbb{Z}$, with $k_n \leq F^n(0) \leq k_n + 1$. Then,

$$\left|\frac{F^n(0)}{n} - \frac{k_n}{n}\right| \leq \frac{1}{n}$$

Let $n \geq 1$,



By the monotonicity of F^n, $F^{2n}(0) \in [2k_n, 2(k_n+1)]$. That is because,

$$k_n \leq F^n(0) \leq k_n + 1 \implies F^n(k_n) \leq (F^{2n}(0)) \leq F^n(k_n+1)$$

$$\implies 2k_n \leq F^n(k_n) \quad \text{ and } \quad F^n(k_n+1) \leq k_n + 1 + k_n + 1$$

In general, for $m \geq 1$, the above argument implies that $mk_n \leq F^{nm}(0) \leq m(k_n+1)$. Thus,

$$\frac{F^{nm}(0)}{nm} \in [\frac{k_n}{n}, \frac{k_n+1}{n}]$$

or,

$$\left| \frac{F^{nm}(0)}{nm} - \frac{k_n}{n} \right| \leq \frac{1}{n}$$

and so,

$$\left| \frac{F^m(0)}{m} - \frac{F^n(0)}{n} \right| \leq \left| \frac{F^m(0)}{m} - \frac{k_m}{m} \right| + \left| \frac{k_m}{m} - \frac{F^{nm}(0)}{nm} \right| + \left| \frac{F^{nm}(0)}{nm} - \frac{k_n}{n} \right| + \left| \frac{k_n}{n} - \frac{F^n(0)}{n} \right|$$
$$\leq \frac{1}{m} + \frac{1}{m} + \frac{1}{n} + \frac{1}{n}$$

In particular $\frac{F^n(0)}{n}$ is a Cauchy sequence (and so convergent).

In the next two lemmas we show that the notion of rotation defined in Proposition 4 is "informative".

Lemma 1. *If a homeomorphism $f : S^1 \to S^1$ has a periodic point $f^N(x) = x \in S^1$, then, $\rho(f)$ is a rational number.*

Proof. For every such lift F and every x' with $\pi(x') = x$ we have $F^N(x') = x' + l$, for some $l \in \mathbb{Z}$. That is because $\pi(F^N(x')) = \pi(x')$.

Let $n = kN + r$ for $k \geq 0$ and $0 \leq r \leq N-1$, then,

$$\lim_{n\to\infty} \frac{F^n(x')}{n} = \lim_{n\to\infty} \frac{F^{kN+r}(x')}{n} = \lim_{n\to\infty} \frac{F^r\left(F^{kN}(x')\right)}{n}$$

$$= \lim_{n\to\infty} \frac{F^r\left(x'+kl\right)}{n} = \lim_{n\to\infty} \frac{F^r(x')+kl}{n} = \lim_{n\to\infty} \frac{kl}{kN+r} = \frac{l}{N}$$

□

Lemma 2. *If f has no periodic points, then $\lim_{n\to\infty} \frac{F^n(x)}{n}$ is an irrational number.*

Proof. Assume that $\lim_{n\to\infty} \frac{F^n(x)}{n} = \frac{p}{q}$, then,

$$\lim_{n\to\infty} \frac{F^{qn}(x')}{n} = q \lim_{n\to\infty} \frac{F^{qn}(x')}{qn} = q\frac{p}{q} = p$$

Thus,

$$\lim_{n\to\infty} \frac{(F^q)^n(x')}{n} = p$$

Note that since f has no periodic points, f^q has no fixed points.

We may choose a lift $G : \mathbb{R} \to \mathbb{R}$ for f^q, such that,

$$\lim_{n\to\infty} \frac{G^n(x')}{n} = 0$$

The map $G : \mathbb{R} \to \mathbb{R}$ cannot have a fixed point as otherwise the fixed point projects under π to a fixed point for f^q. In particular, either,

i) $G(y) > y, \forall y \in \mathbb{R}$,

ii) $G(y) < y, \forall y \in \mathbb{R}$.

Assume we are in case i). There are two possibilities

A) $\exists k > 0$ such that $G^k(0) > 1$,

B) $\forall k \geq 0$ we have $G^k(0) \leq 1$.

If A) occurs, then

$$G^{kn}(0) > n, \forall n \geq 1 \implies \lim_{n\to+\infty} \frac{G^{kn}(0)}{kn} \geq \frac{n}{kn} > \frac{1}{k}$$

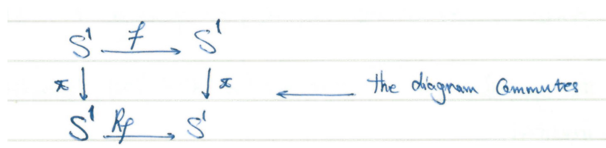
This contradicts the choice of G such that the limit is equal to 0.

If B) occurs, as the sequence $G^n(0) \in [0,1]$ is monotone, the limit $z' = \lim_{n\to+\infty} G^n(0)$ is a fixed point of G. This fixed point projects to a fixed point for f^q.

These contradictions prove that item i) may not occur. The same contradiction may be obtained in case ii) along the above lines. Details are left to the reader. □

We would like to see how a homeomorphism $f : S^1 \to S^1$ with rotation number $\rho(f)$ can be related to a rotation by ρ, i.e. $R_\rho : S^1 \to S^1$, $R_\rho(x) = x + \rho(mod 1)$. In particular we have the following classical result.

Proposition 5. ***(Poincare.)*** *Assume that $f : S^1 \to S^1$ is a minimal homeomorphism with an irrational rotation number $\rho(f)$. Then there is a homeomorphism $\pi : S^1 \to S^1$ such that, $R_\rho \circ \pi = \pi \circ f$.*

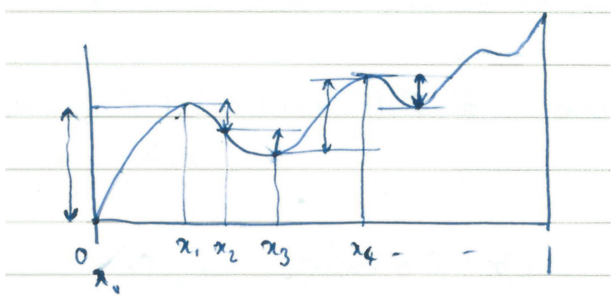


The homeomorphism π in the above proposition is called "topological conjugacy". The map π preserves the order of points in an orbit and the topological properties are preserved. However, metric properties are not preserved under π. For instance, $f^{n_k}(1)$ may converge to a point x on S^1 exponentially fast, but the corresponding sequence $R_\rho^{n_k}(\pi(1))$ may converge to $\pi(x)$ very slowly.

Variation: We say that a continuous function $w : S^1 \to \mathbb{R}$ has bounded variation if,

$$\mathbb{V}ar(w) = \sup\Big\{\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)|; 0 \le x_1 < x_2 < \ldots < x_n = 1\Big\}$$

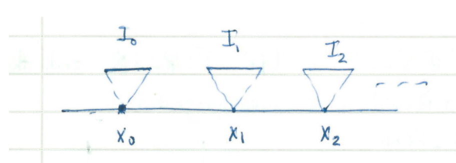
is finite.



Theorem 2. ***(Denjoy).*** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle with irrational rotation $\rho(f) = \rho$. Moreover, assume that $f : S^1 \to S^1$ is continuously differentiable and that $w(x) = \log|f'(w)|$ has bounded variation. Then $f : S^1 \to S^1$ is minimal, and hence, it is topologically conjugate to $R_\rho : S^1 \to S^1$.*

Example 3. ***(Denjoy's Example.)*** *For each irrational ρ, there is a C^1, diffeomorphism $f : S^1 \to S^1$ with rotation number $\rho(f) = \rho$, which is not transitive.*

Let us define $l_n = \frac{1}{(|n|+3)^2}, n \in \mathbb{Z}$.
In particular $\sum_{n\in\mathbb{Z}} l_n \le 2\sum_{n=3}^{\infty} \frac{1}{n^2} \le 2\int_2^\infty \frac{1}{x^2}dx = 1$.
We can "blow-up" the orbit $x_n = R_\rho^n(x), n \in \mathbb{Z}$, to insert intervals I_n of length l_n.
We want to arrange $f(I_n) = I_{n+1}, n \in \mathbb{Z}$.



It is enough to specify f'. Let $I_n = [a_n, a_n + l_n]$ and $h(a,l,x) = 1 - \frac{1}{l}|2(x-a) - l|$.
Set,

$$f'(x) = \begin{cases} 1, & x \notin \bigcup_{n\in\mathbb{Z}} I_n \\ 1 + c_n h(a_n, l_n, x), & x \in \bigcup_{n\in\mathbb{Z}} I_n \end{cases}$$

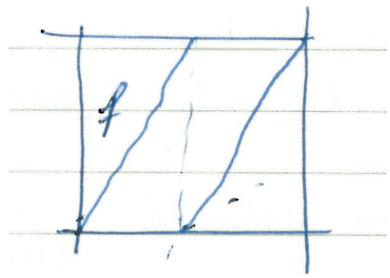
where $c_n = 2\big(\frac{l_{n+1}}{l_n} - 1\big)$. Thus,

$$\int_{I_n} f'(x) = \int \big(1 + c_n h(a_n, l_n, x)\big)dx = l_n + c_n \frac{l_n}{2} = l_{n+1}$$

3 Expanding Maps of the Circle

Let S^1 be the unit circle again. We now consider a different class of dynamical systems on S^1.

Definition 4. ***(Expanding.)*** *A continuously differentiable map $f : S^1 \to S^1$ is called expanding if $|f'(x)| > 1$ for all $x \in S^1$.*



Here, f cannnot be a homeomorphism; since $f' : S^1 \to \mathbb{R}$ is continuous and S^1 is compact, $\exists \beta > 1$ such that $\inf_{x \in S^1} |f'(x)| \geq \beta > 1$.

Definition 5. ***(Degree.)*** *We can define the degree of f, $deg(f)$ to be the number of preimages $f^{-1}(x)$, for any $x \in S^1$ (independent of X).*

Example 4. *Let $m \in \mathbb{N}$, $m \geq 2$, define $f : S^1 \to S^1$ as $f(x) = mx(mod 1)$.*
If we regard $S^1 \subseteq \{z \in \mathbb{C}; |z| = 1\}$, then f can be written as $f(z) = z^m$.

Lemma 3. *If $f, g : S^1 \to S^1$ are expanding maps, then $deg(f \circ g) = deg(f)deg(g)$. In particular, $deg(f^n) = \big(deg(f)\big)^n$.*

Proof. Since for each $y \in f^{-1}(x)$ we have $deg(g)$ preimages $g^{-1}(y)$. Thus $(f \circ g)^{-1}(x)$ has $deg(g)deg(f)$ elements. □

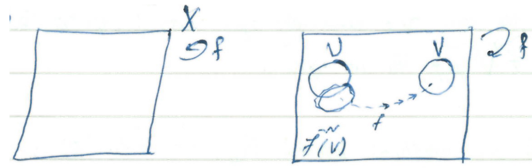
Proposition 6. *If $f : S^1 \to S^1$ is an expanding map of degree $deg(f) = d \geq 2$, the number of periodic points of period n is $(d^n - 1)$.*

Proof. First assume $n = 1$, the number of fixed points of f is equal to the number of points on the intersection of the diagonal with the graph of of f, that is $d - 1$. For arbitrary $n \geq 2$, consider f^n with $deg(f^n) = d^n$. Note that the number of periodic points of period n is equal to the number of fixed points of f^n, $d^n - 1$. □

Proposition 7. *Let $f : X \to X$ be a continuous map of a compact metric space X. The following are equivalent:*

i) f is topologically transitive, that is, there exists $x \in X$ such that $\{f^n(x); n \geq 0\}$ is dense;

ii) if $U, V \subseteq X$ are non-empty open sets, there exists $N \in \mathbb{Z}$ such that $f^{-N}(V) \bigcap U \neq \phi$.



In the following proof one may think of continuous maps of the circle S^1, sphere S^2, or the torus $S^1 \times S^1$.

Proof. **i). $\Longrightarrow$ ii).**

Let $\{f^n(x)\}_{n=1}^{\infty}$ be dense, choose $f^n(x) \in U$ and $m > n$ with $f^m(x) \in V$. Then,

$$f^n(x) \in U \bigcap f^{-(m-n)}V (\neq \phi)$$

ii). $\Longrightarrow$ i).

Let $y \subseteq X$ be a countable dense set $Y = \{y_i\}_{i=1}^{\infty}$. For, instance, when $X = S^1$, one can take all the points with rational coordinate. Let $U_i, i = 1,2,3$ be an open disk centered at y_i, with diameter $1/i$.

Now choose $N_1 \geq 0$ such that $f^{-N_1}(U_2) \bigcap U_1 \neq \phi$. Then choose an open disk V_1 of radius $\leq \frac{1}{2}$ such that,

$$V_1 \subseteq \overline{V}_1 \subseteq U_1 \bigcap f^{-N_1}(U_2)$$

Choose N_2 such that $f^{-N_2}(U_3) \bigcap V_1 \neq \phi$. Choose an open disk V_2 of radius $\leq \frac{1}{4}$ such that,

$$V_2 \subseteq \overline{V}_2 \subseteq V_1 \bigcap f^{-N_2}(U_3)$$

By induction, $V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots$ with radius $V_n \leq \frac{1}{2^n}$.

$$\overline{V}_{n+1} \subseteq V_n \bigcap f^{-N_{n+1}}(U_{n+2})$$

If we let $\{x\} = \bigcap_n^{\infty} \overline{V}_n$, then $f^{N_{n-1}}(x) \in U_n$, for $n \geq 1$.
Therefore $\{f^n(x)\}_{n=1}^{\infty}$ is dense in X. □

Definition 6. *We say that $f : X \to X$, X compact metric space, is topologically mixing if for any two non-empty sets $U, V \subseteq X$, there exists $N \geq 0$ such that $U \bigcap f^{-n}(V) \neq \phi$ for all $n \geq N$.*

Example 5. *For an irrational rotation $R_\alpha : S^1 \to S^1$, ($\alpha \notin \mathbb{Q}$), we have proved that R_α is transitive, but one easily sees that it is not mixing.*

Example 6. *By the above proposition, any mixing transformation is automatically transitive.*

Proposition 8. *An expanding map $f : S^1 \to S^1$ is mixing.*

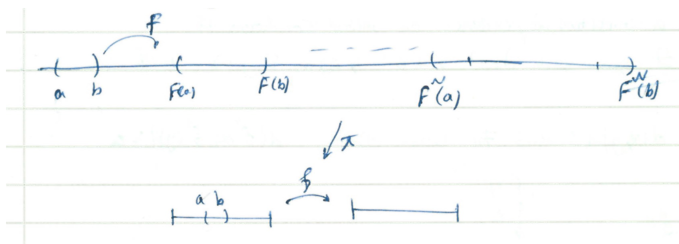
Proof. By definition, there exists a $\beta > 1$ such that $|f'(x)| \geq \beta$ for all $x \in S^1$.

Assume $deg(f) = d \geq 2$. Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. That is, $F : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\pi \circ F(x) = f \circ \pi(x)$ and $F(x+1) = F(x) + d$, for all $x \in \mathbb{R}$.

One can see that $|F'(x)| \geq \beta$, $\forall x \in \mathbb{R}$. Given open sets U in $\mathbb{R}$, choose an interval $(a,b) \subseteq U$.

$$|F(b) - F(a)| = \left|F'(c)(b-a)\right| \geq \beta(b-a)$$

for some $c \in (a,b)$ by mean value property. That is, F increases the length by a factor of β. Similarly, F^n increases the length by a factor of at least β^n. For N sufficiently large so that $\beta^N > \frac{1}{b-a}$ we have that $F^N(a,b)$ has length at least 1.



Thus, $\pi F^N(a,b) = S^1$ and since $\pi \circ F = f \circ \pi$, we have that $f^N(u) \supseteq f^n(a,b) = S^1$. Given V,

$$U \bigcap f^{-N}(V) \neq \phi$$

for any $n \geq N$. □

As a corollary of the above proof, any expanding map of the circle is topologically transitive.

Definition 7. ***(Choatic.)*** *A continuous map $f : X \to X$ of a compact metric space is chaotic if:*

a) f is topologically transitive; and

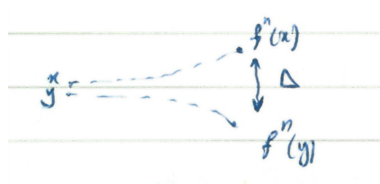
b) the periodic points are dense in X.

Example 7. *Consider the linear expanding map $f : S^1 \to S^1$, defined as $f(x) = mx \pmod 1$, $m \geq 2$. The periodic points take the form $x = \frac{j}{m^n-1}$, $0 \leq j < m^n - 1$. Since,*

$$F^n(x) = m^n\left(\frac{j}{m^n-1}\right) = j\left(\frac{m^n-1}{m^n-1}\right) + \frac{j}{m^n-1} = x \pmod{\mathbb{Z}}$$

These are dense in $[0,1]$ Later, we shall show that any expanding map of the circle is chaotic.

Definition 8. *A continuous map $f : X \to X$ on a compact metric space is said to have a sensitive dependence on initial conditions if $\exists \delta > 0$ such that $\forall x \in X, \forall \varepsilon > 0, \exists y \in X$ with, $d\big(x,y\big) < \varepsilon$ and for some $n \geq 0$, $d\big(f^n(x), f^n(y)\big) \geq \delta$.*



Proposition 9. *Expanding maps from $S^1 \to S^1$ have sensitive dependence on initial condition.*

Proof. By the definition of expansion, there is $\beta > 0$ such that $|f'(x)| \geq \beta$ for all $x \in S^1$. On the other hand, by the continuity of f' and the compactness of S^1, there is $\alpha > 0$ such that $|f'(x)| \leq \alpha$, for all $x \in S^1$. Let x and y be two arbitrary distinct points on S^1. It follows that as long as the distance between the points $f^n(x)$ and $f^n(y)$, for $n = 0, 1, 2, 3, \ldots$ remains small compared to $1/2$, we have the length of the arc $f^n(x, y)$ is at atleast $\beta^n |x - y|$ and at most $\alpha^n |x - y|$. In particular, there is $n \in \mathbb{N}$ such that the distance between $f^n(x)$ and $f^n(y)$ lies between $1/(2\alpha)$ and $1/2$. This implies that we can define $\delta = 1/(2\alpha)$. □

Proposition 10. *A chaotic map $f : X \to X$ is either a single periodic orbit or has sensitive dependence on initial conditions.*

Proof. Let $A := \{x, f(x), f^2(x), \ldots, f^{n-1}(x) = x\}$ and $B := \{y, f(y), f^2(y), \ldots, f^{m-1}(y) = y\}$ be two distinct periodic points. Let

$$\delta := \frac{1}{8} \min \Big(d((f^i(x), f^j(y)); 0 \leq i \leq n-1, 0 \leq j \leq (m-1) \Big)$$

Given arbitrary $z \in X$, either,

$$\begin{cases} \min_{\xi \in A} d\big(z, \xi\big) \geq 4\delta, & or \\ \min_{\eta \in B} d\big(x, \eta\big) \geq 4\delta, \end{cases}$$

Assume that $\min_{\xi \in A} d\big(z, \xi\big) \geq 4\delta$, (the other case is similar), and for $0 < \varepsilon < \delta$. Since periodic points are dense, there is $w = f^N(w) \in B(z, \varepsilon)$.

Let $V = \bigcap_{i=0}^{N} f^{-i} B\big(f^i(x), \delta\big) \neq \phi$. By transitivity of f, $\exists k \in \mathbb{N}$, such that $f^k\big(B(z, \varepsilon)\big) \bigcap V \neq \phi$. That is $\exists \xi \in B(z, \varepsilon)$ with $f^k(\xi) \in V$. Choose j such that $k + N \geq jN \geq k$. Since $f^{jN}(w) = w$ we have,

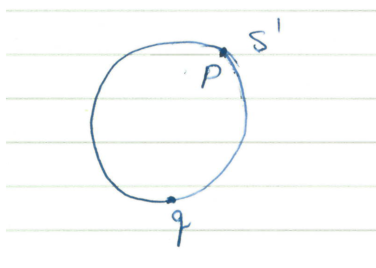
$$\begin{aligned} d\big(f^{jN}(w), f^{jN}(\xi)\big) &= d\big(w, f^{jN}(\xi)\big) \\ &\geq d\big(z, f^{jN}(\xi)\big) - d(w, z) \\ &\geq d\big(z, f^{jN-k}(x)\big) - d\big(f^{jN-k}(x), f^{jN}(\xi)\big) = d(w, z) \\ &\geq 4\delta - \delta - \delta = 2\delta \end{aligned}$$

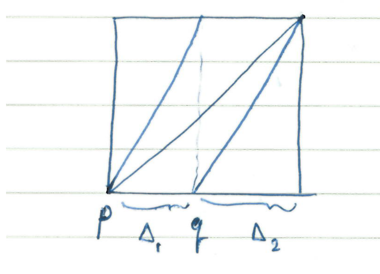
but $\xi, w \in B(z, w)$ where $d\big(f^{jN}(w), f^{jN}(\xi)\big) \geq 2\delta$. Therefore, either $d\big(f^{jN}(w), f^{jN}(z)\big)$ or $d\big(f^{jN}(\xi), f^{jN}(z)\big)$ is bigger than δ. □

4 Symbolic dynamics

Coding expanding maps of the circle

Let $f : S^1 \to S^1$ be an expanding map of degree 2, where S^1 denotes the unit circle.
We know there exists a unique fixed point $p \in S^1$. Let $q \neq p$ be the other preimage of p, i.e. $f(q) = p$.





Recall the sets

$$\Sigma_d = \{(x_0, x_1, x_2, \dots) \mid \forall i \geq 0, x_i \in \{1, 2, \dots, d\}\}.$$

The shift map $\sigma : \Sigma_d \to \Sigma_d$ is defined as $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$.

Let $S^1 = \Delta_1 \bigcup \Delta_2$, where Δ_1 and Δ_2 are closed arcs with end points p and q. Given $x \in S^1$, we want to associate a $w = (w_i)_{i=0}^{\infty} \in \Sigma_2$ such that,

$$f^n(x) \in \Delta_{w_n}, \forall n \geq 0.$$

However, if $f^n(x) \in \Delta_1 \bigcap \Delta_2 = \{p, q\}$ then there are ambiguities. In this case we can finish the sequence $w_n, w_{n+1}, \dots =$ with either, $1, 1, 1, 1, \dots$ or $2, 2, 2, 2, \dots$ if $f^n(x) = p$ and either $2, 1, 1, 1, \dots$ or $1, 2, 2, 2, \dots$ if $f^n(x) = q$.

Example 8. *Let $T : S^1 \to S^1$ be defined as $T(x) = 2x \pmod 1$. Then $p = 0$ and $q = \frac{1}{2}$, $\Delta_1 = [0, \frac{1}{2}]$, $\Delta_2 = [\frac{1}{2}, 1]$. Here, the sequence $w = (w_n)_{n=0}^{\infty}$ associated to x corresponds to a dyadic expansion*

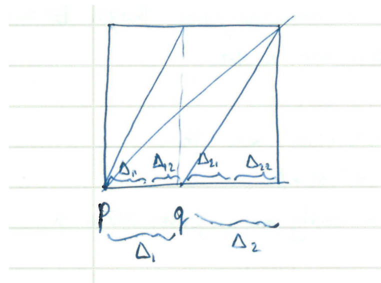
$$x = \sum_{n=0}^{\infty} \frac{w_n - 1}{2^{n+1}}.$$

The coding is similar to the decimal expansion, with similar ambiguities.

Proposition 11. *If $f : S^1 \to S^1$ is an expanding map of degree 2, then f is a factor of $\sigma : \Sigma_2 \to \Sigma_2$. That is, there exists a continuous surjective map $\pi : \Sigma_2 \to S^1$ such that $\pi \circ \sigma = f \circ \pi$.*

Proof. For $w_0, w_1, ..., w_{n-1} \in \{1, 2\}$, we define,

$$\Delta_{w_0,...,w_{n-1}} = \Delta_{w_0} \bigcap f^{-1}\Delta_{w_1} \bigcap f^{-2}\Delta_{w_2} \bigcap ... \bigcap f^{-(n-1)}\Delta_{w_{n-1}}$$



Thus $\Delta_{w_0} \supseteq \Delta_{w_0 w_1} \supseteq \Delta_{w_0 w_1 w_2} \supseteq ... \supseteq ...$ are a nested sequence of closed intervals for every $w = (w_n)_{n=0}^{\infty}$. Also,

$$1 = \int_{\Delta_{w_0 w_1 ... w_{n-1}}} \left|(f^n)'(x)\right| dx = \left|(f^n)'(\zeta)\right| \cdot \text{length}(\Delta_{w_0 w_1 ... w_{n-1}})$$

for some $\zeta \in \Delta_{w_0 ... w_{n-1}}$ (by the intermediate value theorem). Moreover, as $f : S^1 \to S^1$ is expanding, we have $|(f^n)'(\zeta)| \geq \beta^n$ for $n \geq 0$.

Therefore, length$(\Delta_{w_0 w_1 ... w_{n-1}}) \to 0$ as $n \to \infty$ and $\bigcap_{n=1}^{\infty} \Delta_{w_0 w_1 ... w_{n-1}}$ is a single point which we denote by $\pi(w)$.

<u>π is surjective:</u> Given $x \in S^1$, define $w = (w_n)_{n=0}^{\infty}$ such that $f^n(x) \in \Delta_{w_n}$. Therefore $\pi(w) = x$.

<u>π is continuous:</u> If w is close to w', then $w_n = w'_n$ for $n = 0, 1, ..., N$ for large N. This implies that,

$$\left|\pi(w) - \pi(w')\right| \leq \text{length}(\Delta_{w_0 ... w_{N-1}}) \leq \frac{1}{\beta^N}$$

<u>π is semi-conjugacy:</u> This follows immediately from definition;

$$\pi(\sigma w) = \bigcap_{n=1}^{\infty} \Delta_{w_1 ... w_n} = f\Big(\bigcap_{n=0}^{\infty} \Delta_{w_0 ... w_{n-1}}\Big) = f(\pi w).$$

The map $\pi : \Sigma_2 \to S^1$ cannot be a homeomorphism. That is because one of them is connected and the other one is not. □

Proposition 12. *If w and $w' \in \Sigma_2$ ($w \neq w'$) satisfy $\pi(w) = \pi(w') = x$ then there exists $n \geq 0$ such that $f^n(x) = p$. (In particular there are at most countably many points where π fails to be one-to-one.*

We now come to the main classification result for expanding maps $f : S^1 \to S^1$ of degree 2.

Theorem 3. *If $f : S^1 \to S^1$ and $g : S^1 \to S^1$ are two expanding maps of degree 2 then f and g are topologically conjugate. That is there exists a homeomorphism $\pi : S^1 \to S^1$ such that $f \circ \pi = \pi \circ g$. In particular every expanding map of the circle (of degree 2) is conjugate to the linear one.*

Proof. Consider the conjugates $\pi_f : \Sigma_2 \to S^1$ and $\pi_g : \Sigma_2 \to S^1$ associated to the two expanding maps. For $x \in S^1$ let $\pi(x) = \pi_g\big(\pi_f^{-1}(x)\big)$.

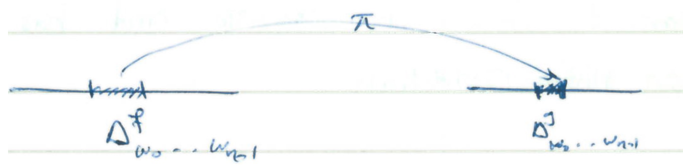
- if $\pi_f^{-1}(x)$ is a single point then $\pi(x)$ is well defined.

- if $\pi_f^{-1}(x)$ is two points, then the sequences end with infinitely many 1's or 2's. But, then $\pi_g\big(\pi_f^{-1}(x)\big)$ is again a single point.

Either way, $\pi_g\big(\pi_f^{-1}(x)\big)$ is a bijection.

Also, it follows from definitions that $\pi \circ f = g \circ \pi$, π is continuous as,

$$\pi : \bigcap_{k=0}^{n-1} f^{-k}\Delta_{w_k} \to \bigcap_{k=0}^{n-1} g^{-k}\Delta_{w_k}$$



□

Exercise 1. *Show that in the above theorem, even if $f : S^1 \to S^1$ and $g : S^1 \to S^1$ are real analytic, π need not even be C^1. (Hint, show that a C^1 conjugacy preserves the derivatives at fixed points.)*

Remark 3. *This theorem extends in an obvious way to expanding maps of degree d, where one projects onto Σ_d.*

Corollary 1. *Let $f : S^1 \to S^1$ be an expanding map of the unit circle. We have,*

i) The periodic points of f are dense in S^1,

ii) $f : S^1 \to S^1$ is mixing.

Proof. i). This follows from the corresponding statement for $\sigma : \Sigma_d \to \Sigma_d$. Since, if $\sigma^p(w) = w$ then $\pi(\sigma^p(w)) = f^p(\pi(w)) = \pi(w)$. That is, the image of a σ- periodic point is a f-periodic point. As σ-periodic points are dense in Σ_d, and π is continuous and surjective, the result follows.

ii). We already proved that every expanding map of the circle is mixing (and hence is transitive). Here is a different proof.

Let $U, V \subseteq S^1$ be non-empty open sets. We can choose $w_0, w_1, ..., w_{m-1}$ and $w'_0, w'_1, ..., w'_{m-1}$ such that,

$$\Delta_{w_0 w_1 ....w_{m-1}} \subseteq U$$

and

$$\Delta_{w'_0 w'_1,...w'_{m-1}} \subseteq V$$

Since $\sigma : \Sigma_d \to \Sigma_d$ is mixing, we can choose $n_0 > 0$ such that,

$$[w_0, w_1, ..., w_{m-1}] \bigcap \sigma^{-n}[w'_0, w'_1, ..., w'_{m-1}] \neq \phi$$

for all $n \geq n_0$. Hence, $\forall n \geq n_0$ there is a $w \in \Sigma_d$ in this intersection. Let $x = \pi(w)$. One can observe that,

$$x = \pi(w) \in \Delta_{w_0 \dots w_{m-1}} \bigcap f^{-n} \Delta_{w'_0 \dots w'_{m-1}}$$

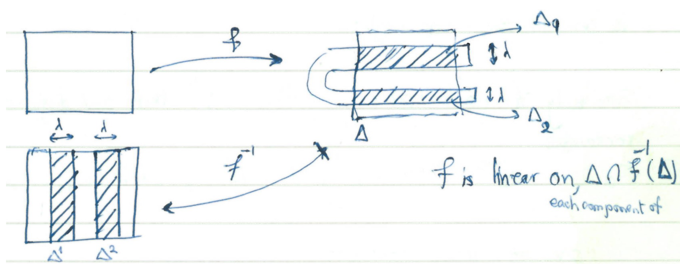
In particular $f : S^1 \to S^1$ is chaotic and has sensitive dependence on initial conditions. □

Coding horseshoe maps

Definition 9. ***(Linear Horseshoes.)*** *Let $\Delta = [0,1] \times [0,1] \subset \mathbb{R}^2$ be a rectangle. Assume $f : \Delta \to f(\Delta)$ is a diffeomorphism onto its image so that $\Delta \bigcap f(\Delta) = \Delta_1 \bigcup \Delta_2$ such that*

i) Δ_0 and Δ_1 are (horizontal) sub-rectangles of height $\lambda \leq \frac{1}{2}$,

ii) the restriction of f on the components of $\Delta \cap f^{-1}(\Delta)$ are linear maps.



One can write $\Delta \bigcap f^{-1}(\Delta) = \Delta^1 \bigcup \Delta^2$ where Δ^1 and Δ^2 are vertical sub-rectangles of width $\lambda \leq \frac{1}{2}$. Define,

$$\Lambda := \bigcap_{n \in \mathbb{Z}} f^{-n} \Delta.$$

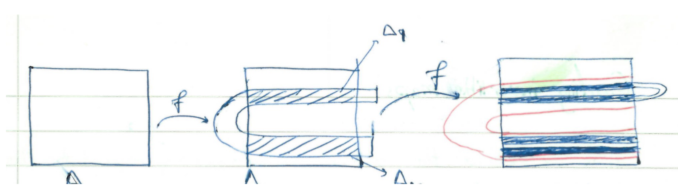
In other words, Λ is the set of points that can be iterated infinitely many times forward and backward under the map f.

Proposition 13. *$f : \Lambda \to \Lambda$ is topologically conjugate to $\sigma : \Sigma \to \Sigma$ where $\Sigma = \{1,2\}^{\mathbb{Z}}$. That is, $\exists$ a homeomorphism $\pi : \Sigma \to \Lambda$ such that $f \circ \pi = \pi \circ \sigma$*

Proof. Observe that $\Delta \bigcap f(\Delta) \bigcap f^2(\Delta)$ consists of four thin rectangles:

$$\Delta_{i,j} = \Delta_i \bigcap f(\Delta_j)$$

for $i, j \in \{1,2\}$.



Continuing inductively, for each $n \geq 1$:

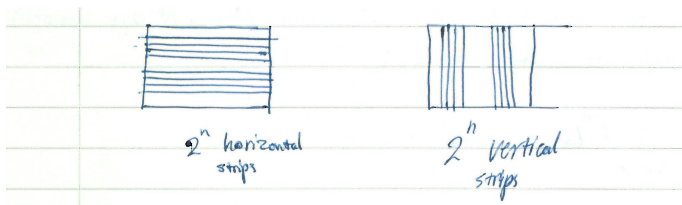
$$\bigcap_{i=0}^{n-1} f^i(\Delta) = \bigcap_{i=0}^{n-1} f^i(\Delta_1 \bigcup \Delta_2)$$

is 2^n thin and disjoint horizontal rectangles. For $w_0, w_1, \ldots, w_{n-1} \in \{1, 2\}$, let

$$\Delta_{w_0 w_1, \ldots, w_{n-1}} := \bigcap_{i=0}^{n-1} f^i(\Delta_{w_i})$$

be one of these rectangles.

Similarly for each $n \geq 1$: $\bigcap_{i=0}^{n-1} f^{-i}(\Delta)$ is 2^n thin and disjoint vertical rectangles. For every finite sequence $w_0, w_{-1}, \ldots, w_{-(n-1)} \in \{1, 2\}$, we let $\Delta^{w_0 w_1, \ldots, w_{n-1}} = \bigcap_{i=0}^{n-1} f^{-i}(\Delta^{w_{-i}})$ be one of these rectangles.



We define,

$$\pi(w) = \Big(\bigcap_{n=0}^{\infty} \Delta_{w_0 w_1 \ldots w_{n-1}}\Big) \bigcap \Big(\bigcap_{n=0}^{\infty} \Delta^{w_0 w_1 \ldots w_{n-1}}\Big)$$

with $\pi : \Sigma \to \Lambda$ and $w = (w_n)_{n \in \mathbb{Z}}$. The image of a cylinder $[w_{-(n-1)}, \ldots, w_0, w_1, \ldots, w_{n-1}]$ is a square $\Delta_{w_0 \ldots w_{n-1}} \bigcap \Delta^{w_0 \ldots w_{-(n-1)}}$ of size $\lambda^{n-1} \times \lambda^{n-1}$. In particular,

1. π is continuous,

2. π is invertible (and a homeromorphism),

3. Λ is a cantor set: perfect, compact, totally disconnected,

4. $\pi \circ \sigma = f \circ \pi$.

□

By the above proposition, $f : \Lambda \to \Lambda$ inherits properties of $\sigma : \Sigma \to \Sigma$. In particular, we obtain the following corollary.

Corollary 2. *Let $f : \Delta \to \mathbb{R}^2$ be a linear horseshoe map. We have,*

i) The periodic points of $f : \Lambda \to \Lambda$ are dense in Λ.

ii) The number of periodic points of $f : \Lambda \to \Lambda$ is 2^n.

iii) $f : \Lambda \to \Lambda$ is topologically mixing.

5 Topological Entropy

We have seen qualitative indications of chaos: transitivity, density of periodic orbits, sensitive dependence on initial conditions. Now,

- We would like to quantify the complexity of f,
- We would like an invariant for (topological) conjugacy between maps $f : X \to X$ and $g : Y \to Y$ (We have seen simple invariants for conjugacy, e.g. the number of periodic points, p_n, $n \geq 1$).

Let $d(x,y)$ be a metric on a compact set X and $f : X \to X$ be continuous. For each $n \geq 1$, we can define new metrics,

$$d_n(x,y) = \max_{0 \leq i \leq n-1} d\big(f^i(x), f^i(y)\big)$$

Let

$$B(x,n,\varepsilon) = \big\{y; d_n(x,y) < \varepsilon\big\}.$$

A finite set $E \subseteq X$ is an (n,ε)-dense set if $X \subseteq \bigcup_{x \in E} B(x,\varepsilon,n)$. This is also called (n,ε)-spanning set. Let $S(n,\varepsilon)$ be the minimum cardinality of any (n,ε)-dense set. (In other words, this is the list of information needed to keep track of all orbits up to ε-error.) We can consider the exponential growth rate,

$$h(f,\varepsilon) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log S(n,\varepsilon).$$

By the definition, for $\varepsilon < \varepsilon'$ we have

$$h(f,\varepsilon) \geq h(f,\varepsilon').$$

We define the topological entropy,

$$h(f) = \lim_{\varepsilon \to 0} h(f,\varepsilon) \geq 0$$

Let X be a compact topological space. Two metrics d and d' on X are equivalent, if the convergence with respect to any of these metrics implies the convergence with respect to the other one.

Lemma 4. *Let X be a compact metric space and $f : X \to X$ be a continuous map. The definition of entropy given above is independent of the (equivalence class of the) metric.*

Proof. By the compactness of X, the identity map $I : (X,d) \to (X,d')$ is a (uniformly) continuous homeomorphism. This implies that, given $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$: $d(x,y) < \delta \implies d'(x,y) < \varepsilon$. Replacing d and d' by d_n and d'_n we can deduce that $d_n(x,y) < \delta \implies d'_n(x,y) < \varepsilon$. In particular, any (d_n,δ)-dense set is also (d'_n,ε)-dense set. Hence,

$$S_d(n,\delta) \geq S_{d'}(n,\varepsilon) \ , n \geq 1.$$

Thus, $h_d(f,\delta) \geq h_{d'}(f,\varepsilon)$,

$$\implies h_d(f) = \lim_{\delta \to 0} h_d(f,\delta) \geq \lim_{\varepsilon \to 0} h_{d'}(f,\varepsilon) = h_{d'}(f)$$

$$\implies h_d(f) \geq h_{d'}(f)$$

By the symmetry of the above argument we also have $h_d(f) \leq h_{d'}(f)$. □

Corollary 3. *Entropy is a conjugacy invariant.*

Proof. Let $\pi : X \to Y$ be a conjugacy between $f : X \to X$ and $g : Y \to Y$, i.e. $\pi \circ g = f \circ \pi$. If d_X is a metric on X, then define d_Y on Y by,

$$d_Y(y,y') = d_X\left(\pi^{-1}(y), \pi^{-1}(y')\right)$$

Thus π becomes an isometry in these metrics. Therefore $h_d(f) = h_{d'}(g) \implies h(f) = h(g)$. $\square$

Example 9. *Consider the expanding map $f : S^1 \to S^1$, $f(x) = dx \pmod 1$, $d \geq 2$. Observe that for any $n \geq 1$, a d_n-ball $B(x,n,\varepsilon)$, has diameter $\frac{2\varepsilon}{d^n}$. Thus we can cover S^1 by $\left(\frac{d^n}{2\varepsilon}\right) + 1$ such balls. Therefore*

$$S(n,\varepsilon) \leq \left(\frac{d^n}{2\varepsilon}\right) + 1, \text{ and } S(n,\varepsilon) \geq \left(\frac{d^n}{2\varepsilon}\right).$$

In particular,

$$h(T) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log(S(n,\varepsilon)) = \lim_{\varepsilon \to 0} (\log d) = \log d.$$

Remark 4. *Any expanding map $f : S^1 \to S^1$ of degree $d \geq 2$ is topologically conjugate to a linear of degree d, and hence has the same entropy* $\log d$.

On the other hand,

Proposition 14. *If $f : X \to X$ is an isometry, then $h(f) = 0$.*

Proof. By definition $d_n(x,y) = d(x,y)$, $\forall n \geq 1$, in particular $S(n,\varepsilon)$ is independent of n. Thus, $h(f) = 0$. $\square$

Example 10. *Consider a rotation $f : S^1 \to S^1$ defined as $f(x) = x + \alpha (mod 1)$. This is an isometry and so $h(f) = 0$.*

Remark 5. *By the Denjoy theorem; certain homeomorphisms are conjugate to rotations, and have zero entropy. In fact, all homeomorphisms of S^1 have zero entropy.*

The above definition is suitable for finding upper bounds for the topological entropy. That is because an example of (n,ε)-dense set provides an upper bound for $S(n,\varepsilon)$. There is an alternative definition of the topological entropy which is more suitable for proving lower bounds. We present this below.

Let $N(n,\varepsilon)$ be the maximal number of points in X whose pairwise d_n distances are at least $\varepsilon > 0$ A set of such points is called an (n,ε)-separated set.

Lemma 5. *We have,*

i) $N(n,\varepsilon) \geq S(n,\varepsilon)$;

ii) $S(n,\varepsilon) \geq N(n,2\varepsilon)$.

Proof. **i).** A (n,ε)-separated set is a (n,ε)-spanning set, since otherwise we could enlarge the separating set by adding a point not already covering. Thus $N(n,\varepsilon) \geq S(n,\varepsilon)$.

ii). Let A be an arbitrary $(n,2\varepsilon)$-separated set, and B be an arbitrary (n,ε)-dense set. We define the map $\phi : A \to B$ as follows. By the definition of (n,ε)-dense set, the set $\cup_{x\in B} B(x,n,\varepsilon)$ covers X. Then, for any $x \in A$, there is $\phi(x) \in B$ such that $d_n(x,\phi(x)) < \varepsilon$.

The map ϕ is one-to-one. That is, because, if $\phi(x) = \phi(y)$, then

$$d_n(x,y) \leq d_n(x,\phi(x)) + d_n(\phi(y),y) < \varepsilon + \varepsilon = 2\varepsilon.$$

However, as A is $(n,2\varepsilon)$-separated, we must have $x = y$.

The injectivity of ϕ implies that the number of elements in A is at least the number of elements in B. Since A and B where arbitrary sets, we must have $S(n,\varepsilon) \geq N(n,2\varepsilon)$. This finishes the proof of the proposition by taking limits as $\varepsilon \to 0$. □

Proposition 15.

$$h(f) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N(n,\varepsilon)$$

Proof. By part i) of the above lemma,

$$\overline{\lim} \frac{1}{n} N(n,\varepsilon) \geq \overline{\lim_{n\to\infty}} \frac{1}{n} \log S(n,\varepsilon) = h(f,\varepsilon)$$

By part ii) of the above lemma,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log N(n,2\varepsilon) \leq \overline{\lim_{n\to\infty}} \frac{1}{n} \log S(n,\varepsilon) = h(f,\varepsilon)$$

Letting $\varepsilon \to 0$, gives the result. □

Proposition 16. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps and $\pi : X \to Y$ be a surjective semi-conjugacy (i.e. $g\pi = \pi f$). Then $h(g) \leq h(f)$.*

Proof. Let d^X and d^Y denote the metrics on X and Y respectively. For $\varepsilon > 0$, choose $\delta > 0$ such that $d^X(x_1,x_2) < \delta \implies d^Y\big(\pi(x_1),\pi(x_2)\big) < \varepsilon$. Thus a d_n^X δ-ball $B(x,n,\delta)$ has an image $\pi\big(B(x,n,\delta)\big) \subseteq B\big(\pi(x),n,\varepsilon\big)$. In particular,

$$S_{d^X}(x,\delta) \geq S_{d^Y}(n,\varepsilon)$$

The result follows. □

This leads to another proof of the following,

Corollary 4. *If $f : X \to X$ and $g : Y \to Y$ are (topologically) conjugate, then $h(f) = h(g)$.*

Review Problems

1. Is $F(x) = x + \frac{1}{2}sin(x)$ the lift of a circle homeomorphism?

2. $F(x) = x + \frac{1}{4x}sin(2\pi x)$ the lift of a circle homeomorphism?

3. Let $F(x) = x + c + bsin(2\pi x)$. Show that if $|2\pi b| < 1$ then this is an orientation preserving homeomorphism. If $|c| < |b|$ show that $\rho(f) = 0$ for the corresponding map $f : S^1 \to S^1$.

4. Let f and g be orientation preserving homeomorphisms of S^1. Prove that $\rho(f) = \rho(g^{-1}fg)$, wehere ρ denotes the rotation number.

5. Let X be a compact metric space with more than one point and $f : X \to X$ be an isometry. Show that f cannot be mixing.

6. Prove that a homeomorphism of $\mathbb{R}$ can have no periodic points with prime period greater than 2

7. Show that $w : [0,1] \to \mathbb{R}$ defined as,

$$\begin{cases} 0, & x = 0 \\ xsin(\frac{1}{x}), & x \neq 0 \end{cases}$$

   is not a function of bounded variation

8. Show that $w : [0,1] \to \mathbb{R}$ defined as,

$$\begin{cases} 0, & x = 0 \\ x^2sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

   is a function of bounded variation

9. Let $Q_c(x) = x^2 + c$. Prove that if $c < \frac{1}{4}$ there is a unique $\mu > 1$ such that Q_c is topologically conjugate to $f_\mu(x) = \mu x(1-x)$ via a map of the form $h(x) = \alpha x + \beta$.

10. Let $f : X \to X$ be a continuous map of a compact metric space. A point $p \in X$ is called topologically recurrent if for any open set V, $p \in V$, there exists an $n > 0$ with $f^n(p) \in V$. Clearly every periodic point is recurrent.
    i). Give an example of a $f : X \to X$ with a non-periodic recurrent point.
    ii). Give an example of a $f : X \to X$ with a non-periodic recurrent point p whose orbit is not dense in X.

11. Show that the horseshow map (Λ, f) is topologically mixing.

12. Prove that the topological entropy of any C^1 (continuously differentiable) map of $S^1 \times S^1$ (torus) is finite.

13. Let $f : S^1 \to S^1$ be a homeomorphism of cricle. Show that $\rho(f^m) = m\rho(f)$ where $\rho(f)$ denotes the rotation number of f.

14. Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ defined as $A(x) = 2x$, observe that A induces a map $f : T^n \to T^n$, where $T^n = S^1 \times S^1 \times ... \times S^1$ (n times) is the n-dimensional torus.
a). Prove that the periodic points of f are dense in T^n.
b). Prove that eventual fixed points, i.e. the points $x \in T^n$ with $f\big(f^n(x)\big) = f^n(x)$, for some n, are dense in T^n.
c). Prove that $f : T^n \to T^n$ is chaotic.

15. Let X be a compact metric space with at least three distinct points. Let $f : X \to X$ be an isomtery.
a). Show that f is not mixing.
b). What if X has only two points?

16. Let $f : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ be defined as,
$$f(x,y,z) = (x, x+y, y+z)(mod 1)$$
Find $h_{top}(f)$.

17. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ and define $f_\lambda : D \to D$, for $\lambda \in [0,1]$, as $f_\lambda(z) = \lambda z^2$.
a). Show that $h_{top}(f_\lambda) \geq \log 2$, when $\lambda = 1$.
b). Show that $h_{top}(f_\lambda) = 0$, when $0 \leq \lambda < 1$.
Therefore, topological entropy does not depend continuously on the map.