# Exam: Dynamical Systems 2015/16

2 hours
Answer 3 out of 4 questions.

**Question 1** *Consider the doubling map $E_2 : [0,1[\to [0,1[$ given by $x \mapsto 2x \mod 1$.*

1. *A point $x$ is called recurrent if there exists a sequence $n_k$ with $\lim_{k\to\infty} \Phi_{n_k}(x) = x$. Does there exists a point which is not recurrent.* ***[4marks]***
2. *Show that the Lebesgue measure $m$ restricted to $[0,1[$ is invariant w.r.t. $E_2$.* ***[4marks]***
3. *Compute $E_2^{-n}([\frac{j}{2^k}, \frac{j+1}{2^k}[)$.* ***[4marks]***
4. *Let $\Phi$ be a map and $\mu$ be a measure. We say that $\mu$ is mixing w.r.t. $\Phi$ iff for any interval $A, B$ of the form $A = [\frac{j}{2^k}, \frac{j+1}{2^k}[$ and $B = [\frac{j'}{2^{k'}}, \frac{j'+1}{2^{k'}}[$ holds that $\lim_{n\to\infty} \mu\left(\Phi_n^{-1}(A) \cap B\right) = \mu(A)\mu(B)$. Show that $m$ is mixing w.r.t. $E_2$.* ***[4marks]***
5. *Show that*
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \sin(2^n \pi x) = 0. \tag{1}$$
***[4marks]***

## Answer

1. Yes, for example any point $x = 2^{-m}$.
2. Let us compute $(E_2)_\# m$, that is we have to compute
$$(E_2)_\# m(A) = m\left(x \in [0,1[: 2x \bmod 1 \in A\right) \tag{2}$$
$$= m\left(x \in [0,1/2[: 2x \in A\right) + m\left(x \in [1/2,1[: 2x-1 \in A\right) \tag{3}$$
Let us consider the two pieces separately. Indeed, that map $f : x \mapsto 2x$ as map from $[0,1/2[\to [0,1[$ is differentiable and
$$m\left(x \in [0,1/2[: 2x \in A\right) = f_\# m(A) \tag{4}$$
Note that $Df(x) = 2$ and $f^1(y) = y/2$. Hence by definition of the push-forward we get that
$$f_\# m(A) = \int_A \frac{1}{|\det(Df(f^{-1}(y)))|} dy = \frac{1}{2} m(A). \tag{5}$$
**[Bookwork]**
3. Using the definition
$$E_2^{-n}([\frac{j}{2^k}, \frac{j+1}{2^k}[) \tag{6}$$
$$= \left\{ x \in [0,1[: \frac{j}{2^k} \le 2^n x < \frac{j+1}{2^k} \mod 1 \right\} \tag{7}$$
$$= \left\{ x \in [0,1[: \exists i \in \mathbb{N}_0 \text{ with } \frac{j}{2^k} \le 2^n x + i \frac{j+1}{2^k} \mod 1 \right\} \tag{8}$$
$$= \left\{ x \in [0,1[: \exists i = 0,\ldots,2^n - 1 \text{ with } \frac{j}{2^{k+n}} + \frac{i}{2^n} \le x < \frac{j+1}{2^{k+n}} + \frac{i}{2^n} \mod 1 \right\} \tag{9}$$
$$= \bigcup_{i=1}^{2^n-1} [\frac{j}{2^{k+n}} + \frac{i}{2^n}, \frac{j+1}{2^{k+n}} + \frac{i}{2^n}[ \tag{10}$$
where the union is disjoint. **[Unseen]**

4. Now we have to compute the intersection

$$E_2^{-n}(A) \cap B = \bigcup_{i=1}^{2^n-1} [\frac{j}{2^{k+n}} + \frac{i}{2^n}, \frac{j+1}{2^{k+n}} + \frac{i}{2^n}[\cap[\frac{j'}{2^{k'}}, \frac{j'+1}{2^{k'}}[ \tag{11}$$

If $n$ is so large that $k+n \geq k'$ then the intervals forming $E_2^{-n}(A)$ either contained in $B$ or disjoint to $B$. Each of the intervals in $E_2^{-n}(A)$ has length $2^{-(k+n)}$ and the gap between the intervals is of the size $2^{-n} - 2^{-(n+k)}$. This means that $\frac{2^{-k'}}{2^{-n}} \pm 1$ can intersect. As each have length $2^{-(k+n)}$ one gets that the length of $E_2^{-n}(A) \cap B$ is $2^{-k'-k} \pm 2^{-(k+n)}$ which tends for $n \to \infty$ to $2^{-k'-k}$ which is just $m(A)m(B)$. **[Unseen]**

5. As the dynamics is mixing it is also ergodic. Consider the testfunction $\varphi(x) := \sin(2\pi x)$ and the convergence follows from Birkhoff's ergodic theorem and the limit is $\overline{\varphi}(x) = \int_0^1 \varphi(y)dy = \int_0^1 \sin(2\pi x)dx = -\frac{1}{2\pi}\cos(2\pi x)\big|_{x=0}^1 = 0$. **[Unseen]**

**Question 2** 1. *Show that if a measure $\mu$ is invariant with respect to $\Phi$ then $\mu$ is invariant w.r.t $\Phi^2$.*

*Show that the converse is not true.* ***[3 marks]***

2. *Let $V = \mathbb{C}$ and consider the cone*

$$C = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_j > 0 \right\} \tag{12}$$

*Show that $C$ is a proper convex cone.* ***[3 marks]***

3. *Compute $\alpha_C$.* ***[3 marks]***

4. *Show that for $v, w \in C$*

$$\Theta_C(v, w) = -\max\left\{ \ln\left(\frac{w_1 v_2}{v_1 w_2}\right), \ln\left(\frac{v_1 w_2}{w_1 v_2}\right) \right\} \tag{13}$$

***[3 marks]***

5. *Identify all linear maps $T : V \to V$ such that $T$ preserves the cone $C$.* ***[4marks]***

6. *Show that the diameter of $T(C)$ is*

$$\min\left\{ \frac{a_{i,j} a_{k,l}}{a_{k,j} a_{i,l}} : i, j, k, l \in \{0, 2\} \right\} \tag{14}$$

***Hint:*** *Prove that $D \geq 1$.* ***[4marks]***

## Answer

1. The invariance follows from the definition. Indeed,

$$\mu(\Phi_2(A)) = \mu(\{x \in \Omega : \Phi_2(x) \in A\}) = \mu(\{x \in \Omega : \Phi(\Phi(x)) \in A\}) \tag{15}$$

$$= \mu(\{x \in \Omega : \Phi(x) \in \Phi^{-1}(A)\}) = \Phi_\# \mu\left(\Phi^{-1}(A)\right) \tag{16}$$

then by invariance of $\mu$ we get

$$= \mu\left(\Phi^{-1}(A)\right) = \Phi_\# \mu(A) = \mu(A). \tag{17}$$

Consider the rotation $\Phi = R_{1/2}$. Then we have $\Phi^2 = R_1$ which is the identity map. So any measure is invariant w.r.t. $\Phi^2$ but not all are invariant w.r.t. $\Phi$. For example, the measure $\delta_{1/2}$ is not invariant,

$$\left(R_{1/2}\right)_{\#} \delta_{1/2}(A) = \delta_{1/2}(R_{1/2}^{-1}(A)) = \begin{cases} 1 & \frac{1}{2} \in R_{1/2}^{-1}(A) \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$= \begin{cases} 1 & R_{1/2}\left(\frac{1}{2}\right) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 1 \in A \\ 0 & \text{otherwise} \end{cases} = \delta_1(A) \tag{19}$$

**[Unseen]**

2. Let $x^{(i)} \in C$ and $\alpha, \beta > 0$ then as $x_j^{(i)} > 0$ and hence $\alpha x_j^{(1)} + \beta x_j^{(2)} > 0$. Let $w \in \mathbb{R}^2$ and $v \in C$, then $w \in \overline{C}^r$ iff $w_j + \alpha v_j > 0$, that is the case iff $w_j \geq 0$. Hence

$$\overline{C}^r = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_j \geq 0 \right\} \tag{20}$$

and

$$\overline{C}^r \cap -\overline{C}^r = \{0\}. \tag{21}$$

**[Modification of exercise]**

3. 

$$\{t \geq 0 : w - tv \in C\} = \{t \geq 0 : w_j - tv_j > 0\} = \left\{ t \geq 0 : t < \frac{w_1}{v_1} \text{ and } t < \frac{w_2}{v_2} \right\} \tag{22}$$

and hence

$$\alpha(v, w) = \max\left\{ \frac{w_1}{v_1}, \frac{w_2}{v_2} \right\} \tag{23}$$

**[Modification of exercise]**

4. From this follows that

$$\Theta_C(v, w) = -\ln \max\left\{ \frac{w_1}{v_1}, \frac{w_2}{v_2} \right\} \max\left\{ \frac{v_1}{w_1}, \frac{v_2}{w_2} \right\} = -\ln \max\left\{ \frac{w_1 v_2}{v_1 w_2}, \frac{w_2 v_1}{v_2 w_1}, 1 \right\} \tag{24}$$

Since either $\frac{w_1 v_2}{v_1 w_2} \leq 1$ or $\frac{w_2 v_1}{v_2 w_1} \leq 1$, we get that

$$\Theta_C(v, w) = -\ln \max\left\{ \frac{w_1 v_2}{v_1 w_2}, \frac{w_2 v_1}{v_2 w_1} \right\} \tag{25}$$

[**Book work**

5. Cone preserving means that for all $v_1, v_2 > 0$ holds that

$$a_{1,1} v_1 + a_{1,2} v_2 > 0 \tag{26}$$
$$a_{2,1} v_1 + a_{2,2} v_2 > 0. \tag{27}$$

Consider now a sequence $\beta_n \downarrow 0$, then for $v_1 = 1$ and $v_2 = \beta_n$ this gives in the limit $n \to \infty$

$$a_{1,1} > 0 \tag{28}$$
$$a_{2,1} > 0. \tag{29}$$

Analogously, one obtains that

$$a_{1,2} > 0 \tag{30}$$
$$a_{2,2} > 0. \tag{31}$$

These conditions are also obviously also sufficient. **[Unseen]**

6. Call

$$D := \min\left\{\frac{a_{i,j}a_{k,l}}{a_{k,j}a_{i,l}} : i,j,k,l \in \{0,2\}\right\} \tag{32}$$

$$\frac{T(w)_1 T(v)_2}{T(v)_1 T(w)_2} = \frac{(a_{1,1}w_1 + a_{1,2}w_2)(a_{2,1}v_1 + a_{2,2}v_2)}{(a_{1,1}v_1 + a_{1,2}v_2)(a_{2,1}w_1 + a_{2,2}w_2)} \tag{33}$$

$$= \frac{a_{1,1}a_{2,1}v_1w_1 + a_{1,1}a_{2,2}v_2w_1 + a_{1,2}a_{2,1}v_1w_2 + a_{1,2}a_{2,2}v_2w_2}{a_{1,1}a_{2,1}w_1v_1 + a_{1,1}a_{2,2}w_2v_1 + a_{1,2}a_{2,1}w_1v_2 + a_{1,2}a_{2,2}w_2v_2} \tag{34}$$

$$\geq \frac{a_{1,1}a_{2,1}v_1w_1 + Da_{2,1}a_{1,2}v_2w_1 + Da_{2,2}a_{1,1}v_1w_2 + a_{1,2}a_{2,2}v_2w_2}{a_{1,1}a_{2,1}w_1v_1 + a_{1,1}a_{2,2}w_2v_1 + a_{1,2}a_{2,1}w_1v_2 + a_{1,2}a_{2,2}w_2v_2} \tag{35}$$

Interchanging $i$ and $k$ one sees that in the minima in Equation 13 which each number also always its inverse appears. Hence $D < 1$. So we can estimate

$$\frac{T(w)_1 T(v)_2}{T(v)_1 T(w)_2} \tag{36}$$

$$\geq \frac{Da_{1,1}a_{2,1}v_1w_1 + Da_{2,1}a_{1,2}v_2w_1 + Da_{2,2}a_{1,1}v_1w_2 + Da_{1,2}a_{2,2}v_2w_2}{a_{1,1}a_{2,1}w_1v_1 + a_{1,1}a_{2,2}w_2v_1 + a_{1,2}a_{2,1}w_1v_2 + a_{1,2}a_{2,2}w_2v_2} \tag{37}$$

$$= D \tag{38}$$

and analogously also

$$\frac{T(v)_1 T(w)_2}{T(w)_1 T(v)_2} \geq D \tag{39}$$

Hence $\Theta_C(T(v), T(w)) \leq \ln(D)$. If we choose $v = (1, t_n)$ and $w = (1, t_n)$ with $t_n \downarrow 0$, then we see that

$$\frac{T(w)_1 T(v)_2}{T(v)_1 T(w)_2} \to \frac{a_{1,2}a_{2,1}}{a_{1,1}a_{2,2}} \tag{40}$$

Looking at the analogous cases we get that the diameter has to be also $\leq D$. **[Unseen]**

**Question 3** *Let $\alpha$ be an irrational number and define*

$$f : [0,1) \times [0,1) \to [0,1) \times [0,1) \text{ as } f(x,y) := (x + \alpha, x + y)(\text{mod } 1)$$

*be a homeomorphism (of the two dimensional torus).*

1. *Define, in terms of orbits, what it means for $f$ to be topologically transitive.* ***[2 marks]***
2. *Prove that for all non-empty open sets $U$ and $V$ in $[0,1) \times [0,1)$ there is a positive integer $n$ with $f^{-n}(U) \cap V \neq \varnothing$. (hint: look at first few iterates of a small square in the domain of the map).* ***[7 marks]***
3. *Prove, using the above statement or directly, that $f$ is topologically transitive.* ***[6 marks]***
4. *When is a continuous map $g : X \to X$, for a compact metric space $X$, is chaotic? Is the above map $f$ chaotic?* ***[5 marks]***

## Answer

Students have not encountered this map in the lectures.

1. The map $f$ is called topologically transitive if it has a dense orbit; i.e. there is a point $x \in [0,1)^2$ with $\{f^j(x)\}_{j \in \mathbb{Z}}$ dense in $[0,1)^2$. **[2pts, definition from lectures]** *Lectures*

2. Given non-empty open sets $U$ and $V$, there are points $(a, b)$ and $(c, d)$ as well as a small constant $\epsilon$ with

*Unseen, a bit difficult to make it precise, please grade easily*

$$V \supseteq (a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon), \text{and}$$
$$U \supseteq (c - \epsilon, c + \epsilon) \times (d - \epsilon, d + \epsilon).$$

Let $\pi_2$ denote the projection onto the second coordinate.

$\pi_2(V)$ is connected and has length $2\epsilon$.

$f(a - \epsilon, b - \epsilon) = (a - \epsilon + \alpha, a + b - 2\epsilon)$ and $f(a + \epsilon, b + \epsilon) = (a + \epsilon + \alpha, a + b + 2\epsilon)$ implies that $\pi_2(f(V))$ has length at least $2\epsilon + 2\epsilon$.

Similarly, $f^2(a - \epsilon, b - \epsilon) = (a - \epsilon + 2\alpha, a + b - 2\epsilon + a - \epsilon + \alpha)$ and $f^2(a + \epsilon, b + \epsilon) = (a + \epsilon + 2\alpha, a + b + 2\epsilon + a + \epsilon + \alpha)$, implies that $\pi_2(f^2(V))$ has length at least $2\epsilon + 2 \cdot 2\epsilon$ [**4pts, for understanding the map in the second coordinate**] .

Repeating this calculations, one concludes that $\pi_2(f^n(V))$ has length at least $2\epsilon + n \cdot 2\epsilon$. Choose $N > 0$ such that $2\epsilon + N \cdot 2\epsilon > 3$. That is, image of $f^k(V)$, for $k > N$, covers the second coordinate at least 3 times. Using the density of orbits in the first coordinate, it follows that $f^k(V) \cap U \neq \varnothing$, for some $k \geq N$. Equivalently, $f^{-k}(U) \cap V \neq \varnothing$[**3pts**].

3. Let $Y = \{y_i\}_{i=1}^{\infty}$ be a countable dense set in $[0, 1)^2$. Let $U_i$, $i = 1, 2, \ldots$, be an open disk centered at $y_i$, with diameter $1/i$.

*Similar to lectures*

Now choose $N_1 \geq 0$ such that $f^{-N_1}(U_2) \cap U_1 \neq \varnothing$. Then choose an open disk $V_1$ of radius less than $1/2$ such that

$$V_1 \subseteq \overline{V_1} \subseteq U_1 \cap f^{-N_1}(U_2).$$

Choose $N_2 \geq 0$ such that $f^{-N_2}(U_3) \cap V_1 \neq \varnothing$. Then choose an open disk $V_2$ of radius less than $1/4$ such that

$$V_2 \subseteq \overline{V_2} \subseteq V_1 \cap f^{-N_2}(U_3)[\mathbf{2pts}].$$

Repeating this process inductively, we obtain open sets, $V_1 \supseteq V_2 \supseteq V_3, \ldots$, with radius $V_n \leq \frac{1}{2^n}$ and

$$\overline{V_{n+1}} \subseteq V_n \cap f^{-N_{n+1}}(U_{n+2})[\mathbf{2pts}].$$

If we let $\{x\} = \cap_{n=1}^{\infty} \overline{V_n}$, then $f^{N_{n-1}}(x) \in U_n$, for $n \geq 1$. Therefore, $\{f^n(x)\}_{n=1}^{\infty}$ is dense**2pts**.

**Question 4** *1. Let $f$ be a continues map of a compact metric space $(X, d)$ to itself. Define what it means for a finite set $E \subseteq X$ to be $(n, \epsilon)$-dense. What does it mean for a set $F \subseteq X$ to be $(n, \epsilon)$-separated.* ***[5 marks]***

2. *Define the topological entropy of the map $f$ in terms of the above sets (both of them).* ***[5 marks]***

3. *Find the topological entropy of the map $f : [0, 1)^2 \to [0, 1)^2$ defined as*

$$f(x, y) = (2x, 3y)(\bmod 1).$$

***[10 marks]***

## Answers

1. For every $n \geq 0$ define the new metric $d_n$ on $X$ as follows.

*Lectures*

$$d_n(x, y) := \max\{d(f^j(x), f^j(y)) \mid 0 \leq j \leq n - 1\}$$

**[1pts]** Let $B(x, n, \epsilon) := \{y \in X \mid d_n(x, y) < \epsilon\}$. A finite set $E \subseteq X$ is $(n, \epsilon)$-dense if

$$X \subseteq \cup_{x \in E} B(x, n, \epsilon)\textbf{[2pts]}$$

A set $F \subseteq X$ is called $(n, \epsilon)$-separated if the $d_n$ distance between any two distinct points in $F$ is greater than $\epsilon$ **[2pts]**.

2. Let $S(n, \epsilon)$ be the minimum of cardinality of all $(n, \epsilon)$-dense sets in $X$**[1pt]**. Define *Lectures*

$$h(f, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log S(n, \epsilon)\textbf{[1pt]}.$$

The topological entropy of $f$ is defined as $h(f) := \lim_{\epsilon \to 0} h(f, \epsilon)$**[1pt]**.

Let $N(n, \epsilon)$ denote the maximal cardinality of all $(n, \epsilon)$-separating sets**[1pt]**. Then,

$$h(f) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon)\textbf{[1pts]}.$$

3. let $d$ denote the maximum metric on $S^1 \times S^1$ i.e. $d((a, b), (c, d)) = \max\{d'(a, c), d'(b, d)\}$ where $d'$ is the angular metric on $S^1$. *Unseen, but not too far from lectures*

Let $\epsilon$ be an arbitrary number less than $1/4$.

For any $n \geq 1$, define the set $E_n$ as follows:

$$E_n := \left\{ \left(\frac{i}{2^n} \frac{\epsilon}{2}, \frac{j}{3^n} \frac{\epsilon}{2}\right) \mid 0 \leq i \leq \lfloor \frac{2 \cdot 2^n}{\epsilon} \rfloor + 1, 0 \leq j \leq \lfloor \frac{2 \cdot 3^n}{\epsilon} \rfloor + 1 \right\}$$

Now, $[0, 1) \times [0, 1) \subseteq \cup_{x \in E_n} B(x, n, \epsilon)$, thus $E_n$ is a $(n, \epsilon)$-spanning set **[3pts, for any optimum $(n, \epsilon)$-spanning set]**. Therefore,

$$h(f, \epsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log 4 \cdot \frac{6^n}{\epsilon^2} = \log 6.$$

Hence, $h(f) \leq \lim_{\epsilon \to 0} h(f, \epsilon) = \log 6$ **[2pts]**.

On the other hand, the set

$$F_n := \left\{ \left(\frac{i}{2^n} \epsilon, \frac{j}{3^n} \epsilon\right) \mid 0 \leq i \leq \lfloor \frac{2^n}{\epsilon} \rfloor, 0 \leq j \leq \lfloor \frac{3^n}{\epsilon} \rfloor \right\}$$

forms an $(n, \epsilon)$-separating set **[3pts, for any optimum $(n, \epsilon)$-separating set]**. Therefore,

$$h(f, \epsilon) \geq \limsup_{n \to \infty} \frac{1}{n} \log \frac{6^n}{\epsilon^2} = \log 6.$$

Hence, $h(f) \geq \lim_{\epsilon \to 0} h(f, \epsilon) = \log 6$ **[2pts]**.

Putting these two inequalities together, we have $h(f) = \log 6$.