

ONE-DIMENSIONAL DYNAMICS IN THE NEW MILLENNIUM

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ABSTRACT. In the early 60's Sarkovskii discovered his famous theorem on the coexistence of periodic orbits for interval maps. Then, in the mid 70's, Li & Yorke rediscovered this result and somewhat later the papers by Feigenbaum and Coulet & Tresser on renormalisation and by Guckenheimer and Misiurewicz on sensitive dependence and existence of invariant measures, kicked off one of the most exciting areas within dynamical systems: iterations in dimension one. The purpose of this paper is to survey some of the recent developments, and pose some of the challenges and questions that keep this subject so intriguing.

One of the appealing aspects of the study of iterations of maps of the interval is that the situation is far from trivial, and yet the theory is remarkably complete. That it is far from trivial is for example clear from the 'period 3 implies chaos theorem' and from the universality found in the periodic doubling bifurcations. It is surprising therefore that, in spite of this complexity, one can prove rather general results and that many of the natural questions have now been resolved.

In this survey we will focus on smooth dynamical systems. To simplify the discussion we will assume throughout this survey that maps are *real analytic*. Many results hold for C^2 maps with non-flat critical points, but to keep the exposition non-technical we will assume throughout that f is real analytic. The aim of this survey is to describe some of the most current results, discuss some of the new ingredients introduced and also to pose questions which may lead to new directions. One very important and exciting topic which was left out of this survey is renormalisation theory. In a new edition of [dMvS93] we will describe the state of the art on this topic, in particular the most recent proofs in this area [McM96], [Lyu99] and recent work of Avila and Lyubich.

As one-dimensional dynamics can be viewed as a model for what one might hope for dynamical systems in higher dimensions, many of the questions will specifically be about the relationship between the one and higher dimensional situation.

1. DESCRIPTION OF ATTRACTORS

Take a *real analytic* map $f: [0, 1] \rightarrow [0, 1]$. It would be great to describe all orbits of this map, but it turns out to be much more fruitful to describe orbits of typical points and in particular the limit sets of these orbits. As usual denote by $\omega(x)$ the set of

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accumulation points of the sequence $x, f(x), f^2(x), \dots$. It turns out that the limit set $\omega(x)$ (the ‘attractor’) of a typical point x can only be of three types:

- (a) $\omega(x)$ is a periodic orbit (whose multiplier has absolute value ≤ 1);
- (b) $\omega(x) = \omega(c)$ where c is a critical point of f (i.e. a point so that $f'(c) = 0$), such that
 - (i) $\omega(c)$ is a Cantor set,
 - (ii) $\omega(c)$ is a *minimal* invariant set for f (i.e. the forward orbit of each point in $\omega(c)$ is dense in $\omega(c)$) and
 - (iii) $\omega(c)$ has zero Lebesgue measure;
- (c) $\omega(x)$ is equal to a finite union of intervals L which contains a critical point and so that $f: L \rightarrow L$ is *topologically transitive* (i.e. there are orbits which are dense in L).

To make this statement precise, we need to specify whether we want to consider the notion of typical from a topological or a metric point of view. In both cases, we will get that the limit sets will be of one of these types, but there is a catch, as we will see.

1.1. Theorem (Topological Attractor).

Let $f: [0, 1] \rightarrow [0, 1]$. Then there exists a set $X \subset [0, 1]$ of 2nd Baire category so that for each $x \in X$ the set $L_x = \omega(x)$ has to be of one of the following three types:

- (a) L_x is a periodic orbit;
- (b) $L_x = \omega(c)$ where c is a critical point of f with $\omega(c)$ a minimal, solenoidal set of zero Lebesgue measure;
- (c) L_x is equal to a finite union of intervals containing a critical point and f acts as a topologically transitive map on this union of intervals.

Moreover, the number of periodic attractors is finite.

That $\omega(c)$ is a *solenoidal set* means that f is *infinitely renormalizable at c* : there exists a sequence of intervals $I_n \ni c$ and integers $s(n) \rightarrow \infty$ so that $I_n, f(I_n), \dots, f^{s(n)-1}(I_n)$ have disjoint interiors and $f^{s(n)}(I_n) \subset I_n$. We should note that the notion of solenoidal used here, requires the map to be renormalizable. In fact, there are examples of non-renormalizable unimodal maps for which the map restricted to $\omega(c)$ is conjugate to an adding machine, see [BKM06]. That f is infinitely renormalizable, implies that $\Lambda := \bigcap_{n \geq 0} \bigcup_{k=0}^{s(n)-1} f^k(I_n)$ is a Cantor set (which, as it turns out, automatically has Lebesgue measure zero).

The simplest non-trivial case is when f is *unimodal*; this means that f has one critical point (an extremum). Quite often in the literature the additional assumption was made that the Schwarzian derivative $Sf(x) = f'''(x)/f'(x) - (3/2)[f''(x)/(f'(x))]^2$ is negative (whenever x is not a critical point). These maps are often called *S-unimodal*, and the quadratic map $f: [0, 1] \rightarrow [0, 1]$ defined by $f(x) = ax(1-x)$, $a \in [0, 4]$ is the simplest example of such a *S-unimodal* map. Another example is $f(x) = a \sin(\pi x)$, $a \in [0, 4]$. We will come back to the assumption about the Schwarzian derivative in Section 2, but let us mention here already that the assumption about the Schwarzian derivative can now be replaced by much simpler and more natural assumptions. When f is a quadratic map $f(x) = ax(1-x)$, then when (a) occurs the critical point $c = 1/2$ is in the immediate basin of the periodic attractor; (b) occurs when f is infinitely renormalizable; (c) occurs

when f (or the first return map of f to some suitable interval) is topologically conjugate to a tent map with slopes $\pm s$, where $s > 1$. Case (c) is often described as *sensitive dependence on initial conditions*, because it implies that there exists $\delta > 0$, so that within any interval there exists x, y so that $|f^n(x) - f^n(y)| \geq \delta$ for some $n \geq 0$.

In the case that f is a S -unimodal map, Theorem 1.1 dates back to [Guc79]. If $f: [0, 1] \rightarrow [0, 1]$ is S -unimodal with $f(0) = 0$ and $f'(0) > 1$, then for a.e. $x \in [0, 1]$ the set L_x is the same. The main difficulty in proving this theorem is in showing the *absence of wandering intervals*: if U is an interval with $U, f(U), \dots$ disjoint then U is in the basin of a periodic attractor. It is now known that real analytic maps of the interval (or circle) do not have wandering intervals. This was proved in increasing generality in [Guc79], [Yoc84], [dMvS89], [BL89], [Lyu89], and [MdMvS92]. In [MdMvS92] it was shown that f has at most a finite number of periodic attractors. The most general (and simplest proof) of the absence of wandering intervals can be found in [vSV04], but some intriguing questions remain (see Question 1.7 below).

A similar classification to Theorem 1.1 exists in terms of Lebesgue measure, but in this case there can be two types of Cantor attractors: *solenoidal and 'wild' ones*.

1.2. Theorem (Metric Attractor).

Let $f: [0, 1] \rightarrow [0, 1]$. Then there exists a set $X \subset [0, 1]$ of full Lebesgue measure so that for each $x \in X$ the set $L_x = \omega(x)$ has to be of one of the following three types:

- (a) L_x is a periodic orbit;
- (b) L_x is equal to $\omega(c)$ where c is a critical point of f with $\omega(c)$ a minimal set of zero Lebesgue measure; L_x can be of two types:
 - $L_x = \omega(c)$ where c is a critical point with $\omega(c)$ a solenoidal attractor (i.e., corresponds to an infinitely renormalizable map);
 - $L_x = \omega(c)$ where c is a critical point so that $\omega(c)$ is a minimal Cantor set which is not of solenoidal type (i.e., f acts as a topologically transitively map on a finite union of intervals containing the Cantor set $\omega(c)$, but Lebesgue almost every point in this union iterates towards $\omega(c)$; in this case we say that $\omega(c)$ is a wild attractor);
- (c) L_x is equal to a finite union of intervals which contains a critical point, and f acts as a topologically transitively map on this union.

Moreover, the number of ergodic components of f is at most equal to the number of critical points of f .

Theorem 1.2 was proved in [BL91] and later on extended in [Mar94], [She04] and [vSV04].

The previous theorem allows for the situation that L is a non-solenoidal Cantor set of the form $L = \omega(c)$ where c is a critical point (so f is *not renormalizable* at c). This situation is rather strange, because $f: L \rightarrow L$ has sensitive dependence, but at the same time, Lebesgue typical points in L are attracted to a Cantor subset of L . It turns out that the situation of a *wild attractor* can actually arise:

1.3. Theorem (Existence of wild attractors).

There exists maps of the form $f(z) = z^d + c$ with $c \in \mathbb{R}$ and d even (and large) so that

- for Lebesgue almost all points x one has that $\omega(x)$ is equal to a Cantor set $L_0 = \omega(c)$ (as in (b) above);
- there exists a set X of 2nd Baire category so that for each $x \in X$ one has that $\omega(x)$ is an interval L ;
- there exist orbits which are dense in L .

The existence of such wild attractors was first shown in [BKNvS96]. Later on, using the same methods but for a larger class in [Bru98]. In fact, similar examples also exist for polynomials of higher degree with only non-degenerate critical points. So wild attractors also occur for multimodal maps for which all critical point are non-degenerate (i.e., of order two). On the other hand,

1.4. Theorem (Non-existence of wild attractors in the quadratic case).

Assume that f is unimodal and has a quadratic critical point. If there exists a set X of positive Lebesgue measure so that for all $x \in X$, $\omega(x)$ is equal to a Cantor set C , then C is a solenoidal attractor and $C = \omega(c)$ for the critical point c .

This was proved in [Lyu94] (building on earlier work in [LM93]), see also [GSS04]. That $C = \omega(c)$ for some critical point follows already from Theorem 1.2, but that C has to be a solenoidal attractor relies on the map being unimodal and quadratic. That this holds relies on the fast decay of a certain (dynamically defined) sequence of nested intervals around c . This was shown in [Lyu94], but an elementary proof can be found in [She06]. In fact, using the similar methods one can show that if f is unimodal with a higher order critical point, then f is infinitely renormalizable at c unless f is a generalised Fibonacci map.

1.1. First return maps, induced transformations and random walks. Let us discuss some aspects of the proof of the previous theorems. It turns out that it is useful to partition the interval $[0, 1]$ into dynamically defined intervals. One could start with taking P to be the set of fixed points, and partition $[0, 1]$ into the components of $[0, 1] - P$. Take a recurrent critical point c , and let I_0 be the component of $[0, 1] - P$ which contains c . Then consider the first return map R_{I_0} to I_0 , and let I_1 be the domain of this first return map which contains c . Next consider the first return map R_{I_1} to I_1 , and let I_2 be the domain of this first return map which contains c . In this way we get a sequence of intervals $I_n \ni c$. If f is non-renormalizable at c , then the return time of R_{I_n} tends to infinity. These open intervals are *nice* in the sense that $f^k(\partial I_n) \cap I_n = \emptyset$ for all $k \geq 1$ (this notion was introduced in Martens' Delft PhD thesis, [Mar94]). That I_n is nice implies that the first return map R_{I_n} has the following property: for each component J of its domain $R_{I_n}(\partial J) \subset \partial I_n$, see Figure 1. Usually one only needs to consider those components of the domain of R_{I_n} which intersect $\omega(c)$ for some critical point c . If $\omega(c)$ is a minimal set (i.e. if all orbits in $\omega(c)$ are dense in this set), then there are only finitely many such components.

The main technical difficulty in the proof of Theorems 1.1 and 1.2 is to control the non-linearity of the branches of R_{I_n} and to estimate the length and position of I_{n+1} in relation to I_n .

This was the set-up chosen in [vSV04] to prove absence of wandering intervals (and a somewhat more precise version of Theorem 1.2). There it was proved that if c has a

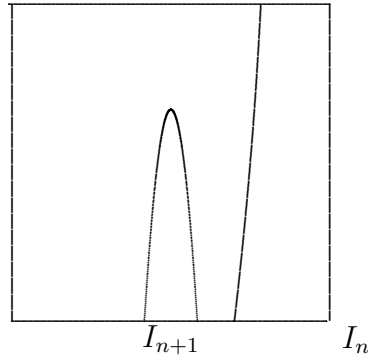


FIGURE 1. Two of the branches of the first return map R_{I_n} to I_n .

non-central return in step n , i.e. $R_{I_{n+1}}(c) \notin I_{n+1}$ then both components of $I_{n+1} - I_{n+2}$ are not small compared to I_{n+2} (that is, I_{n+2} is *well-inside* I_{n+1}). This is what is often called *real bounds*. Earlier instances of such real bounds appeared in many other papers, in particular in the work of Sullivan on renormalisation. (Note that if c is a critical point of odd order, then one has no local symmetry near c , and because of this, proving real bounds in the presence of critical points of odd order is more difficult than if all critical points have even order.) In Section 2 we will discuss the analytic distortion tools to obtain such real bounds.

The idea of the proof Theorems 1.3 and 1.4 is to associate an induced Markov transformation to f , and then consider the corresponding ‘random walk’. In one very important case, the *Fibonacci map* (which is a unimodal map topologically conjugate to a very specific tent map), the strategy can be described as follows. By using the intervals $I_n \ni c$ mentioned above one can construct intervals J_n^\pm near c (with $f(J_n^-) = f(J_n^+)$) so that $\bigcup J_n^\pm = (-1, 1)$ and so that J_{n+1}^\pm is closer to some critical point c than J_n^\pm . The idea is to do this in such a way that there exists a sequence of integer $s(n)$ and $r^-(n)$ and $r^+(n)$ so that for each n ,

$$f^{s(n)}: J_n^\pm \rightarrow \bigcup_{k \geq r^-(n)} J_k^- \cup \bigcup_{k \geq r^+(n)} J_k^+$$

is a surjective diffeomorphism.

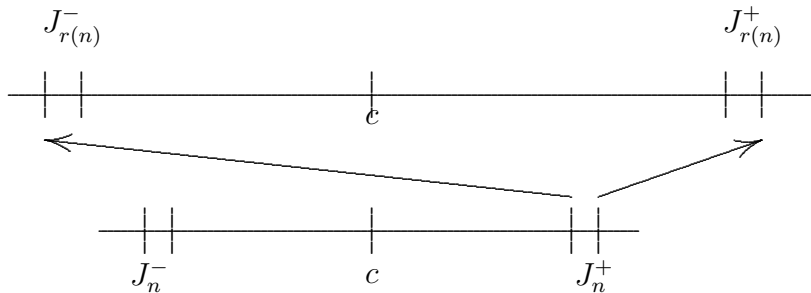


FIGURE 2. The map F on J_n^+ .

Although F is non-linear on J_n^\pm , one can consider this essentially as a random walk in the following way. The interval I can be written as $\bigcup_{n \geq 0} J_n^\pm$ and given $x \in [0, 1]$ one

can define the *state* of x as $n(x) = n$ if $x \in J_n^\pm$. Then $\Delta(x) = n(f(x)) - n(x)$ is the *drift* and tells how many states you drift. Going to a state $n(f(x)) > n(x)$ corresponds to mapping closer to the critical point, and hence ‘mapping closer to the orbit of the critical point’ whereas mapping to a state $n(f(x)) < n(x)$ means mapping away from the orbit of the critical point. To see what typical points x do, we should compute the expected value of the drift, i.e. $E(\Delta) := \int_{J_n} \Delta(x) dx$. If the drift $E(\Delta) > 0$ is positive, then you might expect points typically to move closer to $\omega(c)$ (which corresponds to the wild attractor case) and when $E(\Delta) < 0$ then the opposite holds. To make this precise one needs also to deal with the fact that F is non-linear, and this is exactly what was done in [BKNvS96].

It turns out that in the quadratic case, the lengths of the intervals $|I_n|$ shrink very fast to zero (this was proved in [Lyu94] and [GSS05], but a simpler proof of this fact was given in [She06]). Using an analogous random walk argument to the one mentioned above, and using that $|I_n|$ shrinks fast in the quadratic case, gives that the drift $E(\Delta) < 0$ and that typically points do *not* get attracted to $\omega(c)$.

On the other hand, if the map has critical points of higher order, then for certain combinatorial types the intervals I_n shrink slowly in size, $|J_n|/|J_{n+1}|$ remains close to one, and if the degree is high enough, one can show that $E(\Delta) > 0$ and wild attractors appear.

1.2. The attractor. If $f(A) \subset A$ then we define its basin as $B(A) := \{x; \omega(x) \subset A\}$. Following [Mil85a] and [Mil85b], and given the above results, it makes sense to call a closed set A a *topological attractor* (respectively a *metric attractor*) if $f(A) \subset A$ and

- $B(A)$ contains a residual subset of an open subset of $[0, 1]$ (respectively a set of positive Lebesgue measure);
- there exists no closed forward invariant set $A' \subset A$ with $A' \neq A$ for which $B(A')$ and $B(A)$ coincide up to a meager set (respectively up to a set of Lebesgue measure zero).

If A is a Cantor set, then we say it is a *Cantor attractor*. As before, a Cantor attractor A is called a *wild attractor* if it is a metric attractor and not a topological attractor.

In fact, the Hausdorff dimension of the attractor is uniformly bounded away from 1:

1.5. Theorem (Hausdorff dimension of Cantor attractors).

Assume that f has a Cantor attractor A . Then there exists a constant $\sigma < 1$ which depends only on the number and local order of the critical points of f such that the Hausdorff dimension of A is at most σ .

This was shown in [GK06] in the unimodal case (with non-degenerate critical points) and in the general case in [LS08].

On the other hand, if f has several critical points then f may have several (topological or metric) attractors. It is certainly possible that the basins of these attractors are intermingled: every open interval intersects each basin in a set of positive Lebesgue measure. For example it is possible to have a map with two metric attractors and so that each has a basin which is dense in the interval, see [vS96]:

1.6. Theorem.

There exists a polynomial $f: [0, 1] \rightarrow [0, 1]$ with two critical points with two disjoint invariant Cantor sets Λ_i whose basins both have positive Lebesgue measure, and which are intermingled (in the above sense).

We would like to make the following remark about [vS96]. The proofs in that paper for existence of real polynomials with intermingled basins in the interval rely on [BKNvS96]. In [vS96] we also asserted a similar result for the existence of some rational maps with intermingled basins in the Riemann sphere. For that part of the proof, we relied on a preprint on Julia sets of Fibonacci polynomials of high degree. The status of the result in the latter preprint (which deals with Julia sets in the complex plane) remains unclear. For this reason, the proof in [vS96] has to be modified as follows. Replacing the Fibonacci maps by the recent examples of Buff and Chéritat, see [BC06] and also [IS], we still obtain examples of rational maps with intermingled Julia basins of attractors.

1.3. Questions about attractors and wandering domains.**1.7. Question** (Absence of wandering intervals).

This question indicates a few types of maps for which one would like to know whether they can have wandering intervals.

- (a) *Take a homeomorphism $f: S^1 \rightarrow S^1$ so that f and f^{-1} are both smooth except at a finite number of points at which the map locally is of the form $x \mapsto x^\alpha$ where $\alpha > 0$. Is it possible for f to have wandering intervals?*
- (b) *Take a map $f: [0, 1] \rightarrow [0, 1]$ which is C^∞ except at $c \in (0, 1)$, so that $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Assume that near c the map f takes the form $f(x) = f(c) - |x - c|^a$ for $x < c$ and $f(x) = f(c) - |x - c|^b$ where $a, b > 0$ are not necessarily equal. Is it possible for f to have wandering intervals?*

If one can prove that a map as in (a) cannot have wandering intervals, then it is also likely that one can prove the closing lemma for flows on the torus, using the same methods as those used in [Llo09]. One needs some ‘non-flatness’ condition at the critical point (otherwise one can construct examples of maps with wandering intervals). Continuity is also important: an affine interval exchange transformation can have wandering intervals.

Having a wild attractor is related to the scaling of certain intervals. Therefore we ask the following:

1.8. Question (Hausdorff dimension of wild attractors).

Assume that L is a Cantor set which is a metric attractor and which is not of solenoidal type. Is it necessarily the case that L has positive Hausdorff dimension?

The results in this section for one-dimensional maps also prompts the following type of question.

1.9. Question (Wild attractors for two-dimensional diffeomorphisms).

Are there diffeomorphisms $f: M \rightarrow M$ where for example $M = S^2$ with attractors which are analogous to the wild attractor above? It is well-known that Hénon maps can have a Cantor set as an attractor, but in these cases the Cantor set is of solenoidal type (i.e. correspond to infinitely renormalizable maps), see [GvST89] and [DCLM05].

As Benedicks-Carleson have shown, the closure of an unstable manifold of a saddle point can be an attractor for many Hénon maps, see for example [BC91], [MV93], but it is not clear what attractors can look like in general for maps within the Hénon family. For example, is a topological attractor necessarily a metric attractor?

It is well-known that Hénon maps can have a Cantor set as an attractor, but in these cases the Cantor set is of solenoidal type, see [GvST89] and [DCLM05].

1.10. Question (The Newhouse phenomenon: Coexistence of infinitely many attractors for two-dimensional diffeomorphisms).

Newhouse showed that there exist diffeomorphisms $f: M \rightarrow M$ where for example $M = S^2$ or $M = \mathbb{R}^2$ which have infinitely many periodic attractors, see [New74], [New79] and [PT93]. Such diffeomorphism even exist arbitrarily close to the singular two-dimensional map $H_{a,b}(x, y) = (1 + y + ax^2, bx)$ with $b = 0$. On the other hand, maps of the form $H_{a,0}$ can only have at most periodic attractor. Is it possible to explain the coexistence of periodic attractors using some bifurcation analysis?

A bifurcation analysis of the bifurcations in the two-parameter family of Hénon maps, was given in for example [vS81], [HW84] and [ZRSM00]. Another very interesting question is the *pruning conjecture*, see for example [CGP88], [dC99] and [dCH01].

Another important question is:

1.11. Question (Wandering domains for diffeomorphisms).

Let H be a Hénon map. Is it possible for H to have wandering domains, i.e. is it possible that there exists an open set U so that $U, f(U), \dots$ are all disjoint and so that U is not contained in the basin of a periodic attractor?

More generally, one can ask whether there is Denjoy theory for surface diffeomorphisms. Of course there are many obstacles, see for example [Yoc80], but there are also some results see in particular [NS96], [KM09] and [Kwa09].

2. REAL BOUNDS AND DISTORTION ESTIMATES

Proving the statements in the previous section requires control on the distortion of high iterates of the map f . One way of being able to do this is to assume that the Schwarzian derivative $Sf = f'''/f' - (3/2)(f''/f')^2$ of f is negative. The reason this is a very helpful assumption is that

- (a) $Sf < 0$ implies $Sf^n < 0$ for every $n > 0$;
- (b) $Sg < 0$ implies that $x \mapsto |g'(x)|$ has no strictly positive local minima;
- (c) $Sg < 0$ implies that on each interval T for which $g|_T$ is a diffeomorphism, $g|_T$ satisfies the *Koebe Principle*. This means that for each $\xi > 0$ there exists K (which does not depend on g) so that whenever $J \subset T$ is an interval for which $g(J)$ is ξ -well-inside $g(T)$, one has

$$1/K \leq |Dg(x)|/|Dg(y)| \leq K \text{ for all } x, y \in J.$$

Here an interval J' is said to be ξ -well-inside an interval T' if both components of $T' - J'$ have at least length $\xi|J'|$. Often $g(T) - g(J)$ is referred to as the *Koebe space* around $g(J)$.

Of course the assumption that $Sf < 0$ is rather restrictive (for example, this assumption is not preserved under smooth conjugacy), and now turns out to be unnecessary. One can obtain the same kind of results by estimating distortion of cross-ratios. This approach was developed in [dMvS88], [dMvS89] (building on earlier work in [Yoc84]). Somewhat later, but independently, cross-ratios were also used in [S88]. The idea is the following: let $T \supset J$ be intervals and L, R the components of $T - J$ and define $C(T, J) = (|T||J|)/(|L||R|)$ to be their cross-ratio. Assuming that $f^n|_T$ is a diffeomorphism, one can estimate $A(f^n, T, J) = C(f^n(T), f^n(J))/C(T, J)$ provided f satisfies some smoothness conditions and one has some disjointness for $T, f(T), \dots, f^{n-1}(T)$. Using estimates on the cross-ratio distortion $A(f^n, T, J)$ one can prove the same results as for maps with negative Schwarzian derivative. For more on this, see [dMvS93].

In fact, it turns out that the following two results (which were proved in [vSV04], see also [vSV07]) are sufficient in most proofs which require estimates on the distortion of high iterates. Here, as usual, an open interval I is called *nice* if $f^n(\partial I) \notin I$ for all $n \geq 1$. Moreover, $\mathcal{L}_x(I)$ is defined to be the component of the first entry map to I containing x .

2.1. Theorem (Koebe and negative Schwarzian).

The following properties hold for each real analytic map f .

- (a) [Koebe Principle] *For each $S > 0$, $\delta > 0$ and $\xi > 0$ there exists $K > 0$ such that if $J \subset T$ are intervals, with $f|_T$ a diffeomorphism, $f^n(J)$ ξ -well-inside $f^n(T)$ and either (i) $\sum_{i=0}^{n-1} |f^i(J)| \leq S$ or (ii) $f^n(T) \cap B_0(f) = \emptyset$ and $\text{dist}(f^i(T), \text{Par}) \geq \delta$, $i = 0, \dots, n-1$, then $f|_J^n$ has bounded distortion, i.e. for any $x, y \in J$,*

$$(1) \quad |Df^n(x)|/|Df^n(y)| \leq K.$$

Here $B_0(f)$ is the union of immediate basins of periodic (possibly parabolic) attractors and Par is the set of parabolic periodic points of f .

- (b) [Negative Schwarzian Derivative] *For each critical point c_i which is not in the basin of a periodic attractor, there exists a neighbourhood U_i so that whenever $f^n(x) \in U_i$ for some x and some $n \geq 0$, then the Schwarzian derivative of f^{n+1} at x is negative:*

$$(2) \quad Sf^{n+1}(x) < 0.$$

Clearly the above distortion result, and in particular (1) is extremely useful. Equation (??) implies in particular that there exists a neighbourhood U_i of each critical point c_i , so that if an attracting periodic point p_i is contained in U_i then c_i is in the immediate basin of p_i . The assumption that $f^n(J)$ is ξ -well-inside $f^n(T)$ is often described as having ‘Koebe space’.

Note that if f has no parabolic periodic orbits, then assumption (ii) in the 1st part of Theorem 2.1 is trivially satisfied. The 2nd part of Theorem 2.1 generalizes [Koz00] to the multimodal case. In [GSS01] and [GSS09] it was shown that if f has no parabolic periodic orbits, then f is actually smoothly conjugate to a map with negative Schwarzian derivative.

In order to obtain Koebe space we often need to consider pullbacks of a pair of intervals. This means that we take intervals $T \supset J$ and consider suitable component

of $f^{-k}(T)$ and of $f^{-k}(J)$. Sometimes one says that T_0, \dots, T_s is a *pullback* of T_s if T_i is a component of $f^{-1}(T_{i+1})$ for each $i = 0, 1, \dots, s-1$. In such situations the following theorem is often appropriate (the 2nd part is useful if one needs ‘big space’).

2.2. Theorem (Macroscopic Koebe Principle).

One has the following properties:

- (a) *For each $\xi > 0$, there exists $\xi' > 0$ such that if T is a nice interval, J is a nice interval which is ξ -well-inside T and $x \in T$ and $f^k(x) \in J$ (with $k \geq 1$ not necessarily minimal), then*

the component containing x of $f^{-k}(J)$ is ξ' -well-inside $\mathcal{L}_x(T)$.

- (b) *For each $\xi > 0$ there exists $\xi' > 0$ so that if $J_s \subset T_s$ is ξ -well-inside T_s and $J_s \supset \mathcal{L}_x(T_s)$ for some $x \in J_s$, the following holds. Let $J_i \subset T_i$ be pullbacks of $J_s \subset T_s$. Then J_0 is ξ' -well-inside T_0 . Here $\xi'(\xi) \rightarrow \infty$ when $\xi \rightarrow \infty$.*

The first part of Theorem 2.2 allows one to pullback space and is proved in [vSV04]. So if one finds that one interval is well-inside another one, then one can spread this information. The 2nd part allows one to spread large space as well and is due to Shen, see [She03a] (see also Theorem B in [vSV04]).

2.3. Question.

The results in this section hold for C^3 maps with non-flat critical points, and even for maps with some less smoothness, see [ST05]. However, it is not clear whether the above theorems hold for C^2 maps with non-flat critical points.

3. DENSITY OF HYPERBOLICITY

From Section 1 it is clear that by far the simplest situation is when the attractors of f are all hyperbolic periodic orbits. In this case the map is called hyperbolic. In other words, a map $f: [0, 1] \rightarrow [0, 1]$ is called *hyperbolic* if Lebesgue a.e. point is attracted to a periodic orbit.

By a result by Mañé [Mañ85] (for a simpler proof see also [vS90]), this is equivalent to the following: (i) each critical point of f is in the basin of a periodic attractor and (ii) each periodic orbit is hyperbolic (multiplier is not equal to ± 1).

From the results of the previous section, this is also equivalent to the classical definition: the interval is the union of a repelling hyperbolic set, the basin of hyperbolic attracting periodic points and the basin of infinity.

It would be nice if every map can be approximated by a hyperbolic map. This problem goes back in some form to

- Fatou, who stated this as a conjecture in the 1920’s, see [Fat20, page 73] and also [McM94, Section 4.1].
- Smale gave this problem ‘naively’ as a thesis problem in the 1960’s, see [Sma80].
- Jakobson proved that the set of hyperbolic maps is dense in the C^1 topology, see [Jak71];
- The quadratic case $x \mapsto ax(1-x)$ was proved in a major breakthrough in the mid 90’s independently by Lyubich [Lyu97] and also Graczyk and Swiatek [GŚ97] and [GŚ98].

- Blokh and Misiurewicz [BM00] proved a partial result towards the density of hyperbolic maps in the C^2 topology.
- Shen [She04] then proved the C^2 density of hyperbolic maps.

The general result was proved recently, see [KSvS07a]:

3.1. Theorem (Density of hyperbolicity for real polynomials).

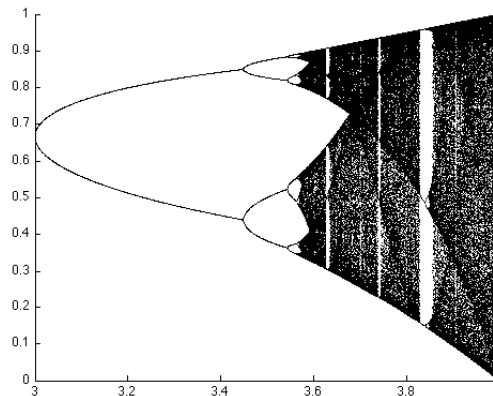
Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.

The above theorem allows us to prove the analogue of the Fatou conjecture in the smooth case, see [KSvS07b], solving the 2nd part of Smale's eleventh problem for the 21st century [Sma00]:

3.2. Theorem (Density of hyperbolicity for smooth one-dimensional maps).

Hyperbolic maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \dots, \infty, \omega$.

For quadratic maps $f_a = ax(1 - x)$, the above theorems assert that the periodic windows are dense in the bifurcation diagram. The quadratic case turns out to be special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$.



Note that every hyperbolic map satisfying the mild transversality condition that critical points are not eventually mapped onto other critical points, is structurally stable.

3.1. Hyperbolicity is dense within generic one-parameter families of one-dimensional maps.

3.3. Theorem (Hyperbolicity is dense within generic families, and so only exceptional families are robustly chaotic).

Near any one-parameter family of smooth interval maps there exists a one-parameter family $\{f_t\}$ of smooth interval maps for which

- *the number of critical points of each of the maps f_t is bounded;*

- *the set of parameters t for which all critical points of f_t are in basins of periodic attractors, is open dense.*

In particular, $\{f_t\}$ is not robustly chaotic.

Here, following [BYG98], a family of maps $\{f_t\}_{t \in [0,1]}$ is said to be robustly chaotic if there exists no parameter $t \in [0,1]$ for which the map f_t has a periodic attractor. The proof of this result follows easily from the theorems in the previous subsection, see [vS09a].

3.2. Connection with the closing lemma.

3.4. Theorem (Pugh's C^1 Closing Lemma).

Let x be a recurrent point of a smooth diffeomorphism f on a compact manifold. Then there exists a smooth diffeomorphism g which is C^1 close to f for which x is periodic.

For the last 30 years any attempt to prove the C^k version of this result has been unsuccessful. However, one of the consequences of our result is the one-dimensional version:

3.5. Theorem (One-dimensional C^∞ Closing Lemma).

Let x be a recurrent point of a C^∞ interval map f . Then there exists a smooth map g which is arbitrarily C^∞ close to f for which x is periodic.

3.3. Comments on the strategy of proof: local and global perturbations. Density of hyperbolicity means that given a map f one can find a g so that g is hyperbolic and so that $g - f$ is 'small' in the C^k topology. It is tempting to add a small 'bump function' h to f , and to consider $g = f + h$. The aim would then be to show that g has a different dynamics from that of f , but in a way for which one has control. The challenge with this approach is that orbits will pass many times through the support of this bump function, and so the dynamics of f and g differs in a way which is hard to control. To ensure that only 'a few points' in the orbit intersect the support of the bump function, one needs to take this set small, but this implies that the norm $|h|$ is also going to be small (how small depends on the degree of differentiability k). Because of these difficulties, it is clear that this approach is unlikely to give the C^k density of hyperbolicity unless k is fairly small. This approach was used successfully in [Jak71] for the C^1 topology, and in [BM00] for the C^2 topology, but with added assumptions on the dynamics of f . In [She04], this approach was used to prove C^2 density, but Shen considered two cases separately: when one has a 'lot of Koebe space' the above approach was used; while for the other cases, he showed that one has 'essentially bounded geometry' and for these cases a form of rigidity was proved (we will discuss the latter approach below).

In [BM09] it is shown that a unimodal case can be perturbed by an arbitrarily C^2 -small perturbation which is localized in an arbitrarily small neighbourhood of one point. On the other hand, an example due to Gutierrez [Gut87] about flows on a punctured torus suggests that in general one might have to consider global perturbations in order to prove density of hyperbolicity.

Another approach to density of hyperbolicity is to show that each non-hyperbolic map is more or less unique, in the following sense: two non-hyperbolic maps which are topologically conjugate, are ‘essentially identical’. In the quadratic case, essentially identical is understood as to be equal ‘up to an affine coordinate change’, and then this statement is equivalent to density of hyperbolicity. In the cubic case, we cannot hope for such uniqueness, because a cubic map may have one critical point which is recurrent and the other may be in the basin of a periodic attractor. In this case, one expects a curve in parameter space of topologically conjugate maps. In the next section we will discuss this rigidity in more detail, and discuss how it is related to density of hyperbolicity.

3.6. Question.

Consider the space \mathcal{A} of real analytic d -modal maps $f: [0, 1] \rightarrow [0, 1]$. Pick $f_0 \in \mathcal{A}$. Does the set of maps $f \in \mathcal{A}$ which are topologically conjugate to f_0 form a real analytic manifold with a finite number of components (and a finite number of singular points)?

If we replace the space \mathcal{A} by the space of real polynomials then the results described in the next section imply that the answer to this question is affirmative, and that the dimension of this manifold is equal to the number of periodic attractors of f_0 . This follows from quasi-symmetric rigidity, see the next section, and arguments based on the Measurable Riemann Mapping Theorem, see [MSS83].

In the unimodal case (i.e. when $d = 1$), the above question was partly answered in [Koz03], [ALdM03], see also [Cla09]. In this case it is also known that the holonomy along the local lamination defined by these real analytic manifolds is not very regular near manifolds corresponding to maps with a parabolic periodic orbit (multiplier equal to one), see [ALdM03]. This question is related to the issues discussed in Sections 9 and 10.

Of course, it is well known that density of hyperbolicity is false in dimension ≥ 2 . Also, it is not clear whether the C^2 closing lemma holds. An interesting list of questions which may focus the approach in the higher dimensional case can be found in [Puj08].

On the other hand, the situation for rational maps on the Riemann sphere is likely to be different. In that context one has the following well-known conjecture (which goes back to Fatou):

3.7. Conjecture (Density of hyperbolicity for rational maps).

Consider the space of rational maps of degree d on the Riemann sphere. Are hyperbolic maps dense within this space?

As was shown in [MSS83], this conjecture is implied by the following

3.8. Conjecture.

If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.

More about this conjecture and related results can be found in [McM94]. In [She03b], [KSvS07a] and finally [KvS09] it was shown that real polynomials (acting on \mathbb{C}) do not carry such invariant line fields. Moreover, real polynomials have Julia sets which are locally connected, see [LvS98] and [KSvS07a] and [KvS09].

Interestingly, any rational map on the Riemann sphere such that the multiplier of each periodic orbit is real, either has a Julia set which is contained in a circle or is a Lattès map, see [EvS08].

4. QUASI-SYMMETRIC RIGIDITY

As mentioned the strategy of the existing proofs of density of hyperbolicity is related to quasi-symmetric rigidity. The most general form can be found in [vS09b], and states:

4.1. Theorem (Quasi-symmetric rigidity).

Assume that $f, g: [0, 1] \rightarrow [0, 1]$ are real analytic and topologically conjugate. Alternatively, assume that $f, g: S^1 \rightarrow S^1$ are topologically conjugate and that f and g each have at least one critical point or at least one periodic point. Moreover, assume that the topologically conjugacy is a bijection between

- the set of critical points and the order of corresponding critical points is the same;
- the set of parabolic periodic points.

Then the conjugacy between f and g is quasi-symmetric.

Here, as usual, a homeomorphism $h: [0, 1] \rightarrow [0, 1]$ is called *quasi-symmetric* (often abbreviated as *qs*) if there exists $K < \infty$ so that

$$\frac{1}{K} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq K$$

for all $x-t, x, x+t \in [0, 1]$. The analogous definition holds for a homeomorphism $h: S^1 \rightarrow S^1$.

We should note that a real analytic interval or circle map can only have finitely many parabolic periodic orbits (see [MdMvS92] or Theorem IV.B in [dMvS93]).

4.1. Why is quasi-symmetric rigidity important? Quasi-symmetric rigidity is a crucial step towards proving the following types of results:

- (a) hyperbolicity is dense, for a discussion see subsection 4.4.
- (b) within certain families of maps, conjugacy classes are connected, see Section 10;
- (c) monotonicity of entropy; for families such as $[0, 1] \ni x \mapsto a \sin(\pi x)$, see Section 10;

see for example [KSvS07a] and [BvS09] for corresponding results for polynomial maps. Theorem 4.1 is part of a joint project with Rempe about density of hyperbolicity of real transcendental maps, see [RvS08] and [RvS10], and builds on earlier joint work with Levin [LvS00], and especially on [KSvS07a] and [KSvS07b].

4.2. Necessity of the assumptions in the quasi-symmetric rigidity Theorem 4.1.

All the assumptions in the theorem are necessary to have a qs conjugacy. In fact they are even necessary for the maps to be Hölder conjugate:

(i) To see that we cannot drop the assumption that the critical points should have the same order, consider Fibonacci maps $f(z) = z^2 + c$ and $g(z) = z^a + c'$ with $a > 2$. Iterates of the critical point of f and g accumulate to 0 at different rates. Indeed, let S_n be the Fibonacci sequence $1, 2, 3, 5, 8, \dots$, then $f^{S_n}(0)$ converges to 0 at a geometric rate, and $g^{S_n}(0)$ at a polynomial rate, see [BKNvS96]. So f and g are topologically conjugate, but not qs conjugate.

(ii) Critical points of *odd order* are *invisible* from a ‘real’ point of view. Yet, these can have rather important consequences for the rate of recurrence. For example, assume that f and g are degree 2 covering maps (one with an odd critical point, and the other one expanding) without periodic attractors or parabolic periodic points. Then f and g are topologically conjugate (because neither of them has wandering intervals). However, these map certainly do not need to be qs conjugate: one can construct the map with a critical point so that it has longer and longer saddle-cascades (see the remarks at the end of this section).

(iii) It is necessary to assume that the topological conjugacy maps parabolic points map to parabolic points, because the local escape rates near a hyperbolic and a parabolic periodic point are completely different (one escapes at a geometric rate and the other polynomially).

4.3. Previous quasi-symmetric rigidity results. In some special cases, a version of Theorem 4.1 was proved before. However, even in the polynomial case the result is new. All previous results in that case still require that there are no parabolic periodic points and no critical points of odd order. Indeed, the most general result which is known about qs-rigidity is then the following:

4.2. Theorem (Quasi-symmetric rigidity for real polynomials with only real critical points).

Let f and g be real polynomials of degree d with only real critical points, which are all even order and without parabolic periodic orbits. If f and g are topologically conjugate (as dynamical systems acting on the real line) and corresponding critical points have the same order, then they are quasi-symmetrically conjugate. (In fact, they are quasi-conformally conjugate on the complex plane. For the definition of the notion of quasi-conformal homeomorphism, see Subsection 4.4.)

This theorem was proved in [KSvS07a]. Prior to that paper, Lyubich [Lyu97] and Graczyk & Świątek [GŚ98] proved this result for real quadratic maps. As we will see their method of proof in the quadratic case does not work if the degree of the map is > 2 .

For real analytic maps which are not of this type, the available previous results only assert the existence of a qs homeomorphism which is a conjugacy restricted to $\omega(c)$. One such result is due to Shen who proved in [She04, Theorem 2, page 345] the following. Let f, g be real analytic topologically conjugate maps with only hyperbolic repelling periodic points, non-degenerate critical points and with *essentially bounded geometry*. Then there exists a qs homeomorphism which is a conjugacy restricted to $\omega(c)$.

For real analytic circle homeomorphisms with critical points a stronger result is known, which is due to Khanin and Teplinsky [KT07]:

4.3. Theorem (C^1 rigidity for critical circle maps).

Any two analytic critical circle homeomorphisms with the same irrational rotation number and the same order of the critical points are C^1 -smoothly conjugate.

Here the presence of critical points is necessary, because for circle diffeomorphisms the analogous statement is false. Indeed, otherwise one can construct maps for which

some sequence of iterates has *almost a saddle-node fixed point*, resulting in larger and larger passing times near these points. This phenomenon is also referred to as *a sequence of saddle-cascades*. It was used by Arnol'd and Herman to construct examples of diffeomorphisms of the circle which are conjugate to irrational rotations, but where the conjugacy is absolutely continuous, nor qs and for which the map has no σ -finite measures, see for example Section I.5 in [dMvS93]. In the diffeomorphic case, to get qs or C^1 one needs assumptions on the rotation number (to avoid these sequences of longer and longer saddle-cascades). We should note that Theorem 4.3 builds on earlier work of de Faria, de Melo and Yampolsky on renormalisation.

In general, one cannot expect C^1 , because having a C^1 conjugacy implies that corresponding periodic orbits have the same multiplier.

For the case of covering maps of the circle the following theorem is known, see [LvS00]

4.4. Theorem (Covering maps of the circle).

Assume that $f, g: S^1 \rightarrow S^1$ are topologically conjugate real analytic covering maps of the circle (positively oriented) both with exactly one critical point (of odd order), so that the orders of these critical points are equal and so that the conjugacy maps the critical point c of f to the critical point \tilde{c} of g , then there exists a qs homeomorphism $h: S^1 \rightarrow S^1$ such that $h(f^k(c)) = g^k(h(c))$ for each $k \geq 0$ and so that $h(c) = \tilde{c}$, provided

- $\omega(c)$ is minimal, or
- $\omega(c)$ is non-minimal and f, g have only repelling periodic orbits.

Note that in Theorem 4.1 we ask the maps f and g to have the same number of critical points. Therefore the set-up is different than in the following theorem by Carsten Petersen [Pet07], which extends weaker results in [dMvS93, Exercise 6.2] and also in the Cornell thesis of L.K.H. Ma (1994):

4.5. Theorem (A linearisation result).

Assume that $f: S^1 \rightarrow S^1$ is a degree $d \geq 2$ map which is a restriction of a Blaschke product, then f is qs conjugate to $z \mapsto z^d$ if

- $\omega(c) \cap S^1 = \emptyset$ for each recurrent critical point, and
- any periodic point in S^1 is repelling.

The assumption that $\omega(c) \cap S^1 = \emptyset$ for any recurrent critical point c ensures that f has no ‘almost saddle-nodes’ on the circle. Perhaps one could replace this assumption by a kind of ‘diophantine’ condition on the itinerary of the critical points of f (so that one does not need to have a globally defined rational map f in this theorem). But in any case, as was also remarked in [LvS00], in general a real analytic covering map of the circle of degree d (with critical points) is *not* qs conjugate to $z \mapsto z^d$ (on the circle $\{z \in \mathbb{C}; |z| = 1\}$). To see this, we can use Theorem C in [LvS00] to see that within any family, near every map without periodic attractors, there exists another map with a parabolic periodic point. Then use the argument given in Section I.5 in [dMvS93].

4.6. Question.

Let $f: S^1 \rightarrow S^1$ be a real analytic covering map. Are there Diophantine condition on

the itinerary of the critical points of f , so that the conclusion of Theorem 4.5 holds even without the condition that $\omega(c) \cap S^1 = \emptyset$?

We should also remark that there are also analogues of these theorems for polynomials in \mathbb{C} , but then one must assume that f is only finitely renormalizable, see for example [KvS09].

4.4. Why is quasi-symmetry relevant to density of hyperbolicity? Quasi-symmetric rigidity turns out to be helpful in proving density of hyperbolicity, because of its connections with quasi-conformal homeomorphisms. To illustrate this, let us consider the following situation. Take a family of real quadratic maps $f_c(z) = z^2 + c$ and consider the set $I(f)$ of real parameters \tilde{c} for which $f_{\tilde{c}}(z) = z^2 + \tilde{c}$ is topologically conjugate to f_c . Assume that we know that for any $c, \tilde{c} \in I(f)$, $f_c, f_{\tilde{c}}$ are quasi-symmetrically conjugate on the real line. Let us outline a famous argument due to Dennis Sullivan which shows that this implies that I is an open interval (if $c \neq \tilde{c}$).

First we use the pullback argument which goes as follows. It is well-known that one can extend a quasi-symmetric homeomorphism on the real line to a *quasi-conformal homeomorphism* H on the complex plane. One definition for this property is that there exists a constant $K < \infty$ such that for Lebesgue almost all $x \in \mathbb{C}$,

$$\limsup_{r \rightarrow 0} \frac{\sup_{|y-x|=r} |H(y) - H(x)|}{\inf_{|y-x|=r} |H(y) - H(x)|} < K.$$

So a circle centered at x is mapped to a curve which fits between two circles both centered at $H(x)$ and of comparable diameter. This extension H is not necessarily a conjugacy between $f, \tilde{f}: \mathbb{C} \rightarrow \mathbb{C}$ but it is a conjugacy between $f, \tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$, i.e. $\tilde{f} \circ H = H \circ f$ on \mathbb{R} . But once one has such a H one can define a sequence of quasi-conformal homeomorphisms H_{n+1} by taking a quasi-conformal extension $H_0: \mathbb{C} \rightarrow \mathbb{C}$ of H and define $H_{n+1}: \mathbb{C} \rightarrow \mathbb{C}$ so that $\tilde{f} \circ H_{n+1} = H_n \circ f$. To see that H_{n+1} exists, one checks inductively in the construction that the critical values of f are mapped by H_n onto the critical values of \tilde{f} (here we use that $\tilde{f} \circ H = H \circ f$ on \mathbb{R}). Since f and \tilde{f} are both conformal, each of the maps H_n is K -quasi-conformal (where K does not depend on n). Now it is well-known that the space of K -quasi-conformal homeomorphisms is compact, so there exists a subsequence H_{n_i} which converges to some K -quasi-conformal homeomorphism H_* . By choosing H so that it is near infinity a conformal conjugacy between f and \tilde{f} , it is not hard to ensure that H_n in fact converges to H_* (in the uniform topology on $\bar{\mathbb{C}}$) and therefore that $\tilde{f} \circ H_* = H_* \circ f$. This is the first step of the argument, and usually referred to as the *pullback argument*.

The second step shows that $I(f)$ is open and relies on the Measurable Riemann Mapping Theorem. This says that one can deform the quasi-conformal homeomorphism H_* (which can be assumed to be normalized so that $H_*(0) = 0$, $H_*(\infty) = \infty$ and has the property that $\tilde{f} = H_* \circ f \circ H_*^{-1}$), in such a way that for the deformation H_t

- $H_0 = id$, $H_1 = H_*$;
- $z \mapsto H_t \circ f \circ H_t^{-1}(z)$ is a conformal map and
- $t \mapsto H_t \circ f \circ H_t^{-1}$ is complex analytic on a neighbourhood of $\{t \in \mathbb{C}; |t| \leq 1\}$.

It follows that $H_t \circ f \circ H_t^{-1}$ is equal to a map of the form $z \mapsto z^2 + c(t)$ where $t \mapsto c(t)$ is complex analytic on a neighbourhood of $\{t \in \mathbb{C}; |t| \leq 1\}$, with $c(0) = c$ and $c(1) = \tilde{c}$. Since one can arrange it so that H is symmetric w.r.t. the real line, $c(t)$ is real when t is real. Since this holds not only for all $t \in [0, 1]$ but even for all t in a neighbourhood $[0, 1]$, it follows that the range of $t \mapsto c(t)$ contains a neighbourhood of $[c, \tilde{c}]$. In particular, $I(f)$ is an open, connected set.

The third step is to show that all this implies density of hyperbolicity. So assume that $f(z) = z^2 + c$ is non-hyperbolic. Using simple topological considerations it is easy to show that this implies that $I(f)$ is a closed set. But since, as we have shown $I(f)$ is also open, it follows that $I(f)$ is the real line. But this is impossible because then all maps $z \mapsto z^2 + c$ would be topologically conjugate.

This argument does not go through for real polynomial maps with more than one (real) critical point, but one can show nevertheless - by somewhat related arguments - that quasi-symmetric rigidity implies density of hyperbolicity, see [KSvS07a, Section 2]. If one deals with real analytic maps then the argument to prove density of hyperbolicity is more subtle, see [KSvS07b].

Note that in deriving density of hyperbolicity from quasi-symmetric rigidity, we have used a complex extension of the interval maps. Is using complex methods really necessary?

4.7. Question.

Assume that \mathcal{A} is a space of C^∞ interval maps for which one has quasi-symmetric rigidity (as in the conclusion of Theorem 4.1). Are hyperbolic maps then dense in \mathcal{A} in the C^∞ topology?

Also, are there other (weaker) types of rigidity which imply density of hyperbolicity:

4.8. Question.

Assume that \mathcal{A} is a space of C^∞ interval maps for which one has Hölder rigidity (topologically conjugate maps are Hölder conjugate). Are hyperbolic maps then dense in \mathcal{A} in the C^∞ topology?

In a similar vein one can ask:

4.9. Question.

Assume that $f_0, f_1: [0, 1] \rightarrow [0, 1]$ are real analytic unimodal maps. Assume that they are quasi-symmetrically conjugate. Does there exist a path f_t connecting f_0 and f_1 of real analytic unimodal maps so that f_t is quasi-symmetrically conjugate to f_0 for all $t \in [0, 1]$?

4.5. How to prove quasi-symmetric rigidity? In this subsection we shall give some reasons why proving quasi-symmetric rigidity may involve extensions to the complex plane. As mentioned, a quasi-symmetric homeomorphism on the real line is always the restriction of a quasi-conformal homeomorphism on the complex plane. It may be even rather convenient to show that a homeomorphism on the real line is quasi-symmetric, by constructing its quasi-conformal extension by successively partitioning the complex plane in finer and finer polygonal pieces.

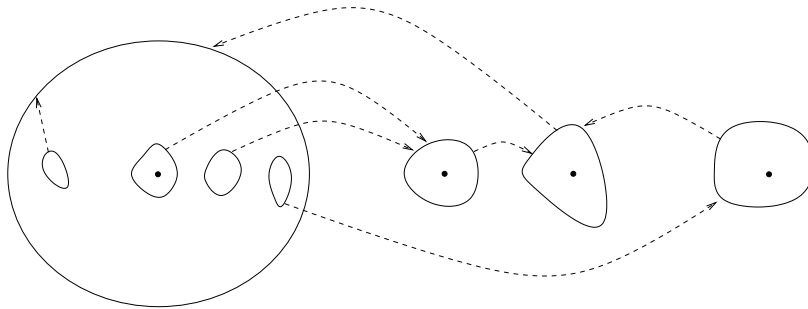


FIGURE 3. A box mapping.

In fact, if the interval maps extend to conformal maps on a neighbourhood of the real line, then one can partially define a quasi-conformal conjugacy near critical points, and then spread the definition to the whole complex plane fairly easily. This method was called the *spreading principle* in [KSvS07a].

The main problem then is to prove that first return maps R_{I_n} as in Subsection 1.1 are quasi-conformally rigid. Assuming the map is non-renormalizable, this can be done by proving:

- (a) R_{I_0} has an extension to a ‘complex box mapping’, see Figure 3 in the multimodal case. Roughly speaking, this is a map $F: U \rightarrow V$ so that each component of U is mapped onto a component of V , and components of U are either compactly contained in a component of V or they are equal to such a component. Components of $F^{-n}(V)$ are called *puzzle pieces*. We also require (roughly speaking) that F is univalent near the boundary of U (slightly more precisely, that there exists an annulus neighbourhood A of ∂V so that $F^{-1}|_A$ is univalent on each component of A and so that $\text{mod}(A)$ is universally bounded from below). The existence of such a map $F: U \rightarrow V$ with the additional property is usually referred to as having **complex bounds**.
- (b) One can then define a sequence of puzzle pieces U_{n_i} called the **enhanced nest**, so that there exists k_i for which $F^{k(i)}: U_{n_{i+1}} \rightarrow U_{n(i)}$ is a branched covering map with degree bounded by some universal number N . This enhanced nest is chosen so that it transfers rather efficiently small scale to large scale, but so that the degree of $F^{k(i)}: U_{n_{i+1}} \rightarrow U_{n(i)}$ remains universally bounded. This enhanced nest was one of the main new ingredients in [KSvS07a] and is used for example in [KvS09], [QY06], [TY09], [PQR⁺09].
- (c) To show that one still has *complex bounds* for the enhanced nest we used ‘bare-hand’ methods in [KSvS07a], but in [KvS09] this was proved in a simpler way using a remarkable new tool due to Kahn and Lyubich, see [KL09]. This tool is about pulling back a thin annulus, and shows that the modulus of the pullback of this annulus is much better than one might expect. In the real case, one can simplify the statement and proof of Kahn and Lyubich’s result as follows, see [KvS09, Lemma 9.1]:

4.10. Lemma (Small Distortion of Thin Annuli).

For every $K \in (0, 1)$ there exists $\kappa > 0$ such that if $A \subset U$, $B \subset V$ are simply

connected domains symmetric with respect to the real line, $F : U \rightarrow V$ is a real holomorphic branched covering map of degree D with all critical points real which can be decomposed as a composition of maps $F = f_1 \circ \dots \circ f_n$ with all maps f_i real and either real univalent or real branched covering maps with just one critical point, the domain A is a connected component of $f^{-1}(B)$ symmetric with respect to the real line and the degree of $F|_A$ is d , then

$$\text{mod}(U - A) \geq \frac{K^D}{2d} \min\{\kappa, \text{mod}(V - B)\}.$$

To prove complex bounds at many levels, one needs to pullback annuli. Because one passes through the critical point, the moduli of these annuli may deteriorate. Using a combinatorial argument, and the above distortion tool, one can show that the annuli cannot get too thin thus giving complex bounds.

- (d) Because of the spreading principle mentioned above, to construct a quasi-conformal conjugacy it then suffices to construct a partial-conjugacy on a puzzle pieces which is ‘natural on the boundary’. Given the above, this can easily be done using the **QC-criterion** from the appendix of [KSvS07a] and bounded geometry of the puzzle pieces. These are very easy to derive from the complex bounds, see [KvS09, Section 10]. One can also proceed as in [AKLS09].

It is of course conceivable that one can prove quasi-symmetric rigidity using entirely real methods. This hinges on questions of the following type:

4.11. Question.

Consider the space \mathcal{A} of maps of the form $z \mapsto |z|^d + c$ where $d > 1$ is not necessarily an integer and where c is real. Does one have quasi-symmetric rigidity for maps within the space \mathcal{A} ?

One of the difficulties with such a real approach is that it is not so easy to know how to use the information that the exponent d is fixed within the family \mathcal{A} : the exponent is not ‘visible’ in the real line. On the other hand, if d is an even integer, and $z \mapsto z^d + c$, then of course the local degree of the map at 0 is different for different values of d . Without imposing a condition on the degree the answer to the question above is definitely negative, see the examples in Subsection 4.2.

5. LEBESGUE TYPICAL MAPS ARE NOT HYPERBOLIC: COLLET-ECKMAN MAPS

The set of hyperbolic polynomials does **not** have full measure within the space of all polynomials. This is a consequence of a theorem of Jakobson from the early 1980’s, see [Jak81]. This theorem states that

5.1. Theorem (Jakobson).

The set of parameters a for which $f_a(x) = ax(1 - x)$ has an absolutely continuous invariant probability measure has positive Lebesgue measure.

In fact,

5.2. Theorem (Benedicks & Carleson).

There exists a set A of positive Lebesgue measure so that for each $a \in A$ the map

$f_a(x) = ax(1-x)$ has the following property:

$$(3) \quad |Df^n(c)| \geq C\lambda^n$$

for some $C > 0$ and $\lambda > 1$ and for all $n \geq 0$.

If (3) holds, then one says that f is a *Collet-Eckman map*. It implies that one has an absolutely continuous invariant probability measure, see Section 6. The above theorem was first proved in [BC91], but several other proofs were given since, see for example Section V.6 in [dMvS93], but also [Luz00]. A similar result holds for multimodal maps: Tsujii, see [Tsu93] and more recently Wang & Young, see [WY06] and also [WY08].

An upshot of the previous two results is that for a randomly chosen coefficient, a real polynomial f is *not* hyperbolic.

In Section 9 we shall discuss remarkable results which describe the dynamics for almost all parameters, and not just for a set of positive Lebesgue measure.

5.1. Topological invariance of the Collet-Eckman condition. It turns out that the Collet-Eckman condition is a topological invariant in the unimodal case, see [NP98], and is also equivalent to uniform hyperbolicity on periodic orbits, see [Now88] and also [KN92].

5.3. Theorem.

Assume that f, g are unimodal maps which are topologically conjugate. Then f satisfies the Collet-Eckman condition if and only if g satisfies the Collet-Eckman condition. In this case, there exists $\lambda > 1$, $C > 0$ so that each periodic point p of f of period n , one has $|Df^n(p)| \geq C\lambda^n$.

The analogous result is not true in the multimodal case (see for example [BvS03]). On the other hand, in the complex setting the situation is much better: there is a topological analogue of the Collet Eckman condition (the Topological Collet-Eckman condition) which is equivalent to a backward version of the Collet-Eckman condition, see [PRLS03].

In the multimodal case, the following holds, see [PRL07]:

5.4. Theorem.

The following are equivalent:

- (a) f admits an absolutely continuous invariant probability measure which has exponential decay of correlations.
- (b) f has uniform hyperbolicity on its set of periodic orbits (i.e. there exists $C > 0$ and λ so that whenever p is a fixed point of f^n then $|Df^n(p)| \geq C\lambda^n$).
- (c) f satisfies the topological Collet-Eckman condition;
- (d) f has exponentially shrinking intervals of monotonicity;

Moreover, Collet-Eckmann and ‘slow recurrence’ is also a topological invariant, see [LW06].

6. EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT PROBABILITY MEASURES

Even if a map has no periodic attractors, one might still hope for a good statistical description of its orbits. For this reason it is natural to consider maps f which have an *absolutely continuous invariant probability measure*. We say that a probability measure μ is invariant if $\mu(f^{-1}(A)) = \mu(A)$ whenever A is a Borel measurable set, and we say that μ is absolutely continuous whenever $\mu(A) = 0$ for any set A of zero Lebesgue measure. (We should point out that because of the real bounds in [vSV04] such a measure automatically has positive metric entropy, see [dMvS93, Exercise 1.4].)

In the late 1970's, Misiurewicz [Mis81] proved that an absolutely continuous invariant probability measure exists for any S-multimodal map which has only repelling periodic orbits and which has the property that all iterates of its critical points stay outside a neighbourhood of the set of critical points. In the 1980's, Collet-Eckman, see [CE83], weakened this assumption and showed a S-unimodal map f satisfying the following condition (the *Collet-Eckmann* condition) has an absolutely continuous invariant probability measure:

$$(CE) \quad \liminf_{n \rightarrow \infty} \frac{\log |(f^n)'(f(c))|}{n} > 0,$$

where c denotes the critical point of f . In fact, in [CE83] another, additional assumption was made on the expansion along the backward orbit of critical points, but Nowicki showed that (CE) implies this other condition.

In the mid 1990's, together with Nowicki we improved this, see [NvS91], by showing that the following summability condition guarantees the existence of an absolutely continuous invariant probability measure for an S-unimodal map:

$$\sum_{n=0}^{\infty} \frac{1}{|(f^n)'(f(c))|^{1/\ell}} < \infty,$$

where ℓ is the order of the critical point c . Moreover, the density of the absolutely continuous invariant probability measure with respect to the Lebesgue measure belongs to L^p for all $p < \ell/(\ell - 1)$.

Much more recently, it was proved that no growth condition is needed at all, see [BSvS03] in the unimodal case and [BRLSvS08] for the multimodal case.

6.1. Theorem (Existence of absolutely continuous invariant probability measure under no growth condition).

There exists a constant $C(f)$ such that if

$$\liminf_{n \geq 0} |(f^n)'(f(c))| \geq C(f)$$

for each critical point c then f has an absolutely continuous invariant probability measure with density in L^p .

In [BLVS03] we showed that a summability condition is enough to have polynomial decay of mixing, but it turns out that this is far from optimal. Indeed, there is a remarkable sequel to Theorem 6.1 and [BLVS03], in which Rivera-Letelier & Shen, see [RLS09] show that under the same conditions one has superpolynomial decay of mixing.

6.1. Strategy of proof of Theorem 6.1 on the existence of absolutely continuous invariant probability measures without growth conditions. To prove the existence of an absolutely continuous invariant probability measure, we only need to show that $|f^{-n}(A)|$ is small when A is a set of small Lebesgue measure.

We say that f satisfies *the backward contracting property with constant r* (abbreviated by $BC(r)$) if the following holds: there exists $\epsilon_0 > 0$ such that for each $\epsilon < \epsilon_0$, any critical points $c, c' \in \text{Crit}(f)$ and any component W of $f^{-s}(B_{r\epsilon}(f(c')))$, $s \geq 1$

$$(4) \quad W \cap B_\epsilon(f(c)) \neq \emptyset \text{ implies } |W| \leq \epsilon.$$

The proof of Theorem 6.1 breaks into the following two propositions.

6.2. Proposition.

If

$$\liminf_{n \geq 0} |(f^n)'(f(c))| \geq C$$

for each critical point c then f satisfies property $BC(r)$ where r depends on C .

The proof of this proposition is based on the usual distortion arguments (but note that we use a one-sided Koebe Lemma here).

6.3. Proposition.

There exists $r(f)$ such that if f satisfies the $BC(r)$ then for each $\kappa \in (0, 1)$ there exists a constant M such that for every Borel set A we have

$$(5) \quad |f^{-n}(A)| \leq M|fA|^{\kappa/\ell_{\max}}.$$

This proposition is harder to prove, but basically uses the $BC(r)$ condition in an inductive fashion. In fact, the proof of Theorem 6.1 in [BRLSvS08] follows a similar strategy to what was done in [NvS91], but in many ways is much better because it uses dynamically defined intervals I_n similar to those mentioned in Subsection 1.1. This makes the combinatorial considerations much more clean and transparent (and allows us to deal with the multimodal and non-summable case).

As mentioned, the conclusion of Proposition 6.3 implies the existence of an absolutely continuous invariant probability measure.

Of course, if f has an absolutely continuous invariant probability measure μ , one can still make rather strong statistical assertions: there exists a set $B(\mu)$ of positive Lebesgue measure with the following property. For any continuous function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ the time and space average agree, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu$$

for all $x \in B(\mu)$. So in this case, even if one cannot forecast what happens long ahead of time, one can give predictions about averages.

7. THE REALLY BAD MAPS WHICH HAVE NO OR NO SENSIBLE PHYSICAL MEASURE

It is well known that not all maps which are topologically mixing have an absolutely continuous invariant probability measure. In fact, there are maps which are really badly

behaved from this point of view. Hofbauer and Keller, see [HK90b], [HK90a] gave several examples of quadratic maps f which have various types of bad behaviour:

- for some bad maps, typical orbits stay most of the time near a repelling fixed point: for a.e. x , $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$ converges to the Dirac measure supported on a repelling fixed point;
- for some other bad maps, for a.e. x , $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$ does not converge in any sense.

In the first case, the orbit of a typical point lingers most of the time near a repelling fixed point p . Of course, as the fixed point is repelling, the iterates eventually get far away from p , but after some time an iterate comes back even closer to p than ever before. So it takes an even longer time before the orbit is again far away. In this case, the ‘attractor’ of the typical point is a repelling fixed point. In the second case, no statistical forecast can be made what so ever.

This kind of behaviour is of course bad. It means that no sort of long-term prediction is possible. Not even about averages.

8. THE PALIS CONJECTURE. IS IT TYPICAL FOR PHYSICAL MEASURES TO EXIST?

One of the main challenges in the theory of dynamical systems is to solve the following:

Question: Is it true that C^k -generically, a diffeomorphism has at least one physical measure (and at most finitely many)?

Here we say that an f -invariant probability measure μ is *physical* or *SRB*, if the set $B(\mu)$ of points x such that for every continuous functions $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu$$

has positive Lebesgue measure.

Definite progress has been made towards this question if we are allowed to make C^1 perturbations. However in the C^k case, $k \geq 2$, no progress seems in sight. In the one-dimensional case there is much more progress as we will see.

We should note that any absolutely continuous invariant probability measure for an interval map f automatically is a physical measure.

9. DO MOST MAPS HAVE A PHYSICAL MEASURE?

In the unimodal case, the Palis conjecture was solved:

9.1. Theorem.

Let ℓ be an even integer. For Lebesgue almost all parameters $c \in \mathbb{R}$, the map $f_c(x) = x^\ell + c$ has a unique physical measure, which is either

- absolutely continuous or
- its support is equal to $\omega(0)$ (the omega-limit set of the critical point 0) and $f|_{\omega(0)}$ is uniquely ergodic.

In fact, for almost all $c \in \mathbb{R}$, one of the following situation holds:

- (a) f_c has a periodic attractor, or
- (b) f_c is a Collet-Eckmann map (and the critical point has a weak recurrent property).

The first part of this theorem was first proved by Lyubich for the case that $\ell = 2$, see [Lyu00] and [Lyu02] and then generalized to $\ell \geq 2$ in [BSvS06]. The second part of the theorem was proved in [AM05c] for $\ell = 2$, and then the first part of this result was proved in [ALS08] for $\ell > 2$. The 2nd part of this result for $\ell > 2$ will be proved in a forthcoming paper by Avila and Lyubich.

This theorem was also proved for smooth unimodal maps, see [ALdM03], [AM03], see also [AM05a] for the case when the critical point has degree two. In [Cla09] this was generalized to the unimodal case where the critical point is degenerate.

One of the main issues when dealing with the smooth case is related to the space M_f of maps which are topological conjugate to a given map f . More precisely, it is related to the question whether M_f is a manifold. In the above papers it is shown that hybrid classes are real analytic manifolds. Here a hybrid class of a polynomial-like map f is the set of all polynomial-like maps g which are conjugate to f via quasi-conformal homeomorphism φ for which $\bar{\partial}\varphi = 0$ a.e. on $K(f)$ (so the hybrid class of f is a subset of M_f).

In fact, for typical parameters, the dynamics of many points is described accurately by the orbit of the critical point. Indeed, in [AM05b] the following remarkable result was shown: for Lebesgue almost every parameter a , the multiplier $|Df_a^n(p)|$ of every n -periodic point p in the attractor of the map is determined by the combinatorial type of f_a , i.e. by the kneading sequence of the critical point.

9.2. Question.

Does the above theorem also hold for multimodal polynomials, i.e. for most polynomials does there exist at least one physical measure?

9.3. Question.

Let \mathcal{A} be the space of real analytic maps and let $f \in \mathcal{A}$. Is the space of maps $M_f \cap \mathcal{A}$ a real analytic manifold or variety? Even in the unimodal case the structure of the space of topological conjugacy classes (rather than about the hybrid class of a map) is not fully understood.

Indifferent periodic points will complicate the description of conjugacy classes. Therefore it may be worth assuming that the maps f have negative Schwarzian derivative. Near indifferent periodic points the holonomy along the leaves (defined by conjugacy classes) is not well-behaved, see [ALdM03], [AM03] and [AM05b]

10. MONOTONICITY OF ENTROPY

In the late 70's, the following question attracted a lot of interest: does the topological entropy of the interval map $x \mapsto ax(1-x)$ depend monotonically on $a \in [0, 4]$? In the mid 80's this question was solved in the affirmative:

10.1. Theorem.

The topological entropy of the interval map $x \mapsto ax(1-x)$ depends monotonically on $a \in [0, 4]$.

There are many proofs for this theorem. In the 80's this was proved using Thurston's rigidity theorem, see [MT88]. Another proof relies on Douady-Hubbard's univalent parametrisation of hyperbolic component, and a third proof is due to Sullivan; for a description of these proofs see [dMvS93]. All of these proofs rely on considering the map $x \mapsto ax(1-x)$ as a polynomial acting on the complex plane. A rather different method was used by Tsujii, [Tsu00]. He showed that periodic orbits bifurcate in the 'right' direction using a calculation on how the multiplier depends on the parameter. Although Tsujii's proof does not use that $z \mapsto z^2 + c$ acts on the complex plane, it turns out that the matrix he considers is related to a matrix used in Thurston's rigidity theorem, and unfortunately his proof does not work for maps of the form $z \mapsto z^a + c$ with a not an integer.

In the early 90's, Milnor (see [Mil92]) posed the more general

Monotonicity Conjecture. The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.

Milnor and Tresser proved this conjecture for cubic polynomials, see [MT00] (see also [DGMT95]). Their ingredients are planar topology (in the cubic case the parameter space is two-dimensional) and density of hyperbolicity for real quadratic maps.

Our methods allow to prove the general case of this conjecture.

More precisely, given $d \geq 1$ and $\epsilon \in \{-1, 1\}$, consider the space P_ϵ^d of real polynomials $f: [0, 1] \rightarrow [0, 1]$ of fixed degree d with $f(\{0, 1\}) \subset \{0, 1\}$, with all critical points in $(0, 1)$ and with the first lap orientation preserving if $\epsilon = 1$ and orientation reversing if $\epsilon = -1$. We call ϵ the *shape* of f . In [BvS09] we proved the general case:

10.2. Theorem (Monotonicity of Entropy).

For each integer $d \geq 1$, each $\epsilon \in \{-1, 1\}$ and each $c \geq 0$,

$$\{f \in P_\epsilon^d; h_{top}(f) = c\}$$

is connected.

To prove this theorem, we relate the class a polynomials P^d to the set of *admissible* stunted sawtooth maps \mathcal{S}_*^d . To define this set, first fix a continuous piecewise linear map $S: [0, 1] \rightarrow \mathbb{R}$ with d turning points and with slope $\pm\lambda$ and with $S(\{0, 1\}) \subset \{0, 1\}$ as in the figure below. Next choose a neighbourhood Z_i of each turning point of S , so that $S(Z_i)$ is constant, and so that

$$T(z) = \begin{cases} S(z) & \text{when } z \notin \cup Z_i \\ S(Z_i) & \text{when } z \in \cup Z_i \end{cases}$$

is continuous. Such maps T are called *stunted sawtooth maps*. An *admissible* stunted sawtooth map is one for which there exists a polynomial $f \in P^d$ with the same kneading invariants. (This corresponds to T not having platforms that act as wandering domains - for a more precise definition see [BvS09].) The map $\Psi: P_\epsilon^d \rightarrow \mathcal{S}_\epsilon^d$ is then defined by requiring that $f \in P_\epsilon^d$ and $\Psi(f) \in \mathcal{S}_\epsilon^d$ have the same kneading. Since f has no wandering intervals, $\Psi(f)$ lies in an admissible sawtooth maps. The main difficulty is then to prove the following two theorems:

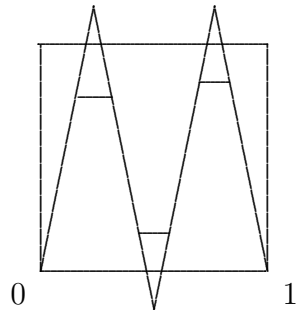


FIGURE 4. A sawtooth and a stunted sawtooth map

10.3. Theorem.

There exists a map $\Psi: P_\epsilon^d \rightarrow \mathcal{S}_\epsilon^d$ such that

- Ψ is ‘almost continuous’, ‘almost surjective’ and ‘almost injective’;
- There exists a connected set $[\Psi(f)] \ni \Psi(f)$ such that the topological entropy of any map $T \in [\Psi(f)]$ is equal to the topological entropy of f ;
- If K is closed and connected then $\Psi^{-1}(K) = \{f; [\Psi(f)] \cap K \neq \emptyset\}$ is connected.

The proof of this theorem relies heavily on quasi-symmetric rigidity (described in Section 3 and 4). Theorem 10.2 then follows from the above theorem and the surprisingly difficult to prove

10.4. Theorem.

The set of admissible sawtooth maps of a given modality and with entropy h is connected.

As mentioned, the proof of Theorem 10.2 relies on quasi-symmetric rigidity, i.e. uses that the maps can be considered as acting on the complex plane. In fact, using the results of Section 4 and using recent joint work with Rempe on transcendental maps we have been able to extend these monotonicity results to include a much wider class of maps, see [RvS10]. For example we have recently been able to prove results of the following type:

10.5. Theorem.

The topological entropy of the map $[0, 1] \ni x \mapsto a \cdot \sin(\pi x) \in [0, 1]$ depends monotonically on a .

However, it is far from clear how to obtain any result on the following type of question/conjecture:

10.6. Question.

Let $f: [0, 1] \rightarrow [0, 1]$ be S -unimodal and symmetric, i.e. $f(1-x) = f(x)$. Does the topological entropy of the map $[0, 1] \ni x \mapsto a \cdot f(x) \in [0, 1]$ depends monotonically on a ? Even if f is of the form $f(x) = [(1/2)^\alpha - (x - 1/2)^\alpha]$ where $\alpha > 0$ is fixed, the answer to this question is not known (unless α is an even integer). Note that if one drops the assumption that f is symmetric then this is no longer true, as was shown by Zdunik, Nusse & Yorke, Kolyada and others.

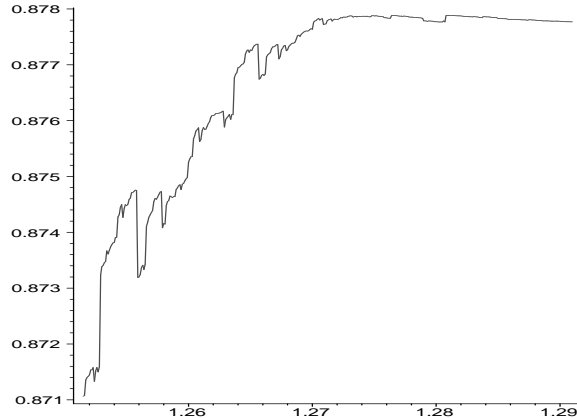


FIGURE 5. Non-monotonicity of entropy for the map $f_b(x) = 2ax^3 - 3ax^2 + b$ with $a = b + 0.515$ with critical values b and 0.515 . Along the horizontal axis b is drawn, and along the vertical axis the topological entropy of f_b .

In fact, it would be enough to show that, under the above assumptions, periodic orbits of $[0, 1] \ni x \mapsto a \cdot f(x) \in [0, 1]$ can never be destroyed as a increases. It should be noted that some partial results towards this can be obtained by applying the notion of rotation number, see [GT92], [Blo94] and [BM97]; periodic orbits with particular types of combinatorics do not disappear as a increases.

10.1. Non-monotonicity in separate variables. It is possible to parametrize the family P^d by critical values. The following example shows that it is not true that topological entropy depends monotonically on each of these parameters. Define $f_{a,b}(x) = 2ax^3 - 3ax^2 + b$ for $a = b + 0.515$. This cubic map has critical points 0 and 1 and critical values $f(0) = b$, and $f(1) = b - a = 0.515$. It is shown in [BvS09] that there are values of b such that the map $a \mapsto h_{\text{top}}(f_{a,b})$ is not monotone. The graph of $a \mapsto h_{\text{top}}(f_{a,b})$ for the family is shown in Figure 5 (the entire graph of the entropy function of the cubic family of polynomials can be found in [BK92]).

This non-monotonicity may be the analogous phenomena to what has been shown in [KKY92] for two-dimensional diffeomorphisms. This motivates the following question

10.7. Question.

Let $H(x, y) = (1 - ax^2 + by, y)$ be the family of Hénon maps. It follows from [KKY92] and [DGY⁺92] that for a fixed b , the set of parameters $\{a; h_{\text{top}}(H_{a,b}) = c\}$ is not connected. However, is it possible that $\{(a, b); h_{\text{top}}(H_{a,b}) = c\}$ is connected?

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