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THE DYNAMICS OF ANALYTIC TRANSFORMATIONS

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ABSTRACT. This is a survey of the theory of iterates of analytic transformations $f: S \to S$ of Riemann surfaces. All the basic aspects of the theory are touched on: local problems, dynamics on hyperbolic Riemann surfaces, the topological dynamics of rational endomorphisms from Fatou and Julia to our day, the connection with functional equations, measurable dynamics on the Julia set, iterates of entire functions.

CONTENTS

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The theory of iterates of analytic transformations created at the beginning of the century in papers of Julia [94] and Fatou ([80]–[82])([1]) has now entered a period of great development, after passing through a fifty-year epoch of stagnation (1930–1980) and several years of stormy revival at the start of the 1980s. This makes it possible to give something of a summary in a relatively calm setting.

The subject of the theory of iterates is the qualitative study of dynamical systems generated by analytic transformations $f: S \to S$ of Riemann surfaces (in the first place by rational endomorphisms $f: z \mapsto P(z)/Q(z)$ of the Riemann sphere $\overline{\mathbf{C}}$; here P and Q are complex polynomials). In this case the dynamical system should be understood to be the semigroup of iterates

$$f^n = \underbrace{f \circ \cdots \circ f}_{n} \qquad (n = 1, 2, \ldots).$$

The basic problem is to picture the phase portrait of such a system, i.e., the typical behavior of its different trajectories $\{f^n z\}_{n=0}^{\infty}$, as well as the character of change of the phase portrait under a smooth deformation of f.

The theory of iterates focuses in itself the ideas and methods of very diverse areas of mathematics. The theory of dynamical systems provides an understanding of the nature of chaos, the fractal property, and structural stability, and is the main source for the formulation of problems. The technique traditional since the times of Julia and Fatou—geometric function theory—has now been enriched by the apparatus of quasiconformal mappings and Teichmüller spaces, the use of which has elevated the whole area to a qualitatively different level. The main role in this has been played by work of Douady and Hubbard [75], Sullivan [136], and Thurston [142]. But in addition the specialist in topology, functional analysis, mathematical physics, or numerical mathematics can find here his own interesting problems, or a field for application of his methods, or nice illustrative examples.

In this survey the authors have tried to present the basic classical and contemporary results in the theory of iterates, while demonstrating the effect of various ideas and methods. The first chapter bears an introductory character. In §1 we outline the vocabulary from complex analysis that is necessary for understanding what follows (the hyperbolic metric, branched cover, Teichmüller space, and so on.) The basic object of investigation—an analytic transformation $f: S \to S$ of a Riemann surface—appears in the short §2, along with the definitions of the simplest concepts connected with its dynamics—periodic points, multipliers, etc.

The next section (§3) already relates directly to the subject of the survey. It presents the local theory: the dynamics of an analytic transformation in a

([1])The basic results were apparently obtained independently by Ritt [126], who published only that part of his investigations that did not overlap the memoirs of Fatou and Julia. The reader interested in the earlier history of the theory of iterates can turn to [11].

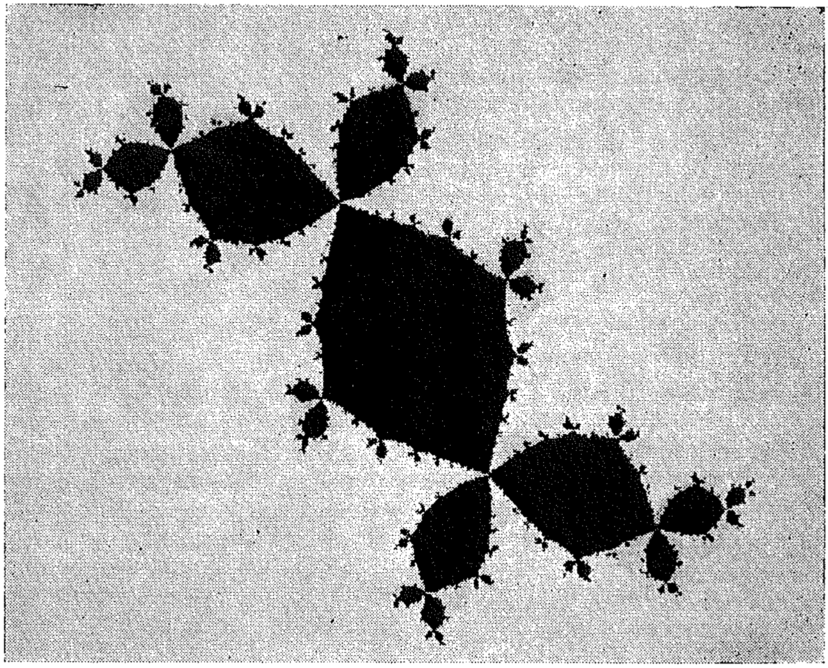

FIGURE 1. "Douady's rabbit" is obtained by iterating the polynomial $z \mapsto z^2 + c$, with $c \approx -0.12 + 0.74i$. The Julia set is the skin of the rabbit.

neighborhood of an equilibrium position. It was these problems (in their connection with certain functional equations) that began the theory of iterates more than a century ago. Nevertheless, there remain delicate unanswered questions here relating chiefly to the problem of stability in a neighborhood of a neutral equilibrium position. Deep results in this circle of problems were very recently obtained by Yoccoz [145].

The first chapter closes with a description of the dynamics on hyperbolic Riemann surfaces. In this case the picture is very transparent, since chaotic phenomena are absent. A prototype for it is the classical Denjoy-Wolff theorem on transformations of the disk.

Chapter 2 is devoted to topological dynamics of rational endomorphisms $f: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of the Riemann sphere. The fascination of this theory has to do with the coexistence of regular and chaotic dynamics, leading to a partition of the sphere into two invariant subsets—the Fatou set $F(f)$ and the Julia set $J(f)$ (Figures 1, 2). The first four sections (§§1–4) contain mostly classical results about the structure of these sets, the classification of periodic points, and the dynamics on the periodic components of the Fatou set. Here we sketch a proof of Sullivan's theorem on the absence of wandering domains, which leads to a complete picture of the dynamics on the Fatou set. This proof is a first illustration of the power of the method of quasiconformal deformations.

In the next five sections (§§5–9) the reader finds many other examples on this theme: quasiconformal surgery becomes the main tool here. We present the theory of Thurston, the theory of structural stability in holomorphic families of rational functions ([26], [27], [115]), and results obtained in the past decade by

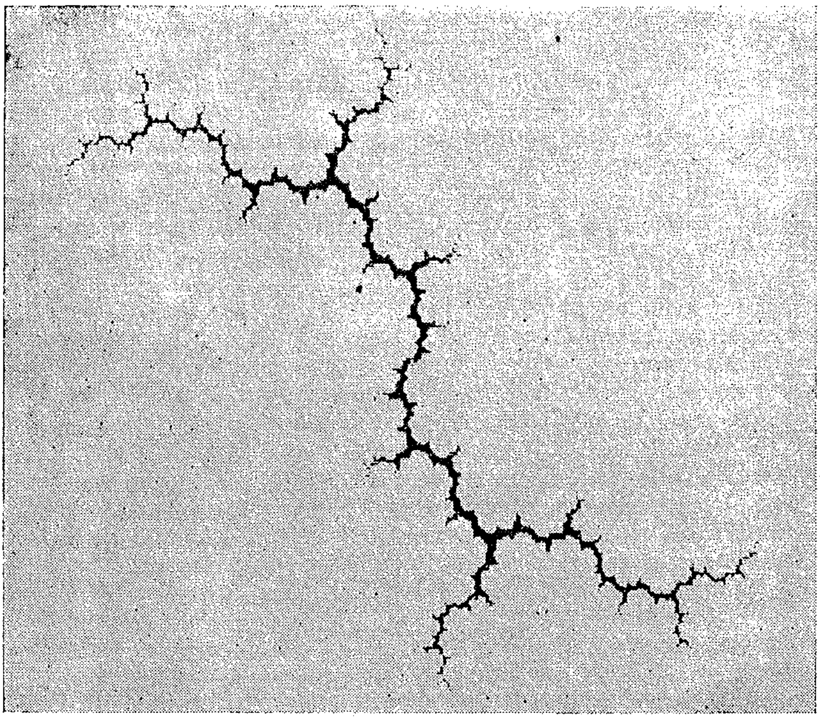

FIGURE 2. The Julia set of the polynomial $z \mapsto z^2 + i$.

Douady and Hubbard, by Shishikura, and by other authors. A special place in this circle of questions is occupied by the investigation of the Mandelbrot set—the bifurcation diagram of the quadratic family $z \mapsto z^2 + c$ (§7). Its beauty was one of the main sources of interest in the theory of iterates that arose at the beginning of the 1980s. The deep penetration of Douady and Hubbard into the intricate structure of this set is one of the most impressive achievements of recent years. We give an initial presentation of the subject, referring the reader wishing to acquaint himself with it more closely to the remarkable paper [76].

In §10 there is a table demonstrating the connection between the theory of iterates and the theory of Kleinian groups. This connection was already understood by the classicists but only after Sullivan's work did it become an effective device that enriched both areas. Unfortunately, we did not have the opportunity to develop this line as thoroughly as it deserves.

Finally, in the last section of Chapter 2 (§11) we present classical results of Julia, Fatou, and Ritt about commuting rational functions and certain functional equations. These problems were an important stimulus for the creators of the theory of iterates.

The material expounded in the first two chapters is close to that in the surveys [29] and [61]. The present survey is supplemented by a number of results obtained in recent years, but here the proofs are of a more sketchy character. Therefore, we recommend that the reader first acquainting himself with the subject turn to [29] and [61] in order to reproduce the omitted proofs or details of them. Moreover, a nice exposition of the classical results with detailed proofs is contained in Brolin's survey [63], which played (along with Montel's book

[36]) a bridging role between the 1920s and the 1980s. Finally, despite the existence of several modern surveys, we recommend that the reader become directly acquainted with the fundamental memoirs [80]–[82] of Fatou.

The material in the last two chapters has not been presented before in the survey literature. Chapter 3 is devoted to dynamics on the Julia set. Since this dynamics bears a chaotic character, its description can be given only in statistical terms. Ergodic theory provides an adequate language for studying such phenomena. The methods of entropy theory, the techniques of unstable manifolds, and the Sinaĭ–Ruelle–Bowen thermodynamic formalism serve as effective tools of investigation. In §1 of Chapter 3 we present the elements of this language and of these tools in a form convenient for later use. It is necessary to have an invariant (or at least quasi-invariant) measure on the Julia set in order to use ergodic theory. There are many such measures, and one of the basic problems is to choose the most important of them, those that have interesting interpretations from the point of view of dynamics, physics, function theory, or geometry.

The first measure of this sort, considered by Brolin [63], was harmonic measure μ on the Julia set of a polynomial (it is the equilibrium measure in the sense of electrostatics). Brolin showed that μ is invariant and gives the asymptotic distribution of the inverse images of a point under the iterations. This measure was later interpreted from a different point of view as the unique maximal entropy measure ([102], [85]).

Another remarkable measure on the Julia set is the conformal measure constructed by Sullivan [138]. On the one hand, this measure is closely connected with the geometry of the Julia set (it coincides with the Hausdorff measure of maximal dimension), and on the other hand, it admits a natural interpretation in the framework of the thermodynamic formalism.

In a recent paper Zdunik [146] proved that, as a rule, the maximal entropy measure is singular with respect to the conformal measure. This result leads to a rigorous proof that the Julia set is fractal (see the end of Chapter 3).

Closely connected with this circle of problems is Makarov's paper [34], in which unexpected geometric properties of harmonic measure were discovered in a very general situation (without the participation of dynamics). However, it is not excluded that these results have a hidden dynamical nature. Very interesting work in this direction has been carried out by Przytycki and his co-authors [123].

Chapter 4 is devoted to iterates of entire transcendental functions. The first work in this area is also due to Fatou [84]. Subsequent development in the 1960s and 1970s is associated with the name of Baker ([53]–[58]). In presenting the foundations of the theory in §§1 and 2, we concentrate attention on the points specific to entire functions. A central place here is occupied by two theorems of Baker that are given with proofs (simpler than Baker's, but based on the same ideas). In §3 we list the main pathologies that can be observed for entire functions (wandering domains, etc.). On the same theme we present elegant examples of Herman. The authors have used approximation techniques to construct pathological examples ([14], [79]). This technique enables us, in particular, to construct a wandering domain whose orbit does not tend to ∞.

A class S of entire functions with dynamics similar to that of rational transformations is described in §4. Here the picture on the Fatou set is exactly the same as for polynomials ([14], [15]).

In the concluding section we consider the exponential family $z \mapsto \exp \lambda z$, $\lambda \in \mathbf{C}^*$, which is a transcendental analogue of the quadratic family $z \mapsto z^2 + c$. Interest in this family arose due to work of Misiurewicz [119], who proved that $J(\exp z) = \mathbf{C}$. Nevertheless, as soon became clear, the chaotic nature of this dynamics is not really so great if one is interested in properties typical with respect to Lebesgue measure ([28], [125]). The dynamical effects demonstrated by the exponential transformation are evidently impossible in the rational case.

The beauty of computer images of various sets generated by iterates is a circumstance of no small importance that has attracted broad attention to holomorphic dynamics. The quality of these images is raised to the highest class in the book [151] of Peitgen and Richter. This book will apparently soon appear in a Russian translation. Until then the reader without access to the book can get some idea of these astounding figures from reproductions of them in the journal "V Mire Nauki" [13].

In conclusion we use this opportunity to express gratitude to E. A. Gorin, who at one time acquainted us with the work of Cowen [66] and related questions. We also thank V. A. Kaĭmanov, who read through the manuscript and made a number of useful remarks, and L. K. Maslov for computer pictures.

CHAPTER 1
INTRODUCTION

§1. Preliminary information from other areas

1.1. Conventions on general topological terminology. A *perfect set* is a complete metric space without isolated points. A *Cantor set* is totally disconnected perfect compact metric space. A *continuum* (in a topological context) is a connected topological space with more than one point. A *curve* is a continuous image of a closed interval. A *Jordan curve* is a homeomorphic image of a circle. A *Jordan domain* is a domain bounded by a Jordan curve. A *disk* (*annulus*) is defined to be a domain homeomorphic to a standard disk (annulus). The connected components of a topological space will be called simply the *components*.

A transformation $f: X \to X$ of a metric space (with metric ρ) is said to be: a) *contracting* if $\rho(fx, fy) \le \rho(x, y)$; b) *strictly contracting* if $\rho(fx, fy) < \rho(x, y)$ for all $x, y \in X$.

Unless stated otherwise, homeomorphisms of Riemann surfaces are assumed to preserve orientation, and the spaces of continuous mappings that arise are assumed to be equipped with the compact-open topology.

1.2. Riemann surfaces ([1], [20]). According to the Riemann-Koebe uniformization theorem, an arbitrary simply connected Riemann surface is conformally equivalent to the Riemann sphere $\overline{\mathbf{C}}$, the complex plane $\mathbf{C}$, or the unit disk $\mathbf{U}$. The sphere $\overline{\mathbf{C}}$ is called an *elliptic* Riemann surface. The plane $\mathbf{C}$, the punctured plane $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$, and the torus $\mathbf{T}^2$ are called *parabolic* Riemann surfaces; their universal coverings are conformally equivalent to the plane. The universal coverings of the remaining Riemann surfaces are conformally equivalent to the disk $\mathbf{U}$. These surfaces are said to be *hyperbolic*. A domain $D \subset \overline{\mathbf{C}}$ is hyperbolic if and only if $\overline{\mathbf{C}} \setminus D$ contains at least three points.

A hyperbolic Riemann surface S is representable as a quotient $\mathbf{U}/\Gamma$, where Γ is a discrete group of conformal automorphisms of $\mathbf{U}$. The group Γ also acts on the circle $\mathbf{T} = \partial\mathbf{U}$, partitioning it into the union of the set R of discontinuity and the limit set Λ (see [20]). The quotient $(\mathbf{U} \cup R)/\Gamma$ is a

Riemann surface with boundary containing S. Its boundary $\partial_I S = R/\Gamma$ is called the *ideal boundary* of S. One says that S is a *surface of finite type* if it is obtained from a compact Riemann surface by deleting some points. An equivalent definition is: the fundamental group $\pi_1(S)$ is finitely generated, and the ideal boundary $\partial_I S$ is empty.

On any Riemann surface S there exists a conformal metric ρ_S of constant curvature. This metric is said to be *hyperbolic* (or a *Poincaré metric*) in the case of a hyperbolic surface S, *Euclidean* in the parabolic case, and *spherical* in the case of $\overline{\mathbf{C}}$.

THE SCHWARZ LEMMA. *Let $f: V \to W$ be an analytic mapping between hyperbolic Riemann surfaces. Then either* a) *f is strictly contracting in the Poincaré metric, in which case $\|Df(x)\| < 1$ for all $x \in V$, or* b) *f is a cover, in which case $\|Df(x)\| = 1$, $x \in V$.*

1.3. Normal families. Montel's Theorem (see [36]). A meromorphic function on a domain $V \subset \overline{\mathbf{C}}$ is an analytic mapping $V \to \overline{\mathbf{C}}$. We introduce the spherical metric on $\overline{\mathbf{C}}$, and the compact-open topology in the space of meromorphic functions $V \to \overline{\mathbf{C}}$.

A family $\{f_n\}$ of meromorphic functions is said to be *normal* if it is precompact in the indicated topology. This is equivalent to each of the following properties holding for an arbitrary compact set $K \subset V$: a) the family $\{f_n\}$ is equicontinuous on K; b) the spherical norm $\|Df_n\|$ of the differential is uniformly bounded on K. The basic test for normality is

MONTEL'S THEOREM. *Assume that there exist three meromorphic functions g_i on V with the following property: the equations $f_n(z) = g_j(z)$ and $g_k(z) = g_j(z)$, $k \neq j$, do not have roots in V. Then the family $\{f_n\}$ is normal.*

1.4. Branched covers. A mapping $f: V \to W$ of two-dimensional surfaces is said to be *interior* (*in the Stoïlow sense*) if for each point $x \in V$ there exists a neighborhood H in which f is topologically reducible to the form $z \mapsto z^k$, where $k \equiv \deg_x f \in \mathbf{N}$. More precisely, there exist homeomorphisms $\varphi: (H, x) \to (\mathbf{U}, 0)$ and $\psi: (fH, fx) \to (\mathbf{U}, 0)$ such that $\psi \circ f \circ \varphi^{-1}: z \to z^k$. If $k > 1$, then x is called a *branch point of index* k (or a *critical point*), and fx is called the *projection of the branch point* (or the *critical value*).

An interior mapping is called a *branched cover* if each $y \in W$ has a neighborhood G such that f reduces to the form $z \to z^k$ in each component (H, x) of the complete inverse image $f^{-1}(G, y)$ (where $k = \deg_x f$ depends on $x \in f^{-1}y$). In this case the points $y \in W$ that are not projections of branch points have the same number of inverse images, and this number is called the degree $d = \deg f$ of the cover f (d can be ∞). Here f is also called a *d-sheeted branched cover.* A branched cover is said to be *regular* if its fibers $f^{-1}(y)$ are orbits of the action of some group of homeomorphisms.

Denote by χ_S the *Euler characteristic* of a surface S.

THE RIEMANN-HURWITZ FORMULA. *Let $f: V \to W$ be a d-sheeted branched cover whose branch points have indices $k_1, \dots, k_n$, where $d < \infty$. Then*

$$\chi_V = d\chi_W - \sum_{j=1}^{n}(k_j - 1).$$

COROLLARY. *Suppose that a domain $D \subset \overline{\mathbf{C}}$ admits a finite-sheeted branched cover $f: D \to D$ having branch points. Then D is simply connected or infinitely connected.*

LEMMA 1.1. *Suppose that V and W are domains on the sphere, and $f: V \to W$ an interior mapping continuous on $\overline{V}$. Then f is a finite-sheeted branched cover if and only if $f(\partial V) \subset \partial W$.*

LEMMA 1.2. *Let $f: V \to W$ be a finite-sheeted branched cover between hyperbolic domains on the sphere, with W simply connected. Assume that all its branch points lie in a single fiber $f^{-1}\alpha$. Then $f^{-1}\alpha$ consists of a single point β, and there exist conformal mappings $\varphi: (V, \beta) \to (\mathbf{U}, 0)$ and $\psi: (W, \alpha) \to (\mathbf{U}, 0)$ such that $\psi \circ f \circ \varphi^{-1}: z \mapsto z^d$.*

1.5. Quasiconformal mappings. The measurable Riemann theorem ([2], [20]). Let V be a Riemann surface, and ω a measurable Riemann metric on V. The metric ω can be reduced locally to the form $\gamma(z)|dz + \beta(z)d\overline{z}|^2$, where γ and β are measurable functions with $\gamma(z) > 0$ and $|\beta(z)| < 1$ a.e. Further, $\beta(z) = k(z)\exp 2i\Theta(z)$, where $(1 + k(z))/(1 - k(z)) = K(z)$ is the ratio of the axes of the infinitesimally small ellipse $|dz + \beta(z)d\overline{z}| = 1$, and $\Theta(z)$ is the direction of its major axis. The function $K(z)$ is defined globally on V and called the *dilatation* of the metric ω. If $\|K(z)\|_\infty < \infty$, then ω is said to have bounded dilatation.

Two metrics ω_1 and ω_2 are *proportional* if there exists a measurable function $\gamma > 0$ such that $\omega_2 = \gamma\omega_1$. A class of proportional Riemannian metrics with bounded dilatation is called a *conformal structure* on V. The *standard conformal structure* (with $K(z) \equiv 1$) on a Riemann surface V will be denoted by σ_V (or simply σ when this does not lead to confusion).

Now suppose that V and W are Riemann surfaces, ρ_V and ρ_W are smooth metrics on them generating the standard conformal structures, $A \subset V$, and $\varphi: A \to W$. For $z \in A$ and $\varepsilon > 0$ denote by $m(z, \varepsilon)$ and $M(z, \varepsilon)$ the infimum and supremum of the quantities $\rho_W(\varphi(z), \varphi(\zeta))$ for $\zeta \in A$ and $\rho_V(z, \zeta) = \varepsilon$. If there are no such points ζ, then let $m(z, \varepsilon) = M(z, \varepsilon) = 1$. The mapping φ is said to be *quasiconformal* (or to have the *Pesin property*) if the quantity

$$P(z) = \overline{\lim_{\varepsilon \to 0}} \frac{M(z, \varepsilon)}{m(z, \varepsilon)}$$

is uniformly bounded on A.

Let $\varphi: V \to W$ be a quasiconformal homeomorphism of Riemann surfaces. Then:

a) φ is a.e. differentiable, and hence φ acts naturally on the measurable Riemannian metrics: $\omega \mapsto \varphi_*\omega$ and on the corresponding conformal structures $\mu \mapsto \varphi_*\mu$;

b) φ can be continuously extended to $V \cup \partial_I V$;

c) if $\varphi_*\sigma_V = \sigma_W$ (i.e., $\overline{\partial}\varphi \equiv \partial\varphi/\partial\overline{z} \equiv 0$ a.e.), then φ is a conformal mapping;

d) for an arbitrary conformal structure μ on V there exist a Riemann surface W and a quasiconformal homeomorphism $\varphi: V \to W$ such that $\varphi_* = \sigma_W$ (Morrey, 1938).

The last theorem is especially important for us in the case $V = \overline{\mathbf{C}}$.

THE MEASURABLE RIEMANN THEOREM (see [2]). *For an arbitrary conformal structure* μ *on the sphere* $\overline{\mathbf{C}}$ *there exists a quasiconformal homeomorphism* $\varphi = \varphi^{\mu}: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ *such that* $\varphi_* \mu = \sigma$. *This homeomorphism is uniquely determined by its values at three points.*

The conformal structure μ on $\overline{\mathbf{C}}$ can be regarded as a point of the unit ball in L^{∞} (identify μ with the function $\beta(z)$). The homeomorphism φ^{μ}, which leaves the points 0, 1, and ∞ fixed, depends continuously on μ.

An interior mapping $f: V \to W$ is said to be *quasiregular* if in a neighborhood of each noncritical point it is a quasiconformal homeomorphism with uniformly bounded dilatation. If μ is a conformal structure on W, then f induces the conformal structure $f^* \mu$ on V.

1.6. Teichmüller spaces (see [1], [2]). Let S be a Riemann surface on which several isolated points are marked. Denote by Z_S the space of conformal structures on S. Two structures μ, $\nu \in Z_S$ are said to be *isotopic* if there exists a continuous family of quasiconformal homeomorphisms $\varphi_t: S \to S$ that are the identity on the ideal boundary $\partial_I S$ and are such that $\varphi_0 = \text{id}$ and $(\varphi_1)_* \mu = \nu$ at the marked points. The *Teichmüller space* T_S of the surface S with marked points is defined to be the space of classes of isotopic conformal structures. The element of T_S corresponding to a conformal structure μ is denoted by $\overline{\mu}$. The space T_S can be endowed with the structure of a complete metric space (the corresponding metric is called the Teichmüller metric) and the structure of a complex analytic manifold. This manifold is finite dimensional if and only if S is a surface of finite type.

Assume that S is a disk or annulus with marked points, and that the rotation group $\mathbf{T}$ acts on it analytically. Then the *equivariant Teichmüller space* is defined in the natural way: in the definition above require that all the conformal structures be $\mathbf{T}$-invariant. In this situation we keep the notation T_S, with the understanding that S remembers the action of the rotation group.

§2. Elementary concepts of conformal dynamics

Let $f: S \to S$ be an analytic transformation (*endomorphism*) of a Riemann surface, and f^n its nth iterate. The *orbit* (*trajectory*) of a point $z \in S$ is defined to be the set $\{f^n z\}_{n=0}^{\infty}$, and the *large orbit* is $\{\zeta: \exists n, m \in \mathbf{N}: f^n z = f^m \zeta\}$. The *limit set* of the orbit of z is denoted by $\omega(z)$.

Let C_f be the set of critical points of f. It follows from the chain rule that $\bigcup_{k=0}^{n-1} f^{-k} C_f$ is the set of critical points of the iterate f^n, and $\bigcup_{k=1}^{n} f^k C_f$ is the set of its critical values. The last circumstance makes it important to study the orbits of critical points.

A point α is said to be *periodic* if $f^p \alpha = \alpha$ for some p, p is called a period of α, and the finite orbit $\{\alpha_n = f^n \alpha\}_{n=0}^{p-1}$ is called the cycle. The smallest period is called the *order* of the periodic point α (of its cycle). A *fixed point* (or *equilibrium position*) is defined to be a periodic point of order 1.

We introduce local coordinates in a neighborhood of the points α_n and define the *multiplier*(*) of the cycle $\{\alpha_n\}_{n=0}^{p-1}$ (or of any point in this cycle) to be $\lambda = f'(\alpha_0) \cdots f'(\alpha_{p-1})$. This definition does not depend on the choice of local coordinates, as is easily seen from the equality $\lambda = (f^p)'(\alpha_0)$. A periodic point (its cycle) is said to be *attracting, repelling*, or *neutral* in the following

(*)*Editor's note.* The term *eigenvalue* is used in [61].

respective cases: $0 < |\lambda| < 1$, $|\lambda| > 1$, $|\lambda| = 1$. If $\lambda = 0$, then the cycle is said to be *superattracting*. In this case one of the points α_n is critical. Nonneutral periodic points are called *hyperbolic* points.

The trajectory of a point $z \in S$ is said to be *Lyapunov stable* (in some metric ρ) if $\forall \varepsilon > 0 \ \exists \delta > 0$

$$\rho(z, \zeta) < \delta \Rightarrow \rho(f^n z, f^n \zeta) < \varepsilon, \qquad n = 1, 2, \ldots.$$

If in addition $\rho(f^n z, f^n \zeta) \to 0$, $n \to \infty$, for all ζ close to z, then the trajectory of z is said to be *asymptotically stable*. In particular, one can speak of the stability of an equilibrium position. This concept does not depend on the choice of metric.

We say that f is a *transformation of finite order* if $f^n = \mathrm{id}$ for some $n > 0$.

A set A is said to be *invariant* if $fA \subset A$, and *completely invariant* if in addition $f^{-1}A \subset A$. We say that the orbit of a point z *is absorbed* by an invariant set if $f^n z \in A$ for some n.

Two transformations $f\colon V \to V$ and $g\colon W \to W$ are said to be *topologically* (*conformally, quasiconformally,* etc.) *conjugate* if there exists a homeomorphism (conformal mapping, quasiconformal mapping, etc.) $\varphi\colon V \to W$ such that $\varphi \circ f = g \circ \varphi$. If in this definition we waive the invertibility of φ, then we arrive at the concept of *semiconjugate* transformations. Under the action of φ the orbits of f pass into the orbits of g, the limit sets into the limit sets, the cycles into the cycles, etc. However, the multipliers of cycles are not topological invariants but only smooth invariants.

§3. Local theory

In this section we describe the dynamics of, and give a classification of, the analytic transformations

$$f\colon z \mapsto \lambda z + az^2 + \cdots \tag{1.1}$$

in a neighborhood of the fixed point 0. The basic functional equations of Schröder, Boettcher, and Abel arise in a natural way in connection with this problem. A large part of the theory extends to the multidimensional case (see [3], [43]), but we shall not dwell on it.

3.1. The hyperbolic case: $|\lambda| \neq 1$. The set of such multipliers is also called the *Poincaré domain*. This is the case of general position, in which all problems are easily solved in entirety.

THEOREM 1.1. *A transformation* (1.1) *with* $|\lambda| \neq 0, 1$ *is conformally conjugate to the linear transformation* $z \mapsto \lambda z$ *in a neighborhood of zero. In other words, in a neighborhood of zero there exists a univalent solution of the Schröder equation* $\varphi(fz) = \lambda\varphi(z)$ (*a Koenigs function*).

The authors know four proofs of Theorem 1.1: two analytic proofs (construction of φ in the form $\lim_{n\to\infty} \lambda^{-n}(f^n z)$([2]) (see [8], §37), or determination of its Taylor expansion), and two geometric proofs ([29], [142]). The last proof follows at once from the next result, which goes back to Julia.

LEMMA 1.3 ([8], [66]). *Let* $f\colon \mathbf{U} \to \mathbf{U}$ *be a univalent endomorphism of the disk. Then* f *is conformally conjugate to* $g|W$, *where* $g\colon \mathbf{C} \to \mathbf{C}$ *is a linear transformation, and* W *is some invariant domain of it.*

([2])In the case $|\lambda| < 1$; the repelling case is reduced to this case by replacing f by f^{-1}.

Since λ is a conformal invariant, Theorem 1.1 gives a conformal classification of germs of analytic transformation in a neighborhood of an attracting fixed point. We note that from the point of view of quasiconformal classification all these transformations are equivalent. The situation is analogous in the repelling case.

We pass to the superattracting case.

THEOREM 1.2 ([4], [8]). *An analytic transformation* $f\colon z \mapsto az^k + \cdots$, $k \geq 2$, $a \neq 0$, *is conformally conjugate to* $z \mapsto z^k$ *in a neighborhood of zero. In other words, in a neighborhood of zero there exists a univalent solution of the Boettcher equation* $\varphi(fz) = (\varphi(z))^k$.

AN ANALYTIC PROOF: $\varphi(z) = \lim_{n\to\infty} \sqrt[k^n]{f^n z}$ ([8], §40). See [29] for a geometric proof based on the measurable Riemann theorem. ●

Attracting and superattracting fixed points are asymptotically stable. It is curious that this property is characteristic.

THEOREM 1.3 [32]. *Zero is an asymptotically stable equilibrium position of the transformation* (1.1) *if and only if* $|\lambda| < 1$.

Thus, in the analytic situation asymptotic stability is a coarse property that depends only on linear approximation. The situation is quite different in the investigation of Lyapunov stability, to which we proceed.

3.2. The problem of stability. This problem is very subtle in a neighborhood of a neutral equilibrium position. Geometric considerations provide little here, and analysis is complicated. But for a beginning we combine these points of view:

THEOREM 1.4 [18]. *The following properties are equivalent for a transformation* (1.1) *with* $|\lambda| = 1$:

a) *Zero is a Lyapunov stable equilibrium position.*

b) f *is topologically conjugate to a rotation in a neighborhood of zero.*

c) f *is conformally conjugate to a rotation in a neighborhood of zero, that is, the Schröder equation* $\varphi(fz) = \lambda\varphi(z)$ *has a univalent solution in a neighborhood of zero.*

Stable neutral equilibrium positions are also called *Siegel points.* The stability of the zero equilibrium position depends first and foremost on the arithmetic properties of the multiplier λ. Let us analyze it in greater detail. A neutral fixed point is said to be a *rational* (or *resonance*) point if λ is a root of unity, and an *irrational* point otherwise. Theorem 1.4 gives us at once the

COROLLARY. *Suppose that zero is a neutral rational fixed point of the transformation* (1.1). *If this transformation has infinite order, then* 0 *is Lyapunov stable.*

A neutral irrational point can also be unstable if its multiplier can be approximated in a pathologically rapid way by rational numbers [43]. The existence of such multipliers can be established from simple Baire category arguments even without explicit estimates:

THEOREM 1.5 ([29], [91]). *Consider the one-parameter family of transformations* $f_\lambda\colon z \mapsto \lambda z + g(z)$, $\lambda \in \mathbf{T}$, *in a neighborhood of zero. Assume that all the* f_λ *have infinite order. Then the set* $\Lambda \subset \mathbf{T}$ *of* λ *such that zero is unstable is a dense* G_δ*-set.*

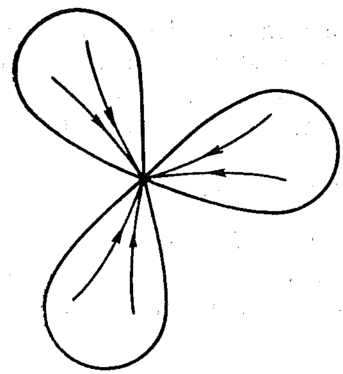

FIGURE 3. Leau Petals.

The simplest family to which Theorem 1.5 is applicable is $f_\lambda\colon z \mapsto \lambda z + z^2$. On the other hand, there are obviously nonlinear transformations with an arbitrary multiplier $\lambda \neq 1$ such that zero is stable, namely, $f(z) = \varphi^{-1}(\lambda\varphi(z))$, where φ is a univalent function in a neighborhood of zero, and $\varphi(0) = 0$. It is remarkable that for some λ (more precisely, for almost all $\lambda \in \mathbf{T}$) stability always holds, independently of the higher terms:

SIEGEL'S THEOREM [18]. *Suppose that $\lambda = e^{2\pi i\theta}$ and assume that there exist constants $C > 0$ and $k > 0$ such that for all $m, n \in \mathbf{N}$*

$$|\theta - m/n| \geq C/n^k. \tag{1.2}$$

Then the transformation $z \mapsto \lambda z + az^2 + \cdots$ is conformally conjugate to a rotation.

We remark that Bryuno found [147] a weaker arithmetic condition for linearization of transformations $z \mapsto e^{2\pi i\theta} z + az^2 + \cdots$ with arbitrary nonlinear additional terms. Very recently Yoccoz proved that this condition is necessary for linearization of the quadratic polynomial $z \mapsto e^{2\pi i\theta} z + z^2$ ([145], [148]).

3.3. Dynamics in a neighborhood of a neutral equilibrium position. We assume that f has infinite order. In a neighborhood of a Siegel point the picture is obvious: each orbit is densely imbedded in a closed analytic curve. The complete picture can also be described in the opposite case of a neutral rational point. This was done by Leau (1897) and Fatou ([80], Chapter 2).

Suppose that the multiplier λ is a qth root of unity. Then it turns out that $l = qs$ *Leau petals* L_k touch the fixed point $\alpha = 0$. These are disjoint Jordan domains of arbitrarily small diameter, having the following properties (Figure 3):

a) Away from zero the boundary ∂L_k is an analytic curve, and has a break at zero with angle π/l.

b) The petal L_k is obtained from L_1 by a rotation through the angle $2\pi(k-1)/l$.

c) The transformation f permutes the petals L_k, breaking them up into cycles of order q.

d) If $z \in L_k$, then $f^n z \to \alpha$.

e) $\exists \delta > 0$ such that if $|f^n z| \leq \delta$ for all $n \in N$, then $z \in \bigcup_{k=1}^{l} L_k$.

The *Leau flower* of the inverse function f^{-1} is obtained from the Leau flower of f by a rotation through the angle π/l. The number of petals is determined by the Taylor expansion of f, namely, if $f^q z = z + a_{l+1}z^{l+1} + \cdots$, $a_{l+1} \neq 0$, then l is the number of petals. By Lemma 1.3, the transformation $f^q|L_k$ can be linearized. It turns out that in this case $g\colon z \mapsto z+1$, and W is the right half-plane. In other words, there exists a conformal mapping $\varphi\colon L_k \to \{z\colon \operatorname{Re} z > 0\}$ that solves the *Abel equation* $\varphi(f^q z) = \varphi(z) + 1$.

As always, the *Abel function* φ can be found analytically [80] and geometrically ([29], [142]). Fatou determined an asymptotic expression for φ from which it follows that each f^q-orbit in L_k lies on an invariant analytic curve that is a bisector at zero.

We note in conclusion that neutral rational points admit complete classification, both topological (the multiplier and the number of Leau petals form a complete invariant) and conformal [10].

There has been little study of the dynamics in a neighborhood of an unstable irrational equilibrium position. Even the question of whether it can attract some orbit is open. The only result on this theme known to the authors was obtained by Fatou:

THEOREM 1.6 ([82], p. 241). *Suppose that* $|\lambda| = 1$ *and* $\lambda \neq 1$. *Assume that the domain* V *has the property that* $fV \cap V \neq \varnothing$. *Then the orbit* $\{f^n V\}_{n=0}^{\infty}$ *cannot converge uniformly to zero.*

§4. Analytic transformations of hyperbolic Riemann surfaces

THEOREM 1.7 (see [140]). *Let* $f\colon V \to V$ *be an analytic transformation of a hyperbolic Riemann surface. Then one of the following possibilities holds*:

a) f *has an attracting or superattracting fixed point* $\alpha \in V$ *to which all orbits* $\{f^n z\}_{n=0}^{\infty}$ *converge.*

b) *All orbits tend to infinity, i.e.,* $\rho_V(\alpha, f^n z) \to \infty$, $n \to \infty$, *for every point* $\alpha \in V$.

c) f *is conformally conjugate to an irrational rotation of* (i) *a disk,* (ii) *a punctured disk, or* (iii) *an annulus.*

d) f *is a conformal automorphism of finite order, i.e.,* $f^p = \mathrm{id}$ *for some* p.

COROLLARY. *Suppose that* V *is a hyperbolic Riemann surface, and* $f\colon V \to V$ *is an analytic transformation of infinite order. Then* f *has at most one fixed point. If this point is neutral, then* f *is conformally conjugate to an irrational rotation of the disk* $\mathbf{U}$.

Theorem 1.7 can be refined in part b) when there is additional information about the Riemann surface V and about the boundary properties of f:

THEOREM 1.8. *Let* V *be a hyperbolic domain on the sphere, and* $f\colon V \to V$ *an analytic transformation continuous up to the boundary. Assume that the set of fixed points of* f *on* ∂V *is totally disconnected. Then in case* b) *of Theorem* 1.7 *there exists a fixed point* $\alpha \in \partial V$ *such that* $f^n z \to \alpha$, $n \to \infty$, *for all* $z \in V$.

In the case when V is bounded by finitely many Jordan curves, no assumptions are needed in Theorem 1.8 about the boundary properties of f. The corresponding result for $V = \mathbf{U}$ is called the *Denjoy-Wolff theorem* ([8], §43), and the point α to which all orbits converge is called the *Denjoy-Wolff point.*

An invariant set V is said to be *absorbing* if it absorbs all orbits. A set D is said to be *fundamental* if it intersects each orbit in a point.

THEOREM 1.9 [66]. *Let $f: \mathbf{U} \to \mathbf{U}$ be analytic endomorphism that does not reduce to a rotation. Assume that its Denjoy-Wolff point α is not superattracting. Then f has a simply connected absorbing domain V to which the restriction of f is univalent. Further, the set $D = V \setminus fV$ is fundamental.*

We remark that if $\alpha \in \mathbf{U}$, then V is simply a small disk about α, and D is an annulus.

CHAPTER 2
TOPOLOGICAL DYNAMICS OF RATIONAL ENDOMORPHISMS

§1. The Julia set: initial notions

1.1. Definition and simplest properties. Let us consider an analytic endomorphism $f: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of the Riemann sphere. It is given by a rational function $z \mapsto P(z)/Q(z)$. Denote by $d = \deg f$ the degree of f, which is equal to $\max(\deg P, \deg Q)$. Each point of the sphere has d inverse images, counting multiplicity.

If $d = 1$, then $f: z \mapsto (az + b)/(cz + d)$ is a Möbius transformation. The classification (hyperbolic, parabolic, and elliptic cases) and investigation of the dynamics of such transformations are elementary and well known (see [12], [29]). Further, we assume unless otherwise stated that $d > 1$.

The *Fatou set* $F = F(f)$ (or *set of normality*) is defined to be the maximal open set on which the family $\{f^n\}$ of iterates is normal. It follows immediately from the definition that $F(f)$ is completely invariant and $F(f^n) = F(f)$ for $n \in \mathbf{N}$. If D is an invariant hyperbolic domain on the sphere, then it follows from Montel's theorem that $D \subset F(f)$.

The *Julia set* $J = J(f)$ is defined to be the complement of the Fatou set. We note at once that attracting cycles lie on the Fatou set, and repelling cycles lie on the Julia set.

THEOREM 2.1 (Fatou, Julia). *The Julia set $J(f)$ is a nonempty perfect completely invariant compact set, and $J(f^n) = J(f)$ for $n \in \mathbf{N}$. Either $J(f)$ is nowhere dense, or $J(f) = \overline{\mathbf{C}}$.*

To illustrate how Montel's theorem works, we prove the last assertion. Suppose that $J(f)$ contains a domain D. Then, by Montel's theorem, $\overline{\bigcup_{n=0}^{\infty} f^n D}$ is the whole sphere, possibly less two exceptional points. But $J(f) \supset \overline{\bigcup_{n=0}^{\infty} f^n D} = \overline{\mathbf{C}}$.

1.2. Examples of Julia sets with simple structure. Julia sets have an extremely varied and complicated structure. Numerous confirmations of this will be given below. However, we now dwell on exceptional examples that nevertheless serve as good models for the dynamics on $J(f)$.

EXAMPLE 2.1. $f_d: z \mapsto z^d$. In this case $J(f) = \mathbf{T}$, the unit circle, and $F(f)$ consists of the two components $\mathbf{U}$ and $\overline{\mathbf{C}} \setminus \mathbf{U}$. In the first of them the orbits tend to zero, and in the second they tend to ∞. The dynamics on $J(f)$ bear a complicated chaotic character, which can be described only in statistical terms (see [46]).

EXAMPLE 2.2. The *Tchebycheff polynomials* $P_d: z \mapsto \cos(d \arccos z)$. They satisfy the functional equation $\cos dz = P_d(\cos z)$. With the help of this equation it can be shown that $J(P_d) = [-1, 1]$. The set $F(P_d)$ consists of a single component, in which all orbits tend to ∞. But chaos is observed on the interval

$[-1, 1]$. This phenomenon for $P_2: z \mapsto 2z^2 - 1$ was discovered by Ulam and von Neumann on one of the first computers (1947).

The first example of a function for which $J(f) = \overline{\mathbf{C}}$ is usually ascribed to Lattès (1918), although it was already known to Boettcher [4]:

EXAMPLE 2.3. Let $\mathfrak{P}$ be the Weierstrass elliptic function with periods $\{1, \tau\}$. Then there exists a rational solution of the functional equation (see [12]) $\mathfrak{P}(nz) = R_{n,\tau}(\mathfrak{P}(z))$, $n \in \mathbf{N}$, $n \geq 2$. From the point of view of dynamics this equation means that the function $\mathfrak{P}$ semiconjugates the transformation $z \mapsto nz$ of the torus and the rational endomorphism $R_{n,\tau}$. Since the repelling cycles of the transformation $z \mapsto nz$ are dense on the torus, the repelling cycles of the endomorphism $R_{n,\tau}$ are dense in $\overline{\mathbf{C}}$. Consequently, $J(R_{n,\tau}) = \overline{\mathbf{C}}$.

1.3. The exceptional set. The *exceptional set* E of a rational endomorphism f is defined to be the largest finite completely invariant set.

THEOREM 2.2 (Fatou, Julia).

a) *The exceptional set exists* (E *can be* $\varnothing$) *and consists of at most two points.*

b) *If* $\operatorname{card} E = 1$, *then* f *is conformally conjugate to a polynomial.*

c) *If* $\operatorname{card} E = 2$, *then* f *is conformally conjugate to* $z \mapsto z^{\pm d}$.

d) $E \subset F(f)$ *and, what is more, the exceptional points are superattracting periodic points* (*of order* 1 *or* 2).

e) *If* V *is a neighborhood of a point* $z \in J(f)$ *disjoint from* E, *then* $\bigcup_{n=0}^{\infty} f^n V = \overline{\mathbf{C}} \setminus E$.

1.4. Denseness of repelling cycles. Fatou showed that a rational endomorphism can have only finitely many nonrepelling cycles. We discuss this result in greater detail in the next section, but now we assume it and prove a theorem that can serve as an alternative definition of the Julia set (and was such for Julia; the approach we are following is due to Fatou).

THEOREM 2.3. (Fatou, Julia). *The Julia set is the closure of the set of repelling periodic points.*

PROOF. Let $a \in J(f)$. We want to approximate a by repelling periodic points. Since $J(f)$ is a perfect set, it can be assumed that a is not itself periodic and not a critical value. Consequently, a has two different inverse images a_1 and a_2 with $a_i \neq a$, and in a neighborhood $V \ni a$ there exist single-valued branches g_1 and g_2 of the function f^{-1} such that $g_i(a) = a_i$. Let $g_0(z) \equiv z$. By Montel's theorem, in every neighborhood of a there is a root α of some equation $f^p(z) = g_i(z)$. The point α is periodic. Application of the theorem on the finiteness of the number of nonrepelling cycles concludes the proof. ●

1.5. Denseness of inverse images and mixing. The next result is easy to derive from Theorem 2.3 and Montel's theorem.

THEOREM 2.4 (Fatou, Julia). *Suppose that* $a \in J(f)$, V *is a neighborhood of* a, *and* K *is a compact set not containing exceptional points. Then there exists an* N *such that* $K \subset f^n V$ *for* $n \geq N$.

COROLLARY 1. *The orbit* $\{f^n z\}_{n=0}^{\infty}$ *is Lyapunov stable if and only if* $z \in F(f)$.

PROOF. The stability of the orbit of a $z \in F(f)$ follows from the equicontinuity of the family $\{f^n\}$ in a neighborhood of z, and the stability of the orbit of a $z \in J(f)$ follows from Theorem 2.4. ●

Note that this corollary can also be taken as the definition of the Fatou set and the Julia set.

COROLLARY 2. *Suppose that* b *is not an exceptional point, and let* $\varepsilon > 0$. *Then there exists an* N *such that for* $n \geq N$ *the complete inverse image* $f^{-n}b$ *is an* ε*-net for the Julia set.*

A continuous transformation f of a set X is said to be *topologically mixing* if for each two neighborhoods U, $V \subset X$ there exists an N such that $f^{-n}U \cap V \neq \varnothing$ for $n \geq N$. Baire category arguments lead to the conclusion that a topologically mixing transformation has a dense orbit (if X is a complete separable metric space).

COROLLARY 3. *A rational endomorphism is topologically mixing on the Julia set.*

The facts presented above give an initial idea of the chaotic unstable nature of the dynamics on the Julia set. A more complete understanding emerges if we look at the problem from the point of view of ergodic theory, to which the next chapter is devoted.

§2. Dynamics on the Fatou set

In this section we present the most complete part of the theory: a description of the dynamics on the Fatou set. A classification of periodic components is given in 2.2–2.5. In 2.6 the theorem on finiteness of the number of such components and the number of nonrepelling periodic points is discussed. In 2.7 a proof of Sullivan's theorem is sketched, and a summary is given in the concluding subsection.

2.1. The components of the Fatou set. Let D be a connected component of the Fatou set. We observe first of all that D is a hyperbolic domain, since its complement $\overline{\mathbf{C}} \setminus D$ contains the continuum Julia set. Further, since $F(f)$ is invariant, fD is contained in some component D_1 of D, and since $J(f)$ is invariant, it follows that $f(\partial D) \subset \partial D_1$. By Lemma 1.1, $f\colon D \to D_1$ is a branched cover. Consequently, f maps D surjectively onto D_1.

THEOREM 2.5 (Fatou, Julia). *Let* $V \neq \varnothing$ *be a completely invariant union of components of the set* $F(f)$ *(in particular, it is possible that* $V = F(f)$*). Then* V *consists of one, two, or countably many components.*

In the examples above, the Fatou set consists of one or two components, but later it becomes clear that $F(f)$ consists of a countable number of components as a rule (for the simplest example $z \mapsto z^2 - 1$ see 2.3).

The next result is closely connected with Theorem 2.5.

THEOREM 2.6 (Fatou, Julia). *Suppose that the set* $F(f)$ *has a completely invariant component* D. *Then*

a) *all the remaining components of* $F(f)$ *are simply connected*;

b) $J(f) = \partial D$. *If there are two completely invariant components* D_1 *and* D_2, *then* $F(f) = D_1 \cup D_2$.

The following conjecture was formulated by Makienko by analogy with a theorem of Abikoff [52] relating to Kleinian groups. Let D_k be the components of $F(f)$. If $J(f) = \bigcup_k \partial D_k$, then among these components there is a completely invariant component.

2.2. Schröder domains and Boettcher domains. We consider an attracting or superattracting cycle $\alpha = \{\alpha_k\}_{k=0}^{p-1}$ of order p. The set $\Delta(\alpha)$ of points whose orbits tend to α is called the *domain of attraction of the cycle.* The domain of attraction is open, but is not connected in general, i.e., is not a domain in the usual understanding. It is not hard to show that $\Delta(\alpha)$ is the union of certain components of the Fatou set. By Theorem 2.5, the number of these components is equal to 1, 2, or ∞.

The components $D(\alpha_k)$ of the set $\Delta(\alpha)$ containing the points α_k will be called *Schröder domains* if α is an attracting cycle, and *Boettcher domains* if α is a superattracting cycle. By the corollary to Theorem 1.7, the Schröder domains (respectively, Boettcher domains) $D(\alpha_k)$ are pairwise distinct. These domains can be constructed with the help of the following iteration construction. Let B_0 be a small disk about α_k, and let B_n be the component of the complete inverse image $f^{-pn}B_0$ containing α_k. Then $B_0 \subset B_1 \subset \cdots$, and $\bigcup_{n=0}^{\infty} B_n = D(\alpha_k)$.

The union $\bigcup_{k=0}^{p-1} D(\alpha_k)$ is called the *domain of immediate attraction* of the cycle α. We formulate a result that, despite its simplicity, is one of the central points in the classical theory.

THEOREM 2.7 (Fatou, Julia). *The domain of immediate attraction $D(\alpha)$ of an attracting cycle α contains a critical point whose orbit is not absorbed by the cycle α.*

PROOF. *Method* 1 gives a proof of only the first part of the theorem. If $D(\alpha)$ does not contain critical points, then $f^p\colon D(\alpha_k) \to D(\alpha_k)$ is an unbranched cover. Consequently, f^p is a locally isometric transformation in the Poincaré metric of the domain $D(\alpha_k)$. On the other hand, $\|Df^p(\alpha_k)\| < 1$.

Method 2. We use an iteration construction of the Schröder domain: $D(\alpha_k) = \bigcup_{n=0}^{\infty} B_n$. If the conclusion of the theorem does not hold, then in view of Lemma 1.2 the mapping $f^{pn}\colon B_n \to B_0$ is an analytic isomorphism. Let us consider the inverse mapping $f^{-pn}\colon B_0 \to B_n$. By Montel's theorem, it is normal, contradicting the fact that $\|Df^{-pn}(\alpha_k)\| \to \infty$, $n \to \infty$.

See [29] about *Method* 3, which uses a Koenigs function. ●

The Riemann-Hurwitz formula gives us

COROLLARY 1. *Schröder domains and Boettcher domains are either simply connected or infinitely connected.*

COROLLARY 2. *A rational endomorphism $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of degree d has at most $2d-2$ attracting and superattracting cycles.*

The question of whether this estimate is sharp, which remained open for almost 60 years, was recently solved by Shishikura [153].

EXAMPLE 2.4. Let p and $d-1$ be relatively prime positive integers, and let $\lambda = \exp(2\pi i/p)$. We consider the family of functions

$$f_\varepsilon(z) = z\frac{(1+\varepsilon)+z^{d-1}}{1+(1-\varepsilon)\lambda z^{d-1}}, \qquad \varepsilon > 0.$$

Obviously, the functions f_ε commute with the rotation $z \mapsto z\exp(2\pi i/(d-1))$. A simple analysis shows that if ε is sufficiently small, then f_ε has $d-1$ attracting fixed points near zero, and $d-1$ attracting cycles of order p near ∞. Thus, f_ε has $2d-2$ attracting cycles in all.

The analogue of Theorem 2.7 for Boettcher domains is

THEOREM 2.8 (see [29], §1.9). *Let* $D(\alpha) = \bigcup_{k=0}^{p-1} D(\alpha_k)$ *by a cycle of Boettcher domains. Then the following alternative holds:*

a) *the transformation* $f^p : D(\alpha_k) \to D(\alpha_k)$ *is conformally conjugate to the transformation* $z \mapsto z^n$ *of the disk* $\mathbf{U}$, $n \geq 2$;

b) *one of the domains* $D(\alpha_k)$ *contains a critical point whose orbit is not absorbed by the cycle* α.

With each attracting cycle $\alpha = \{\alpha_k\}_{k=0}^{p-1}$ we can associate a torus with marked points. Namely, let S be the space of large orbits lying in the set $\Delta_0(\alpha) = \Delta(\alpha) \setminus \bigcup_{n=0}^{\infty} f^{-n}\alpha$. The space S is equipped with the natural structure of a Riemann surface such that the projection $\pi : \Delta_0(\alpha) \to S$ is an analytic branched cover. The branch points of this cover are critical points of the functions f^n, $n \in \mathbf{N}$. Consequently, there are only finitely many projections of the branch points, and these we mark on S. To understand the topology of S we observe that S is obtained from a fundamental annulus (see Chapter 1, §4) by gluing together the boundary components. Thus, S is a torus.

The construction presented here is completely analogous to the construction of the Riemann surface associated with a Kleinian group. In the case of superattracting cycles it does not work, because in Boettcher domains the large orbits are nondiscrete. It is easy to see with the help of Theorem 1.2 that there exists in each Boettcher domain a foliation with singularities (at critical points of the functions f^n) such that each large orbit densely fills in a countable number of leaves. Nevertheless, Sullivan [140] associated with each cycle of Boettcher domains a Riemann surface (a small annulus included between two leaves, or a punctured disk) with marked points (the traces of the large orbits of critical points) and an additional structure determining a foliation: the action of the rotation group.

2.3. The domain of attraction of infinity for a polynomial. A polynomial is a rational endomorphism for which $f^{-1}\infty = \{\infty\}$. Consequently, ∞ is a superattracting fixed point that is exceptional in the sense of 1.3. The entire specific nature of polynomial endomorphisms is connected with the existence of such a point.

THEOREM 2.9. (Fatou, Julia). *Let* f *be a polynomial. Then the Boettcher domain* $D(\infty)$ *is a complete invariant and coincides with the domain of attraction* $\Delta(\infty)$. *Further,* $J(f) = \partial D(\infty)$, *and all the bounded components of the Fatou set are simply connected.*

PROOF. Complete invariance holds because each component of the complete inverse image $f^{-1}D(\infty)$ must contain a pole. The remaining assertions follow from Theorem 2.6. ●

For a polynomial, it is easy to determine from the behavior of the orbits of critical points whether the Julia set is connected.

THEOREM 2.10 (Fatou, Julia). *The Julia set of a polynomial is connected if and only if the orbits of all finite critical points do not tend to* ∞. *Further, the transformation* $f|D(\infty)$ *is conformally conjugate to the transformation* $z \mapsto z^d$ *of the disk* $\mathbf{U}$.

PROOF. The connectedness of the Julia set is equivalent to the simple connectedness of the domain $D(\infty)$. Applying the Riemann-Hurwitz formula and Theorem 2.8 to this domain, we get what is needed. ●

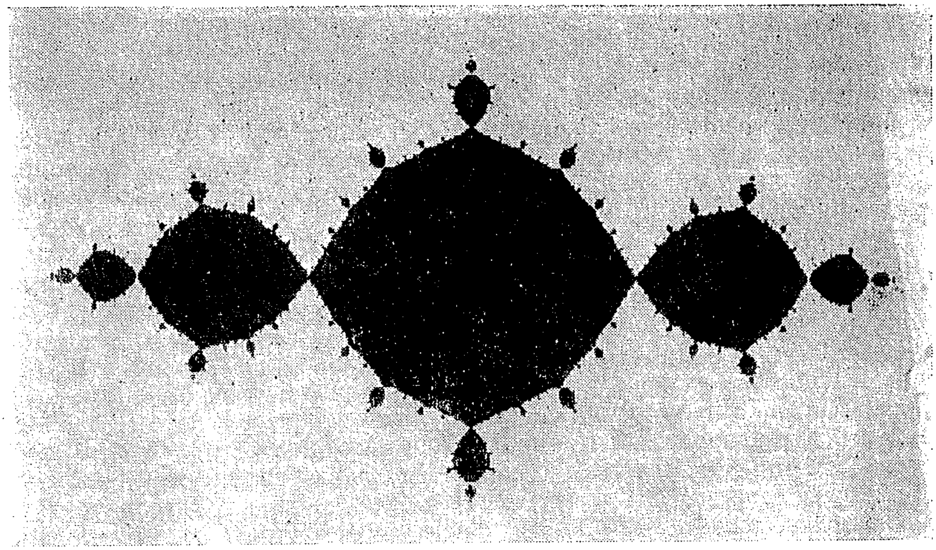

FIGURE 4. Dynamics of the polynomial $z \mapsto z^2 - 1$

EXAMPLE 2.5. $f\colon z \mapsto z^2 - 1$ (Figure 4). The points 0 and -1 form a superattracting cycle of second order. Consequently, the Julia set is connected. The Boettcher domains $D(0)$ and $D(-1)$ are simply connected, and f maps $D(-1)$ univalently onto $D(0)$. Consequently, there exists a component $D \neq D(-1)$ that is also mapped univalently onto $D(0)$. There are also two inverse images for the component D—the components D_1 and D_2, and so on. Thus, the domain of attraction of the cycle $\alpha = \{0, -1\}$ consists of countably many components. It will be clear later that the Fatou set $F(f)$ is the union $\Delta(\alpha) \cup D(\infty)$ (see §3, Theorem 2.19).

2.4. The Leau bouquet. First of all we note that the neutral rational cycles lie on the Julia set, since they are not Lyapunov stable (see the corollaries to Theorems 1.4 and 2.4). With each point of a cycle it is possible to associate a Leau flower in such a way that their union is invariant. This union will be called a *Leau bouquet.* The local dynamics in a neighborhood of the cycle can be described in a natural way with the help of the Leau bouquet.

The domain of attraction $\Delta(\alpha)$ of a neutral rational cycle is defined to be $\{z\colon f^n z \to \alpha,\ n \to \infty\} \setminus \bigcup_{n=0}^{\infty} f^{-n}\alpha$. It follows from the local picture that the orbit of an arbitrary point $z \in \Delta(\alpha)$ is absorbed by the Leau bouquet. Consequently, $\Delta(\alpha)$ is nonempty, open, and contained in the Fatou set (this is the reason the neutral rational cycles are important for understanding the dynamics on $F(f)$). The components of the set $\Delta(\alpha)$ containing the Leau petals will be called *Leau domains*, and their union $D(\alpha)$ will be called the domain of immediate attraction of the cycle α. Each Leau domain contains one Leau petal.

Associated with each cycle V of Leau domains is a Riemann surface S (the space of large orbits of the restriction $f|V$) with marked points (the orbits of the critical points). The natural projection $\pi\colon V \to S$ is a branched cover that branches at the critical points and their inverse images of all orders. Linearization with the help of an Abel function (see Chapter 1, 3.3) shows that $S = \mathbf{C}^*$.

THEOREM 2.11 ([81], §30). *Each cycle of Leau domains contains a critical point of f.*

PROOF. In the contrary case $\pi: V \to S$ is an unbranched cover, which is impossible, since V is hyperbolic and S parabolic. ●

The following result, which follows from Theorem 1.6, gives an important refinement of Theorem 1.8.

THEOREM 2.12. *Let f be a rational endomorphism, and D a periodic component of $F(f)$ in which all orbits tend to the cycle of a periodic point $\alpha \in \partial D$. Then α is a neutral rational point, and D is its Leau domain.*

Neutral rational cycles can be removed by a small perturbation of the function. However, their study is important for bifurcation theory (see §7).

2.5. Siegel disks and Arnol′d-Herman rings. Let α be a neutral irrational cycle. The terms *Siegel cycle* and *Siegel periodic point* do not need additional explanation (see Chapter 1, 3.2). It follows from Theorem 2.4 that α is a Siegel cycle if and only if $\alpha \in F(f)$. The components of $F(f)$ containing the points α_k are called the *Siegel disks* $D(\alpha_k)$. We use the corollary to Theorem 1.7.

THEOREM 2.13. *The Siegel disks $D(\alpha_k)$ are simply connected. The transformation $f^p: D(\alpha_k) \to D(\alpha_k)$ is conformally conjugate to an irrational rotation of the disk* $\mathbf{U}$.

It remains for us to consider one more type of periodic domain, which is related to Siegel disks. A periodic domain D of order p is called an *Arnol′d-Herman ring* if it is doubly connected, and the transformation $f^p: D \to D$ is conformally conjugate to an irrational rotation of the standard annulus $\mathbf{A}(r, 1)$. Siegel disks and Arnol′d-Herman rings are also called *singular domains.* In contrast to the types of domains considered above, Arnol′d-Herman rings are not connected with periodic points. This makes their study more difficult. We remark also that in view of Theorem 2.9 polynomials do not have Arnol′d-Herman rings.

EXAMPLE 2.6 (Herman [91]). $f: z \mapsto e^{2\pi i\theta} z^2(1-\bar{a}z)/(z-a)$, $|a| < 1$. The circle $\mathbf{T}$ is f-invariant, and for small $|a|$ the restriction of $f|\mathbf{T}$ is a diffeomorphism. It follows from a theorem of Arnol′d ([3], §12) that for suitable θ and a there exists an invariant neighborhood V of $\mathbf{T}$ on which the transformation $f: V \to V$ is conformally conjugate to a rotation of an annulus. The component of $F(f)$ containing V is an Arnol′d-Herman ring.

A Riemann surface is also naturally associated with the cycle of a singular domain: a disk or annulus with marked points (the traces of large orbits of critical points) and with the action of the rotation group ([140]; see also [29]).

THEOREM 2.14 (see [29], §1.14). a) *Let D be a singular domain. Then ∂D is contained in $\bigcup \omega(c)$, where c runs through the set of critical points lying in $J(f)$.*

b) *If α is a non-Siegel neutral cycle, then there exists a critical point $c \in J(f)$ such that $\alpha \subset \omega(c) \setminus \bigcup_{n=0}^{\infty} f^n c$.*

COROLLARY ([81], §30). *If the closure of the union of the orbits of the critical points does not separate the plane, then there are no singular domains.*

Theorem 2.14 can evidently be refined as follows:

a) each component of the boundary of an invariant singular domain with multiplier satisfying (1.2)([3]) contains a critical point;

([3])Herman [149] showed that this condition is essential.

b) if α is a non-Siegel neutral cycle, then there exists a critical point c whose orbit tends to α.

THEOREM 2.15 [92]. *Suppose that* $f\colon z \mapsto z^d + c$ *and* $\{D_k\}_{k=0}^{p-1}$ *is a cycle of Siegel disks with multiplier satisfying* (1.2) ([3]). *Then* $0 \in \bigcup \partial D_k$.

2.6. The finiteness theorem. Theorems 2.7 and 2.11 show that the total number of cycles of Schröder domains, Boettcher domains, and Leau domains does not exceed $2d-2$. Fatou established that, in fact, the total number of non-repelling cycles is finite ([81], §30). With this goal he perturbed the function in such a way that half its neutral cycles became attracting. Sullivan [140] proved the finiteness theorem for Arnol'd-Herman rings by quasiconformal deformation techniques. However, the question of sharp estimates for the number of cycles of various types remained open until very recently, when it was answered by Shishikura [153].

Let N_A be the number of attracting cycles, N_L the number of cycles of Leau domains (there can be more of them than of neutral rational cycles), N_I the number of neutral irrational non-Siegel cycles, N_S the number of cycles of Siegel disks, and N_{AH} the number of cycles of Arnol'd-Herman rings.

THEOREM 2.16 [153]. *The following estimates hold*:

$$N_A + N_L + N_I + N_S + 2N_{AH} \leq 2d - 2, \qquad N_{AH} \leq d - 2.$$

Each collection of numbers satisfying these estimates can be realized for some rational function.

We discuss a proof of Shishikura's theorem in §9.

2.7. The absence of wandering components. A set X is said to be *wandering* if $f^n X \cap f^m X = \varnothing$ for $n > m \geq 0$. The theoretical possibility of the existence of wandering components of the Fatou set $F(f)$ was the main difficulty in investigating the dynamics of $F(f)$. Sullivan removed this obstacle at the beginning of the 1980s.

A *quasiconformal deformation* of an endomorphism f is defined to be a family of rational endomorphisms $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$ such that $f_{t_0} = f$, and $\varphi_t\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a family of quasiconformal homeomorphisms that depend continuously on the point t in some manifold (perhaps infinite-dimensional).

THEOREM 2.17 ([136], [139]). *The Fatou set* $F(f)$ *of a rational endomorphism does not have wandering components.*

SKETCH OF THE PROOF. Let D be a wandering component, and let $\mathscr{O} = \bigcup_{n=0}^{\infty} f^{-n}(\bigcup_{m=0}^{\infty} f^m D)$. We consider the space S of large orbits lying in $\mathscr{O}$. With one exception (which we discuss below) S can be equipped with the structure of a Riemann surface in such a way that the neutral projection $\pi\colon \mathscr{O} \to S$ is a branched cover. It can be shown that S is a surface of infinite conformal type. Consequently, the Teichmüller space T_S is infinite-dimensional, and hence for each N there exists a continuous family of conformal structures μ_t on S $(t \in \mathbf{R}^N)$ such that the mapping $\mu_t \mapsto \overline{\mu}_t \in T_S$ is injective.

Each structure μ_t can be lifted to an f-invariant conformal structure on $\mathscr{O}$, and then it can be extended to the whole Riemann sphere in the standard way. We obtain a family ν_t of f-invariant conformal structures on $\overline{\mathbf{C}}$. By the measurable Riemann theorem, there exists a continuous family of quasiconformal homeomorphisms $\varphi_t\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that $(\varphi_t)_* \nu_t = \sigma$. Then the endomorphisms $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$ preserve the standard structure σ, and thus are rational.

We have constructed a quasiconformal deformation f_t, $t \in \mathbf{R}^N$, of the endomorphism f. If $N > 2d-2$, then there must exist a continuum $X \subset \mathbf{R}^N$ such that $f_t = f_\tau$ for $t, \tau \in X$. It remains to show that there are no such continua. This follows easily from the lemma below. ●

The exception mentioned in the beginning is a wandering annulus D such that $\deg(f^n|D) \to \infty$, $n \to \infty$. Fortunately, a contradiction can be reached by elementary means in this case ([139]; see also [29], [142]).

In conclusion we formulate the lemma promised above. Let G_J be the group of homeomorphisms of $J(f)$ that commute with f.

LEMMA 2.1. *The group* G_J *is totally disconnected.*

2.8. The complete picture of the dynamics.

THEOREM 2.18 [136]. *An arbitrary orbit in the Fatou set is absorbed by some cycle of Schröder domains, Boettcher domains, Leau domains, Siegel disks, or Arnol′d-Herman rings. There exist only finitely many cycles of components.*

PROOF. By Theorem 2.17, an arbitrary orbit in the Fatou set is absorbed by some cycle of components $\{f^n D\}_{n=0}^{p-1}$. Theorem 1.7 is applicable to the transformation $f^p\colon D \to D$. In case a), D is a Schröder domain or a Boettcher domain. In case b), D is a Leau domain, as follows from Theorems 1.8 and 2.12. In cases (i) and (iii), D is a singular domain. The case (ii) is excluded, because the Julia set is perfect, and d) is excluded for obvious reasons.

Thus, a rational endomorphism on the Fatou set admits five different types of dynamics. ●

§3. Axiom A

In this section we present initial notions about rational endomorphisms satisfying Axiom A. The importance of such endomorphisms has to do with the fact that they are "often encountered" and "cannot be removed by small perturbations" (see §6). The study of these endomorphisms is made easier by the fact that many ideas and methods in the general theory of dynamical systems are directly applicable to them. This theme will be developed more deeply in Chapter 3.

3.1. Definition. A rational endomorphism f is said to be *expanding* on an invariant compact set if there exist a neighborhood $V \supset X$ and constants $C > 0$ and $\lambda > 1$ such that:

(i) $f^{-1}X \cap V = X$;

(ii) $\|Df^n(z)\| \geq C\lambda^n$, $z \in X$, $n \in \mathbf{N}$.

We discuss this concept from more general points of view in Chapter 3.

THEOREM 2.19 ([81], §31). *The following properties of a rational endomorphism* f *are equivalent*:

a) f *is expanding on the Julia set.*

b) *The orbits of its critical points tend to attracting or superattracting cycles.*

Further, the orbits of all points in the Fatou set tend to attracting or superattracting cycles.

PROOF. The implication a)⇒ b) and the concluding part of the theorem follow from the description of the dynamics on the Fatou set (although the complete picture is not necessary, of course: Fatou did not know it). To prove the implication b)⇒a) we remove the orbits of the critical points from the sphere, along with invariant neighborhoods of the attracting cycles. The domain S so obtained is hyperbolic and f^{-1}-invariant. ●

The multivalued mapping $f^{-1}: S \to S$ can be lifted to a single-valued mapping of the universal covering $\mathbf{U}$. Using Schwarz's lemma, we get that $f|J$ is expanding in the hyperbolic metric ρ_S, and hence in the spherical metric.

One says that a rational endomorphism f satisfies *axiom* A, or is *hyperbolic*, if properties a) and b) of Theorem 2.19 hold. This terminology was taken from the general theory of dynamical systems (see [38]).

3.2. Symbolic dynamics on a Cantor Julia set. Symbolic dynamics is an encoding of orbits by sequences of symbols. We dwell in greater detail on this topic in Chapter 3; here, however, (and in the next subsection) we consider the simplest situations, which were known already to Fatou ([80], p. 252).

Let Σ_d^+ be the space of one-sided sequences $\{\beta_i\}_{i=0}^{\infty}$ in d symbols. Equipped with weak topology, it becomes a Cantor set. The (one-sided topological) *Bernoulli shift* is defined to be the transformation $\sigma: (\beta_0\beta_1\cdots) \mapsto (\beta_1\beta_2\cdots)$ of Σ_d^+.

THEOREM 2.20 ([81], [63]). *Assume that a rational endomorphism f satisfies axiom A and that the Fatou set $F(f)$ consists of a single component. Then the Julia set $J(f)$ is a Cantor set, and the restriction $f|J$ is topologically conjugate to the Bernoulli shift σ.*

The idea of the proof is to find an f^{-1}-invariant neighborhood Δ of J such that $f^{-1}\Delta = \bigcup_{i=1}^{d}\Delta_i$, where $\overline{\Delta}_i \cap \overline{\Delta}_j = \varnothing$, $i \neq j$, and f maps Δ_i univalently onto Δ. We now encode the trajectory of an arbitrary point $z \in J$ by a sequence of d symbols as follows: $z \mapsto (\beta_0\beta_1\cdots)$ if $f^n z \in \Delta_{\beta_n}$, $n \in \mathbf{N}$. This encoding gives the desired conjugacy of $f|J$ and σ.

In particular, for quadratic polynomials $f_c: z \mapsto z^2+c$ the Julia set is either a Cantor set or a connected set, depending on whether or not the orbit $\{f_c^n(0)\}_{n=0}^{\infty}$ tends to ∞ (see 2.3). If c is real, then it is easy to determine which situation is realized: for $c \in [-2, 1/4]$ the Julia set $J(f_c)$ is connected, and outside this interval it is a Cantor set (further, if $c < -2$, then $J(f_c)$ is "linear", i.e., lies on the line $\mathbf{R}$, while if $c > 1/4$, then $J(f_c)$ is a "planar" Cantor set). And what is the structure of the set of complex values c for which $J(f_c)$ is connected? See §7 about this.

3.3. Invariant quasicircles. A *quasicircle* is defined to be a Jordan curve that is a quasiconformal image of the circle $\mathbf{T}$.

THEOREM 2.21 ([82], [51]). *Let $U \subset V$ be two annuli bounded by smooth Jordan curves, and let $f: U \to V$ be a d-sheeted analytic cover. Then the set $J = \{z: f^n z \in U,\ n \in \mathbf{N}\}$ is a quasicircle, and $f|J$ is topologically conjugate to the transformation $z \mapsto z^d$ of the circle $\mathbf{T}$.*

It can be deduced from this theorem that if f satisfies axiom A and has a completely invariant component of its Fatou set, then the remaining components are bounded by quasicircles. In particular, if f consists of two components, then $J(f)$ is a quasicircle. The simplest illustration is the polynomial $z \mapsto z^2 + c$ for small $|c|$. How small? See §7.

In conclusion we mention that the Julia set can contain a continuum of components, each a Jordan curve:

EXAMPLE 2.7 (McMullen [109]). $f\colon z \mapsto z^2 + \varepsilon z^{-3}$. For small $|\varepsilon|$ the set $J(f)$ is homeomorphic to the product of a Cantor set and the circle.

3.4. Quasihyperbolic endomorphisms. A rational endomorphism is said to be *quasihyperbolic* if the orbit of each critical point tends to an attracting cycle or is absorbed by a repelling cycle. The simplest nonhyperbolic examples are the Tchebycheff polynomials and the Lattès endomorphisms (§1.2). The next result is a justification for the name.

THEOREM 2.2. ([76], [142]). *Let f be a quasihyperbolic rational endomorphism. Then in a neighborhood of the Julia set $J(f)$ there exists a Riemannian metric γ with singularities at a finite number of points x_j such that $\|Df(z)\|_\gamma \geq \lambda > 1$ for $z \in J(f) \setminus \{x_j\}$. All orbits on the Fatou set tend to attracting cycles.*

An important class of quasihyperbolic endomorphisms will be considered in more detail in §5.

§4. The boundaries of invariant components of the Fatou set

In this section D is an *invariant simply connected component of the Fatou set,* and $\psi\colon \mathbf{U} \to D$ is a conformal mapping. The study of the boundary ∂D is equivalent to the study of the boundary properties of ψ.

4.1. Local connectedness. By Carathéodory's theorem ([12], [76]), the following conditions are equivalent:

a) *The boundary ∂D is locally connected.*

b) *The conformal mapping $\psi\colon \mathbf{U} \to D$ is continuous up to the boundary.*

c) *∂D is a closed curve.*

THEOREM 2.23 (Fatou, 1923). *Let D be an invariant simply connected component of the Fatou set $F(f)$. If the endomorphism $f|\partial D$ is expanding, then ∂D is a curve.*

This curve can have an extremely complicated structure, as shown by the example $J(f) = \partial D(\infty)$ for a polynomial f (Figures 1, 2), although there are also cases of a simple topology (see 3.3).

Theorem 2.23 can be extended to the quasihyperbolic case. However, it is false in the general case.

EXAMPLE 2.8 ([29], [73]). $f\colon z \mapsto e^{2\pi i\theta} z + z^2$, where θ is such that the neutral fixed point 0 is not a Siegel point. Then the Julia set $J(f) = \partial D(\infty)$ is not locally connected.

4.2. Blaschke products. The endomorphism $f\colon D \to D$ is conformally conjugate to the finite-sheeted branched cover $B = \psi^{-1} \circ f \circ \psi\colon \mathbf{U} \to \mathbf{U}$. Such covers are easy to describe: they are the so-called *Blaschke products.*

$$B(z) = \lambda \prod_{k=1}^{d} \frac{z - a_k}{1 - \bar{a}_k z}, \qquad |\lambda| = 1, \ |a_k| < 1.$$

We arrive naturally at the problem of classifying Blaschke products from the point of view of dynamics. Fatou devoted the whole third chapter of the first memoir [80] to this problem.

Since the domains $\mathbf{U}$ and $\overline{\mathbf{C}} \setminus \overline{\mathbf{U}}$ are B-invariant, we have $J(B) \subset \mathbf{T}$, and the set $F(B)$ consists of one or two components. By the Riemann-Hurwitz formula, each of $\mathbf{U}$ and $\overline{\mathbf{C}} \setminus \overline{\mathbf{U}}$ contains $d-1$ critical points. Thus, all the critical points are contained in the Fatou set. According to Theorem 1.8, the orbits of a point $z \in \mathbf{U}$ tend to a fixed point $\alpha \in \overline{\mathbf{U}}$. By the symmetry principle, the orbits of points $z \in \overline{\mathbf{C}} \setminus \overline{\mathbf{U}}$ tend to $\overline{\alpha}^{-1}$. The following cases are possible (under the assumption that $d > 1$):

1) $\alpha \in \mathbf{U}$ is an attracting point. Then $J(B) = \mathbf{T}$, and $F(B)$ consists of two completely invariant components. Further, B satisfies axiom A.

2) $\alpha \in \mathbf{T}$ is an attracting point. Here $F(B)$ consists of a single component, and B satisfies axiom A. By Theorem 2.20, $J(B)$ is a Cantor set on the circle.

3) $\alpha \in \mathbf{T}$ is a neutral point. Since the transformation $B: \mathbf{T} \to \mathbf{T}$ preserves orientation, $B'(\alpha) = 1$. The Leau flower of the point α can consist of one or two petals. In the first case $J(B)$ is a Cantor set on the circle, and in the second $J(B) = \mathbf{T}$.

4.3. Nonsmoothness. If the boundary ∂D is an analytic curve (closed or not), then the conformal mapping $\psi: \mathbf{U} \to D$ is analytic in a neighborhood of $\overline{\mathbf{U}}$. Fatou showed that ψ actually extends to a meromorphic function on $\mathbf{C}$ and, investigating the equation $f \circ \psi = \psi \circ B$ in the class of meromorphic functions, proved the following theorem.

THEOREM 2.24 [82]. *If the boundary of an invariant component of the Fatou set is an analytic curve, then this curve is a circle or an arc of a circle. In the first case the endomorphism $f: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is conformally conjugate to the Blaschke product, and in the second it is conformally conjugate to a Tchebycheff polynomial.*

THEOREM 2.25 [82]. *Let α be a repelling periodic point on the boundary of an invariant simply connected component D of $F(f)$. If f is not an exceptional function in Theorem 2.24, then ∂D does not have a tangent at the point α.*

PROOF. It can be assumed that α is a fixed point. We linearize f in a neighborhood of α, using Theorem 1.1. A piece of ∂D passes into a continuum invariant under the transformation $z \mapsto \lambda^{-1} z$, where $|\lambda| > 1$. It is easy to see that if such a continuum has a tangent at zero, then it is a closed line segment. Consequently, ∂D contains an analytic arc. Then the whole boundary ∂D is an analytic curve, and Theorem 2.24 is applicable. ●

COROLLARY. *If the Julia set contains an isolated arc of a smooth curve, then f is one of the exceptional functions mentioned in Theorem 2.24.*

The requirement that the arc be isolated is essential. For example, if the polynomial $f_a: z \mapsto az(1-z)$, where $a \in (0, 4)$, does not have an attracting cycle (and this very often happens (see [65])), then $J(f_a) \supset [0, 1]$.

The last corollary can be strengthened in the case when f satisfies axiom A, and $J(f)$ is a Jordan curve different from a circle: at every point this curve does not have a tangent, and, in particular, is not rectifiable (see [63]).