Attractors of quadratic polynomials with an irrationally indifferent fixed point

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Introduction

$$P_\alpha(z) = e^{2\pi\alpha\mathbf{i}} z + z^2 : \mathbb{C} \to \mathbb{C}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}$$
$$P_\alpha(0) = 0, \;\; P'_\alpha(0) = e^{2\pi\alpha\mathbf{i}},$$

The study of local, semi-local, and global dynamics of P_α has a rather rich history. This usually involves the arithmetic of α.

Still, the dynamics of some P_α are not understood at all.

Here, we look at the measurable dynamics of P_α on its Julia set.
This is partly motivated by

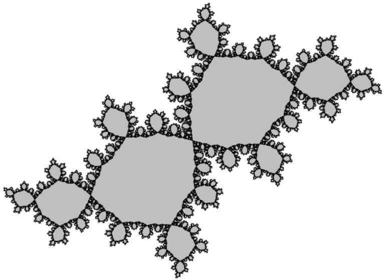
Theorem (Buff-Cheritat 2005)

$\exists\alpha$ *(of Brjuno and non-Brjuno type) s.t.* $J(P_\alpha)$ *has positive area.*

linearizable maps

P_α is called linearizable at 0, if there exists a conformal coordinate ϕ near 0 s.t. $\phi \circ P_\alpha \circ \phi^{-1}(z) = e^{2\pi\alpha\mathbf{i}} z$.

Example: $\alpha = [1, 1, 1, \dots]$



When P_α is linearizable, the maximal domain of linearization is called the Siegel disk of P_α, and α is called a Brjuno number.

Main results

To understand the long term behaviour of the orbits in the Julia set, one needs to understand the orbit of the critical point, and the iterates of the map on and near it.

The post-critical set of P_α is $\mathcal{PC}(P_\alpha) := \overline{\bigcup_{i\geq 1} P_\alpha^i(c.p.)}$

With $\alpha := [a_0, a_1, a_2, \dots]$, we show that

Theorem

$\exists N$, such that for α with all $a_i > N$, $\mathcal{PC}(P_\alpha)$ has zero area. Moreover, $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$ is non-uniformly porous, where Δ_α is the maximal domain of linearization.

Although P_α on $\mathcal{PC}(P_\alpha)$ is not minimal,

Theorem

For α with all $a_i > N$, $\omega(z) = \mathcal{PC}(P_\alpha)$, for a.e. z in $J(P_\alpha)$.

Basic tools

- Inou-Shishikura Renormalization:

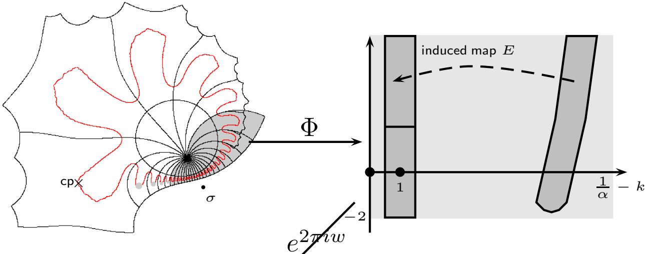
  Indeed, the compactness of a sequence of normalized return maps define on a sector containing the critical point.

- Good estimates on the perturbed Fatou coordinates:

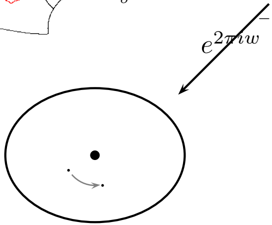
  How perturbed Fatou coordinates degenerate and converge to the Fatou coordinates.

Renormalization: Inou-Shishikura

$h_0(z) = e^{2\pi\alpha\mathbf{i}} z + \cdots$

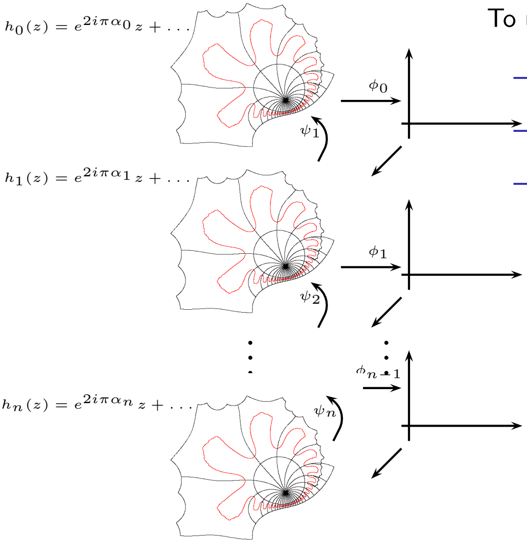


$h_1 := \mathcal{R} h_0$



$E(z+1) = E(z) + 1$

Renormalization tower



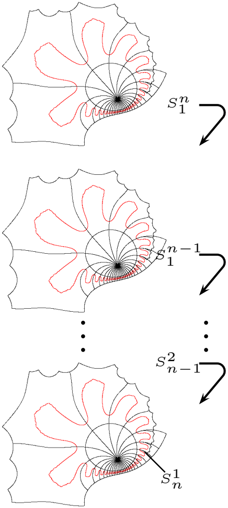
To use the tower, one needs

- Estimates on Φ_i
- Combinatorics of the tower
- Arithmetic of α

Domains where many iterates are understood

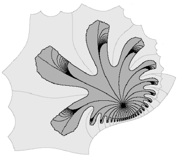
Using the tower, one introduces a nest of topological disks

$$\Omega^0 \supset \Omega^1 \supset \Omega^2 \supset \cdots \supset \mathcal{PC}(P_\alpha).$$

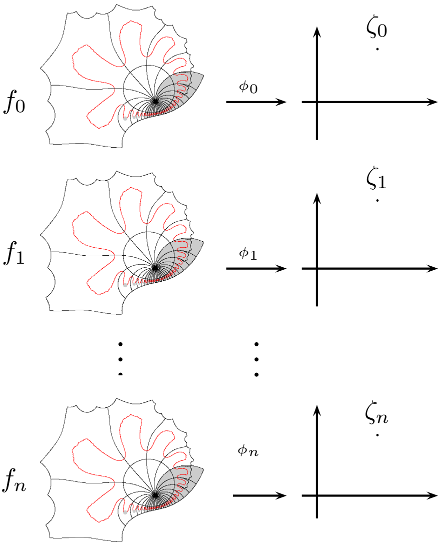


$$\Omega^0 := \bigcup_{j=0}^{a_0+k} P_\alpha^j(S_0^1)$$

$$\Omega^n := \bigcup_{j=0}^{q_{n+1}+\ell q_n} P_\alpha^j(S_1^n)$$



Proof of the main theorem



– Given $z \in \cap\Omega^n$, we obtain a sequence of points ζ_n in the tower

-$\{\mathrm{Im}\,\zeta_n\}_n$ highly depends on the arithmetic of α

Two types of points in the tower

For some constant D, define

$$L := \{z \in \cap_{n=0}^{\infty} \Omega_0^n \cap \mathcal{PC}_\alpha \mid \exists\ \infty\text{-ly many } m \text{ with } \operatorname{Im} \zeta_m < \frac{D}{\alpha_m}\},$$

$$\Gamma := \{z \in \cap_{n=0}^{\infty} \Omega_0^n \cap \mathcal{PC}_\alpha \mid \exists K \text{ s.th. } \forall m \geq K, \operatorname{Im} \zeta_m \geq \frac{D}{\alpha_m}\}.$$

Proposition

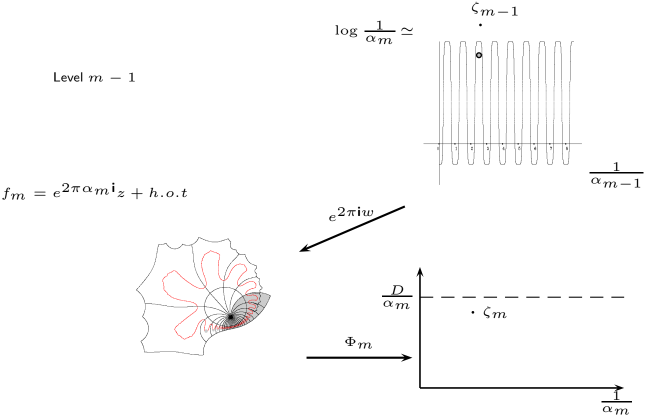
(1) $\mathcal{PC}(P_\alpha)$ *is non-uniformly porous at every point in* L.

(2) Γ *has zero area.*

Proposition

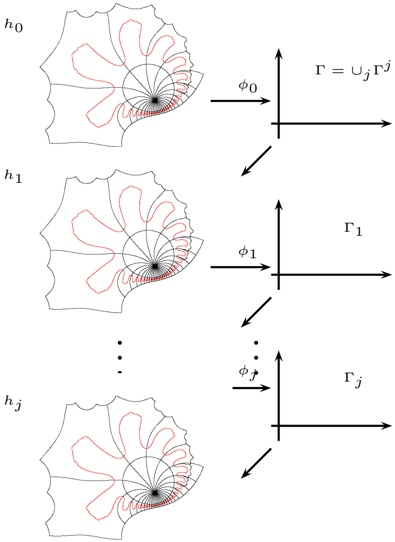
When $0 \in \mathcal{PC}(P_\alpha)$, $\mathcal{PC}(P_\alpha) \setminus \{0\} \subseteq L$.
When $0 \notin \mathcal{PC}(P_\alpha)$, $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha \subseteq L$, *where* Δ_α *is the Siegel disk of* P_α.

On the set L



ζ_{m-1}
$\log \frac{1}{\alpha_m} \simeq$
Level $m-1$
$\frac{1}{\alpha_{m-1}}$
$f_m = e^{2\pi\alpha_m \mathbf{i}} z + h.o.t$
$e^{2\pi \mathbf{i} w}$
$\frac{D}{\alpha_m}$
ζ_m
Φ_m
$\frac{1}{\alpha_m}$

On the set Γ



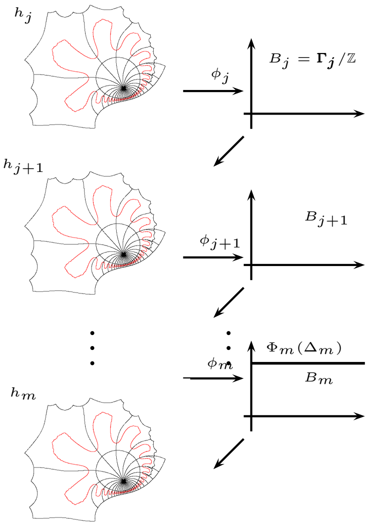
$\Gamma := \cup_{j=0}^{\infty} \Gamma^j$, where

$$\Gamma^j = \{z \in \mathcal{PC}_\alpha \mid \forall m \geq j, \operatorname{Im} \zeta_m \geq \frac{D}{\alpha_m}\}$$

Γ^j projects to Γ_j on level j

$\operatorname{area} \Gamma^j = 0$ iff $\operatorname{area} \Gamma_j = 0$

On the set Γ_j



$\text{area}\,\Gamma_j = 0$

define $b_i := \sup_{w,w' \in B_i} \mathrm{Im}(w - w')$

claim: $\sup_i b_i < \infty$

proof: step 1 $b_{i-1} \le \alpha_i b_i + err$

step 2 $b_m \le \log \frac{1}{\alpha_{m+1}} + \alpha_{m+1} \log \frac{1}{\alpha_{m+2}} + \ldots$

$\Rightarrow b_l \le \alpha_l \alpha_{l+1} \cdots \alpha_m (\log \frac{1}{\alpha_{m+1}} + \alpha_{m+1} \log \frac{1}{\alpha_{m+2}} + \ldots) + ERR$

$b_l \le$ (the tail of the Brjuno sum for α_l) $+ ERR$

The estimate

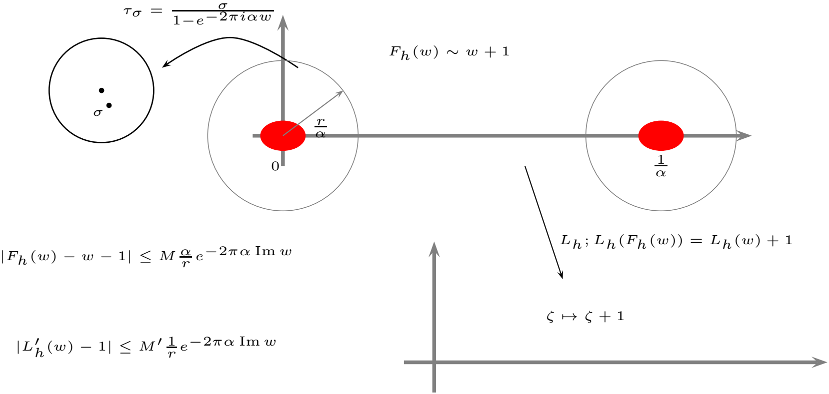
Proposition

$\forall D > 0$, $\exists C$ s.t. $\forall h$ in the Inou-Shishikura class, and $\forall w$ with $\operatorname{Im} w > D/\alpha$, we have

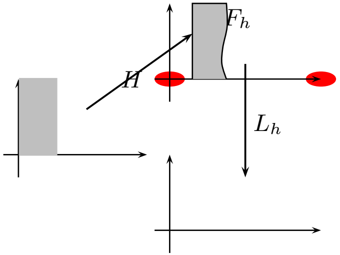
$$|\chi_h'(w) - \alpha| \leq C\alpha e^{-2\pi\alpha \operatorname{Im} w}$$

The estimate on the perturbed Fatou coordinates

Given $h(z) = e^{2\pi\alpha\mathbf{i}}z + \cdots$,



From estimate on F_h to estimate on L_h



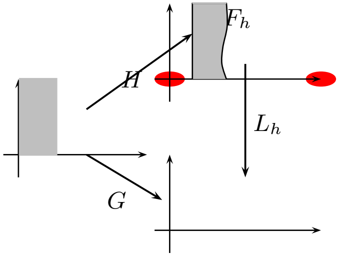
$H : (0,1) \times (0,\infty) \to \mathbb{C}$ is C^2
$\forall t \in (0,\infty), \quad F(H(0,t)) = H(1,t)$
$|\partial_z H(z) - 1| \preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} z}$
$|\partial_{\bar z} H(z)|, |\partial_{z\bar z} H(z)| \preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} z}$

$$H(s,t) := A + \int_0^s X(\ell,t)\, d\ell + \mathbf{i}\Big(t + \int_0^s Y(\ell,t)\, d\ell\Big), \text{ where}$$

$$X(s,t) := a_0(t) + a_1(t)\sin(\pi s) + a_2(t)\cos(\pi s) + a_3(t)\sin(2\pi s) + a_4(t)\cos(2\pi s),$$
$$Y(s,t) := b_0(t) + b_1(t)\sin(\pi s) + b_2(t)\cos(\pi s) + b_3(t)\sin(2\pi s) + b_4(t)\cos(2\pi s),$$

$$a_0(t) = \operatorname{Re}(F_h(A+\mathbf{i}t) - A) + \operatorname{Re} F_h''(A+\mathbf{i}t)/\pi, \ldots$$
$$b_0(t) = \operatorname{Im}(F_h(A+\mathbf{i}t) - A) - t + \operatorname{Im}(F_h''(A+\mathbf{i}t))/\pi, \ldots$$

From estimate on F_h to estimate on L_h



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$|\partial_z H(z) - 1| \preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} z}$

$G := L_h \circ H : \mathbb{C} \to \mathbb{C}$ is C^2 with
$G(z+1) = G(z) + 1$, and
$G(z) = z + z_0$, if $\operatorname{Im} z < -1$
$|\partial_{\bar z} G(z)|, |\partial_{z\bar z} G(z)| \preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} z}$

G induces the map $K : \mathbb{C} \to \mathbb{C}$ that

$$|\partial_{\zeta\bar\zeta} H(\zeta)| \preceq \frac{\alpha}{r} |\zeta|^{\alpha-1}$$

By general Cauchy Integral formula, one can show that

$$|\partial_\zeta K(\zeta) - \partial_\zeta K(0)| \preceq \frac{1}{r} |\zeta|^{\alpha}$$