Typical Orbits of Quadratic Polynomials with a Neutral Fixed Point

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Seillac, France, May 3-8, 2010

The map

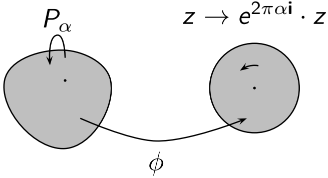
Let $P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2 : \mathbb{C} \to \mathbb{C}$, with α irrational.
0 is fixed with rotation

$$\alpha := \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}}$$

whose best rational approximants are

$$\frac{p_n}{q_n} := [a_0, a_1, \dots, a_{n-1}].$$

Local dynamics



Local and global dynamics

Theorem (Brjuno-Yoccoz 1965)

The quadratic polynomial P_α is locally linearizable at 0, iff

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty.$$

Such values of α are called Brjuno.

Global dynamics:
The Julia set of P_α, $J(P_\alpha)$, for non-Brjuno α (and for some Brjuno α) is known to have complicated topology.

Theorem (Buff-Cheritat 2005)

There exist α, both of Brjuno and non-Brjuno type, such that $J(P_\alpha)$ has positive area.

Measurable dynamics

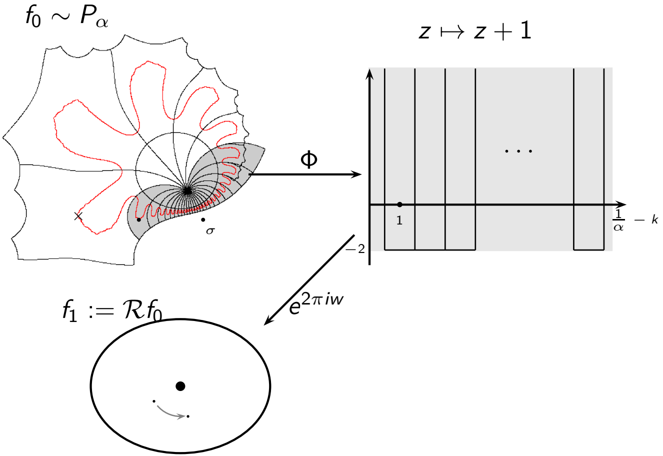
Assume $J(P_\alpha)$ has positive area for some α.
Can we describe typical orbits on the Julia set?
It is important to understand the structure of the postcritical set of P_α

$$\mathcal{PC}(P_\alpha) := \overline{\bigcup_{i \geq 1} P_\alpha^i(c.p.)}.$$

and the iterates near it.

To address the problem, we use a renormalization technique developed by H. Inou and M. Shishikura!

Inou-Shishikura Renormalization in 1 min.



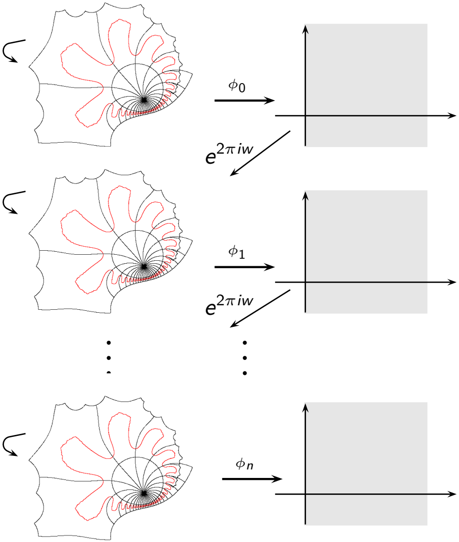
One time iterating f_1 corresponds to a_0 times iterating f_0.

Renormalization tower

$$P_\alpha(z) = e^{2i\pi\alpha_0} z + \text{ h.o.t}$$

$$\mathcal{R}P_\alpha(z) = e^{2i\pi\alpha_1} z + \text{ h.o.t}$$

$$\mathcal{R}^n P_\alpha(z) = e^{2i\pi\alpha_n} z + \text{ h.o.t}$$



Main results

Theorem (Inou-Shishikura 2005)

There exists a compact class of maps such that for every map h in this class all the renormalizations $f_n := \mathcal{R}^n(h)$ are well defined and belong to this class.
In particular, there exists a constant $N > 0$ such that the renormalization tower is defined for P_α, provided all $a_i > N$.

By controlling how ϕ_n behaves as the rotation number goes to zero, using quasi-conformal methods, we can show

Theorem

If α is a non-Brjuno number with all $a_i > N$, then the orbit of almost every point in the Julia set of $P_\alpha(z) = e^{2\pi\alpha \boldsymbol{i}}z + z^2$ accumulates at the fixed point 0.

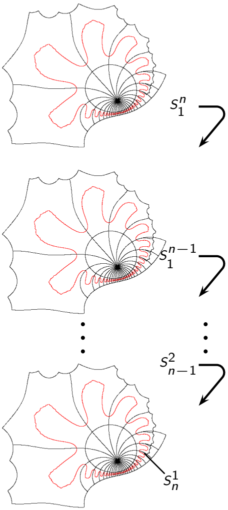
Theorem

If α is Brjuno with all $a_i > N$, then the orbit of almost every point in the Julia set of $P_\alpha(z) = e^{2\pi\alpha \boldsymbol{i}}z + z^2$ accumulates on the boundary of the Siegel disk.

Ideas of the proof

Using the tower, one introduces a nest of topological disks

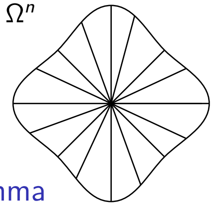
$$\Omega^0 \supset \Omega^1 \supset \Omega^2 \supset \cdots \supset \mathcal{PC}(P_\alpha).$$



$$\Omega^0 := \bigcup_{j=0}^{a_0+k} P_\alpha^j(S_0^1)$$

$$\Omega^n := \bigcup_{j=0}^{q_{n+1}+\ell q_n} P_\alpha^j(S_1^n)$$

Size of the sectors



Main technical lemma

There exists a constant C such that at every level $n \geq 1$, there exists a sector whose diameter is less than

$$C \cdot \alpha_0 \cdot \alpha_1^{\alpha_0} \cdot \alpha_2^{\alpha_0\alpha_1} \cdot \alpha_3^{\alpha_0\alpha_1\alpha_2} \ldots \alpha_{n-1}^{\alpha_0\ldots\alpha_{n-2}}.$$

Taking log, we obtain

$$\sum_{i=1}^{n-1} \alpha_0\alpha_1 \cdots \alpha_{i-1} \log \alpha_i,$$

which goes to ∞ iff

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} = \infty.$$

A corollary of the proof

Theorem

If α is non-Brjuno with all $a_i > N$, then P_α and every map in the Inou-Shishikura class, with the fixed point of multiplier $e^{2\pi\alpha\boldsymbol{i}}$ at 0, is not linearizable at 0.

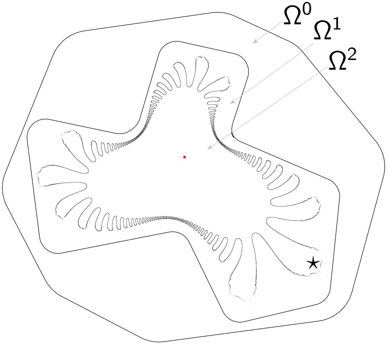
In particular, there exists a Jordan domain U containing 0 such that if f is a map of the form $e^{2\pi\alpha\boldsymbol{i}}h(1+h)^2$, where $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of the Riemann sphere satisfying

- $h(0) = 0$, $h'(0) = 1$, *and*
- *no critical value of h belongs to U,*

then f is not linearizable at zero.

This gives rational maps of arbitrarily large degree for which the Brjuno condition is sharp for the linearizability.
These examples do not have Douady-Hubbard quadratic-like restriction.

Further properties



Theorem

If α is non-Brjuno with all $a_i > N$, then $\mathcal{PC}(P_\alpha)$ has measure zero.

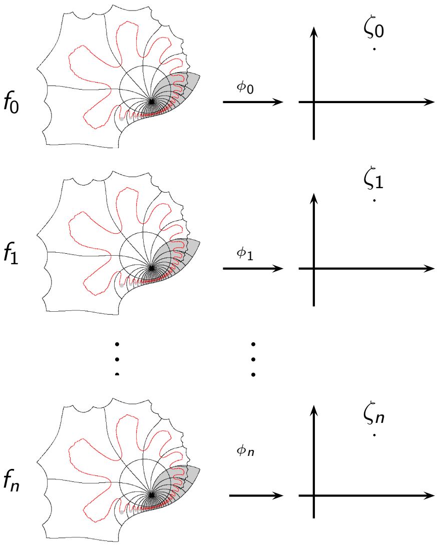
Corollary

With α as above, a.e. $z \in J(P_\alpha)$ is non-recurrent. In particular, there is no finite absolutely continuous invariant measure on the Julia set.

Theorem

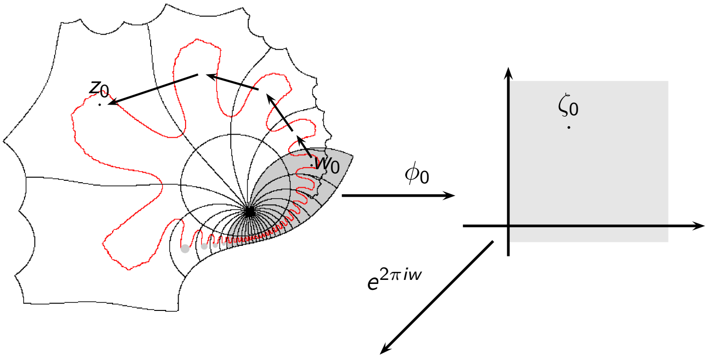
If α is non-Brjuno with all $a_i > N$, the orbit of a.e. $z \in J(P_\alpha)$ accumulates on the critical point.

Idea of the proof



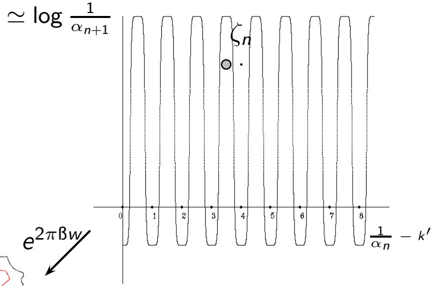
Given $z \in \cap \Omega^n$, we can go down to deep levels of the tower

Going down the tower

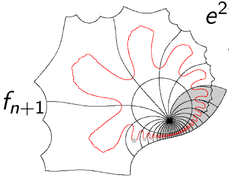


What we see on level n

Level n



$e^{2\pi ßw}$



Proposition: There are infinitely many n with

$$\operatorname{Im} \zeta_n \leq \log \frac{1}{\alpha_{n+1}}$$

Going up the tower