Measurable Dynamics of Quadratic Polynomials with a Neutral Fixed Point

Davoud Cheraghi

University of Warwick

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Basic holomorphic dynamics

Let $P(z) = \lambda z + z^2 : \mathbb{C} \to \mathbb{C}$,

$$\mathcal{O}(z) := z, P(z), P^{\circ 2}(z), P^{\circ 3}(z), \ldots$$

0 is fixed under P, there is a critical point at $-\lambda/2$,

$$K(P) := \{z \in \mathbb{C} \mid \mathcal{O}(z) \text{ is bounded}\},$$
$$J(P) := \partial K(P) := \overline{\{\text{repelling periodic points of } P\}},$$

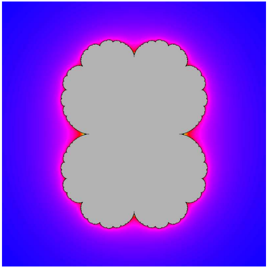
We only consider the case $|\lambda| = 1$, so $\lambda = e^{2\pi\alpha\mathbf{i}}$, where

$$\alpha := [a_0, a_1, a_2, \ldots] := \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}}, \text{ and } \frac{p_n}{q_n} := [a_0, a_1, \ldots, a_{n-1}].$$

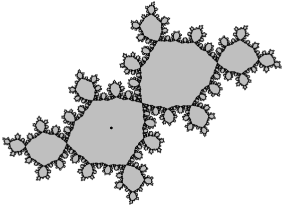
Basic holomorphic dynamics

Examples:

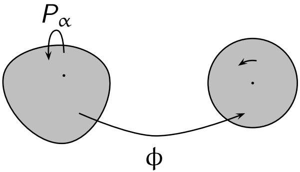
$\alpha = 1,\ P(z) = z + z^2$



$\alpha = [1, 1, 1, \ldots]$



P_α is called linearizable at 0, if there exists a change of coordinate near 0 which conjugates P_α to the linear map $z \to e^{2\pi\alpha\mathbf{i}} z$, i.e. $e^{2\pi\alpha\mathbf{i}} \cdot \phi = \phi \circ P_\alpha$



Local dynamics near an irrationally indifferent fixed point

Theorem (Brjuno-Yoccoz)

P_α is linearizable at 0 iff

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty.$$

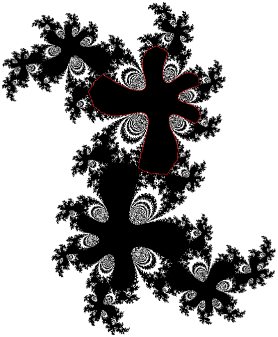
We say that α is Brjuno, and write $\alpha \in \mathcal{B}$, if the above sum is finite.
If the fixed point is not linearizable then it is called Cremer.

R. Perez Marco extended these ideas to structurally stable polynomials.
Unfortunately, these results do not provide any picture of the local dynamics near a Cremer fixed point.

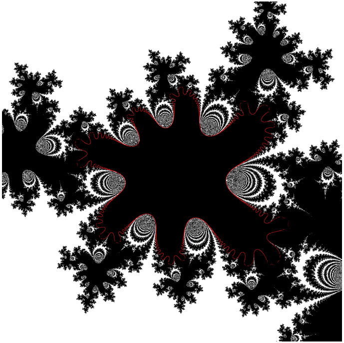
The Julia set of P_α, with $\alpha \in \mathcal{B}^c$ and even some $\alpha \in \mathcal{B}$, has complicated topology and geometry.

Examples of Siegel disks

$\alpha = [2, 2, 10^5, 1, 1, \ldots]$



$\alpha = [2, 3, 300, 10^9, 1, 1, \ldots]$



The Problem

Theorem (Buff-Cheritat)

There exist α, both of Brjuno and non-Brjuno type, such that $J(P_\alpha)$ has positive area.

Assume $J(P_\alpha)$ has positive area for some α.
Can we describe the dynamics on the Julia set?
For example:

- Are there particular typical behaviours on such Julia sets?
- Can we describe the geometry and topology of the measure theoretic attractor?
- Can the Julia set be decomposed into two invariant pieces each with positive area? (Ergodicity?)

Global dynamics

The post-critical set of P_α is defined as

$$\mathcal{PC}(P_\alpha) := \overline{\bigcup_{i \geq 1} P_\alpha^i(c.p.)}.$$

Lemma (Lyubich-1984)

The orbit of Lebesgue almost every point in the Julia set accumulates on the post-critical set.

Theorem (Mane)

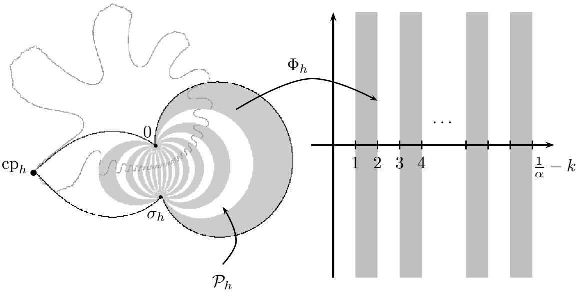
For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the critical point is recurrent and accumulates at the boundary of the Siegel disk or the Cremer fixed point, whichever is present.

This is useful if $\mathcal{PC}$ has simple structure, and one can understand the iterates near the critical orbit.

To address the problem, we use a renormalization technique developed by H. Inou and M. Shishikura!

Basic dynamics near the fixed point

Let $h : \text{Dom } h \to \mathbb{C}$ be close to P_0, with $h(0) = 0$, and $h'(0) = e^{2\pi\alpha\mathbf{i}}$.



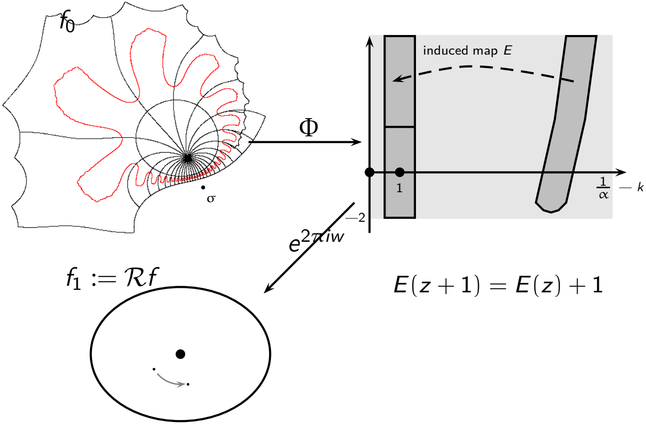
a. $\mathcal{P}_h$ contains cp_h, 0, and σ_h on its boundary.

b. $\Phi_h(\overline{\mathcal{P}_h}) = \{w \in \mathbb{C} \mid 0 \leq \mathrm{Re}(w) \leq \lfloor 1/\alpha \rfloor - k\}$.

c. Φ_h satisfies $\Phi_h(h(z)) = \Phi_h(z) + 1$, for $z, h(z) \in \mathcal{P}_h$.

d. The map Φ_h depends continuously on h.

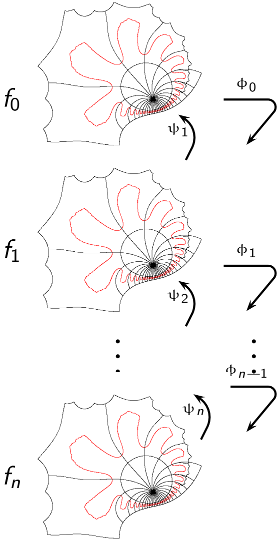
Inou-Shishikura Renormalization



$$E(z+1) = E(z) + 1$$

One time iterating f_1 corresponds to a_0 times iterating f_0.

Renormalization tower



$f_0(z) = e^{2i\pi\alpha_0} z + \text{h.o.t}$

$f_1(z) = e^{2i\pi\alpha_1} z + \text{h.o.t}$

$f_n(z) = e^{2i\pi\alpha_n} z + \text{h.o.t}$

One time iterating f_n corresponds to q_{n+1} times iterating f_0.

Main results

Can we build such a tower?

Theorem (Inou-Shishikura)

There exists a constant $N > 0$ such that if $a_i > N$, for all $i \geq 0$, all the renormalizations $f_n := \mathcal{R}^n(P_\alpha)$ are well defined and belong to a compact class of maps.

This controls f_n's. To transfer information from level n to level 0 one needs to control ϕ_n's and ψ_n's.

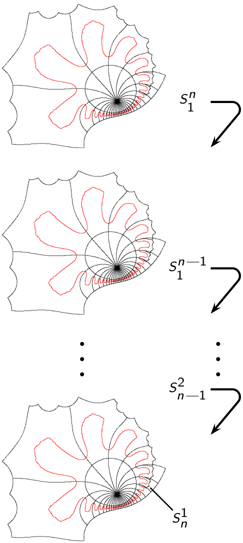
Theorem

If α is a non-Bruno number with all $a_i > N$, then the orbit of almost every point in the Julia set of $P_\alpha(z) = e^{2\pi\alpha \boldsymbol{i}} z + z^2$ accumulates at the 0 fixed point.

Theorem

If α is Bruno with all $a_i > N$, then the orbit of almost every point in the Julia set of $P_\alpha(z) = e^{2\pi\alpha \boldsymbol{i}} z + z^2$ accumulates on the boundary of the Siegel disk.

Controlling the orbits



$$\Omega^{(0)} := \bigcup_{j=0}^{a_0+k} P_\alpha^j(S_0^1)$$

$$\Omega^{(n)} := \bigcup_{j=0}^{q_{n+1}+\ell q_n} P_\alpha^j(S_1^n)$$

Size of the sectors

Lemma

1. *Ω^n's form a nest;* $\Omega^{(0)} \supset \Omega^{(1)} \supset \Omega^{(2)} \dots$
2. *For every $n \geq 0$, $\mathcal{PC}(P_\alpha) \subset \operatorname{int} \Omega^{(n)}$*

Lemma

There exists a constant C such that for every $n \geq 1$, there exists a positive integer $\gamma(n) \leq q_{n+1} + \ell q_n$, with

$$\operatorname{diam}\left(P_\alpha^{\gamma(n)}(S_1^n)\right) \leq C \cdot \alpha_0 \cdot \alpha_1^{\alpha_0} \cdot \alpha_2^{\alpha_0 \alpha_1} \cdot \alpha_3^{\alpha_0 \alpha_1 \alpha_2} \dots \alpha_{n-1}^{\alpha_0 \dots \alpha_{n-2}}.$$

Taking log we obtain: $\sum_{n=1}^{\infty} \alpha_0 \alpha_1 \cdots \alpha_{n-1} \log \alpha_n$,

which is equivalent to $-\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n}$.

A corollary of the proof

Theorem

If α is non-Bruno with all $a_i > N$, then P_α and every map in the Inou-Shishikura class, with the fixed point of multiplier $e^{2\pi\alpha\boldsymbol{i}}$ at 0, is not linearizable at 0.

In particular, there exists a Jordan domain U containing 0 such that if f is a map of the form $e^{2\pi\alpha\boldsymbol{i}}h(1+h)^2$, where $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of the Riemann sphere satisfying

- $h(0) = 0$, $h'(0) = 1$, *and*
- *no critical value of h belongs to U,*

then f is not linearizable at zero.

$\bigcap_{n\geq 0} \Omega^{(n)}$ is invariant under P_α, $\rightsquigarrow \exists$ a continuum of invariant subsets.

If these invariant sets have positive area, then the map can not be ergodic and we can't have a.e. $z \in J(P_\alpha)$ accumulates at the critical point.

Further properties

Theorem

If α is a non-Bruno number with all $a_i > N$, then $\bigcap_{n\geq 0} \Omega^{(n)}$ has measure zero. Hence, the postcritical set has measure zero.

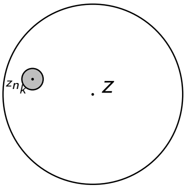
Corollary

With α as above, a.e. $z \in J(P_\alpha)$ is non-recurrent. In particular, there is no finite absolutely continuous invariant measure on the Julia set.

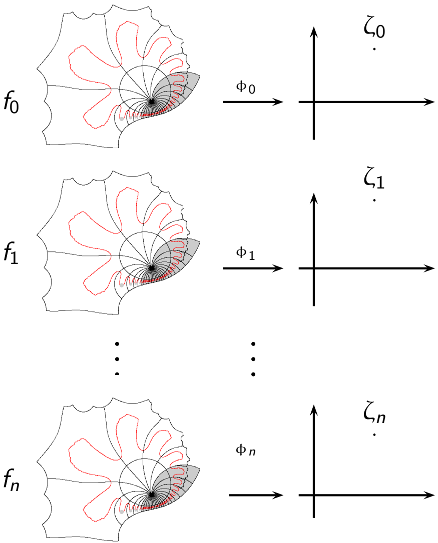
Ideas of the proof of the theorem:

Given $z \in \bigcap_{n\geq 0} \Omega^{(n)}$, there exists a sequence $z_{n_k} \longrightarrow z$, and real numbers $r_{n_k} \longrightarrow 0$, with

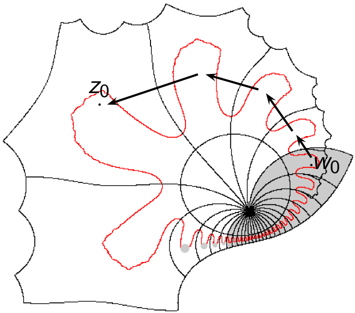
- $\Omega^{(n_k+1)} \bigcap B(z_{n_k}, r_{n_k}) = \emptyset$, and
- $\frac{r_{n_k}}{d(z, z_{n_k})} > \eta > 0$, for some constant η.



Idea of the proof

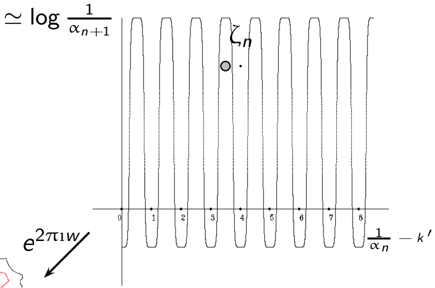


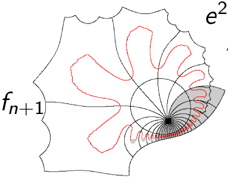
Idea of the proof



Idea of the proof

Level n





corollaries of the contraction

Theorem

If α is non-Brjuno with all $a_i > N$, the orbit of almost every point in $J(P_\alpha)$ accumulates on the critical point.

Corollary

For α with all $a_i > N$, the postcritical set of P_α is connected.

Corollary

There are positive constants M and $\mu < 1$ such that for every α with all $a_i \geq N$ and every $z \in \Omega_0^{n+1}$ we have

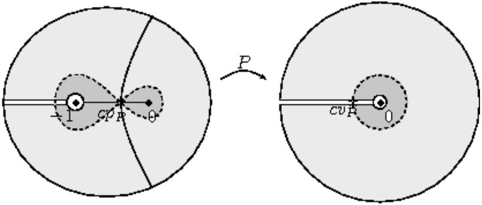
$$\| P_\alpha^{q_n}(z) - z \| \leq M\mu^n.$$

In particular this holds on the post-critical set.

Inou Shishikura class

Let $P(z) = z(z+1)^2$,

$$\mathcal{IS} := \left\{ P \circ \varphi^{-1} \colon U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi\colon U \to U_f \text{ is univalent, } \varphi(0) = 0,\ \varphi'(0) = 1, \textit{ and} \\ \phi^{-1} \text{ extends onto } \bar{U}_f \text{ as a continuous function} \end{array} \right\}.$$



For a positive real number α_*, consider the class

$$\mathcal{IS}[\alpha_*] := \{e^{2\pi\alpha \mathbf{i}} \cdot f \mid f \in \mathcal{IS}, \text{ and } \alpha \in [0, \alpha_*]\}.$$