Near-Parabolic renormalization; hyperbolicity and rigidity

Davoud Cheraghi

Imperial College London

Simons semester at IMPAN, Sept 2015

There are several notions of renormalization operators in holomorphic dynamics:

- Polynomial-like renormalization
- Commuting-pair renormalization
- Cylinder renormalization
- Sector renormalization
- Near-parabolic renormalization

On circle:

- renormalization of critical circle maps
- renormalization of critical circle covers
- renormalization of Henon maps

There are several notions of renormalization operators in holomorphic dynamics:

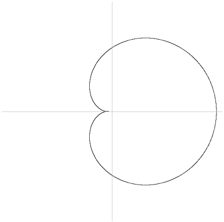
- Polynomial-like renormalization
- Commuting-pair renormalization
- Cylinder renormalization
- Sector renormalization
- Near-parabolic renormalization

On circle:

- renormalization of critical circle maps
- renormalization of critical circle covers
- renormalization of Henon maps

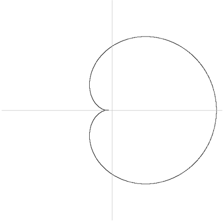
Near-parabolic renormalizations unifies several of these notions. We focus on this renormalization operator.

There is an explicit Jordan domain $U \subset \mathbb{C}$ bounded by an analytic curve:



$0 \in U$, $-1 \notin U$, $-8/9 \in U$

There is an explicit Jordan domain $U \subset \mathbb{C}$ bounded by an analytic curve:



$0 \in U$, $-1 \notin U$, $-8/9 \in U$

Let

$$P(z) = z(1+z)^2.$$

- $P(0) = 0$ and $P'(0) = 1$,
- $P'(-1) = P'(-1/3) = 0$; $P(-1) = 0$ and $P(-1/3) = -4/27 \in U$.

$P : U \to P(U)$ has a particular covering structure.

Let $\mathcal{F}$ be the set of maps

$$h = P \circ \psi^{-1}$$

where

- $\psi : U \to \mathbb{C}$ is univalent and has quasi-conformal extension onto $\mathbb{C}$,
- $\psi(0) = 0$ and $\psi'(0) = 1$.

Let $\mathcal{F}$ be the set of maps

$$h = P \circ \psi^{-1}$$

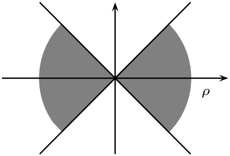
where

- $\psi : U \to \mathbb{C}$ is univalent and has quasi-conformal extension onto $\mathbb{C}$,
- $\psi(0) = 0$ and $\psi'(0) = 1$.

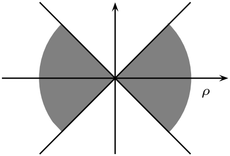
It follows that

- h is defined on $\psi(U)$,
- $h(0) = 0$, $h'(0) = 1$,
- h has a critical point at c.p. $= \psi(-1/3)$ which is mapped to $-4/27$,
- $h : \psi(U) \to P(U)$ has the same covering structure as the one of P.

Let $A_\rho = \{\alpha \in \mathbb{C} \mid 0 < |\alpha| \le \rho, |\operatorname{Im}\alpha| \le |\operatorname{Re}\alpha|\}$,



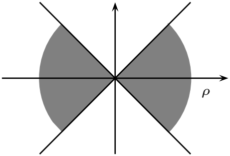
Let $A_\rho = \{\alpha \in \mathbb{C} \mid 0 < |\alpha| \leq \rho, |\operatorname{Im}\alpha| \leq |\operatorname{Re}\alpha|\}$,



For $\alpha \in A_\rho$ and $h \in \mathcal{F}$, let

$$(\alpha \ltimes h)(z) = h(e^{2\pi i\alpha} z).$$

Let $A_\rho = \{\alpha \in \mathbb{C} \mid 0 < |\alpha| \leq \rho, |\operatorname{Im}\alpha| \leq |\operatorname{Re}\alpha|\}$,



For $\alpha \in A_\rho$ and $h \in \mathcal{F}$, let

$$(\alpha \ltimes h)(z) = h(e^{2\pi i\alpha} z).$$

Set

$$A_\rho \ltimes \mathcal{F} = \{(\alpha \ltimes h) \mid \alpha \in A_\rho, h \in \mathcal{F}\}$$

We equip $A_\rho \ltimes \mathcal{F}$ with the topology of uniform convergence on compact sets.

We equip $A_\rho \ltimes \mathcal{F}$ with the topology of uniform convergence on compact sets.

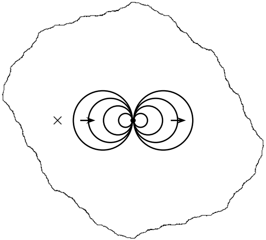
Since

$$\mathcal{F} \hookrightarrow \{\phi : \mathbb{D} \to \mathbb{C} \mid \phi(0) = 0, \phi'(0) = 1\}$$

by Koebe distortion theorem, $\mathcal{F}$ forms a pre-compact class of maps.

Dynamics of a map $h \in \mathcal{F}$;

h has a parabolic fixed point at 0; the orbit of c.p. tends to 0.



If ρ is small enough, $\alpha \ltimes h$ has two preferred fixed points at 0 and $\sigma = \sigma(\alpha \ltimes h)$. $|\sigma| = O(|\alpha|)$.

We have

$$(\alpha \ltimes h)'(0) = e^{2\pi i\alpha}, \qquad (\alpha \ltimes h)'(\sigma) = e^{2\pi i\beta}$$

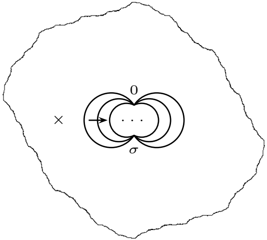
where β is a complex number with $-1/2 < \operatorname{Re}\beta \leq 1/2$.

If ρ is small enough, $\alpha \ltimes h$ has two preferred fixed points at 0 and $\sigma = \sigma(\alpha \ltimes h)$. $|\sigma| = O(|\alpha|)$.

We have

$$(\alpha \ltimes h)'(0) = e^{2\pi i \alpha}, \qquad (\alpha \ltimes h)'(\sigma) = e^{2\pi i \beta}$$

where β is a complex number with $-1/2 < \operatorname{Re}\beta \leq 1/2$.



There is a simply connected region

$$\mathcal{P}_{\alpha\ltimes h} \subset \mathrm{Dom}(h)$$

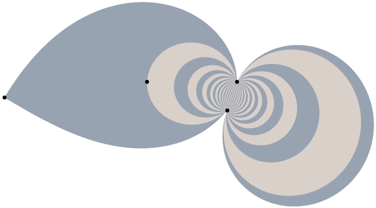
which is bounded by analytic curves landing at 0, σ, and c.p.,

as well as a univalent map

$$\Phi_{\alpha\ltimes h} : \mathcal{P}_{\alpha\ltimes h} \to \mathbb{C}$$

such that

$$\Phi_{\alpha\ltimes h}((\alpha \ltimes h)(z)) = \Phi_{\alpha\ltimes h}(z) + 1, \text{ on } \mathcal{P}_{\alpha\ltimes h}, \quad \Phi_{\alpha\ltimes h}(\text{c.p.}) = 0.$$



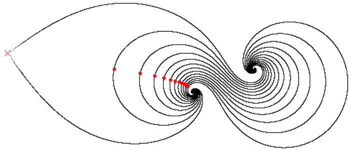
Proposition (Ch. 2009)

There is a constant k_1 independent of α and h such that one may choose $\mathcal{P}_{\alpha\ltimes h}$ and $\Phi_{\alpha\ltimes h}$ with

$$\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h}) = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z \leq \operatorname{Re} \frac{1}{\alpha} - k_1\}$$

and for $y \geq 0$,

$$\arg \Phi^{-1}_{\alpha\ltimes h}(iy) \simeq -2\pi y \operatorname{Im} \alpha + \arg \sigma + C_{\alpha\ltimes h}.$$



We drop the subscripts $\alpha \ltimes h$ from $\mathcal{P}_{\alpha\ltimes h}$ and $\Phi_{\alpha\ltimes h}$, ...

Define

$$A = \{z \in \mathcal{P} : 1/2 \leq \mathrm{Re}(\Phi(z)) \leq 3/2\,,\ 2 \leq \mathrm{Im}\,\Phi(z)\}$$

$$C = \{z \in \mathcal{P} : 1/2 \leq \mathrm{Re}(\Phi(z)) \leq 3/2\,,\ -2 \leq \mathrm{Im}\,\Phi(z) \leq 2\}$$

We drop the subscripts $\alpha \ltimes h$ from $\mathcal{P}_{\alpha \ltimes h}$ and $\Phi_{\alpha \ltimes h}$, ...

Define

$$A = \{z \in \mathcal{P} : 1/2 \leq \mathrm{Re}(\Phi(z)) \leq 3/2\,,\ 2 \leq \mathrm{Im}\,\Phi(z)\}$$

$$C = \{z \in \mathcal{P} : 1/2 \leq \mathrm{Re}(\Phi(z)) \leq 3/2\,,\ -2 \leq \mathrm{Im}\,\Phi(z) \leq 2\}$$

It follows from the work of Inou-Shishikura that there are chains

$$A^k \xrightarrow[1-1]{\alpha \ltimes h} A^{k-1} \xrightarrow[1-1]{\alpha \ltimes h} \dots \xrightarrow[1-1]{\alpha \ltimes h} A^1 \xrightarrow[1-1]{\alpha \ltimes h} A$$

and

$$C^k \xrightarrow[1-1]{\alpha \ltimes h} C^{k-1} \xrightarrow[1-1]{\alpha \ltimes h} \dots \xrightarrow[1-1]{\alpha \ltimes h} C^1 \xrightarrow[2-1]{\alpha \ltimes h} C$$

where A^k and C^k are contained in $\mathcal{P}$.

We drop the subscripts $\alpha \ltimes h$ from $\mathcal{P}_{\alpha\ltimes h}$ and $\Phi_{\alpha\ltimes h}$, ...

Define

$$A = \{z \in \mathcal{P} : 1/2 \le \mathrm{Re}(\Phi(z)) \le 3/2\,,\ 2 \le \mathrm{Im}\,\Phi(z)\}$$

$$C = \{z \in \mathcal{P} : 1/2 \le \mathrm{Re}(\Phi(z)) \le 3/2\,,\ -2 \le \mathrm{Im}\,\Phi(z) \le 2\}$$

It follows from the work of Inou-Shishikura that there are chains

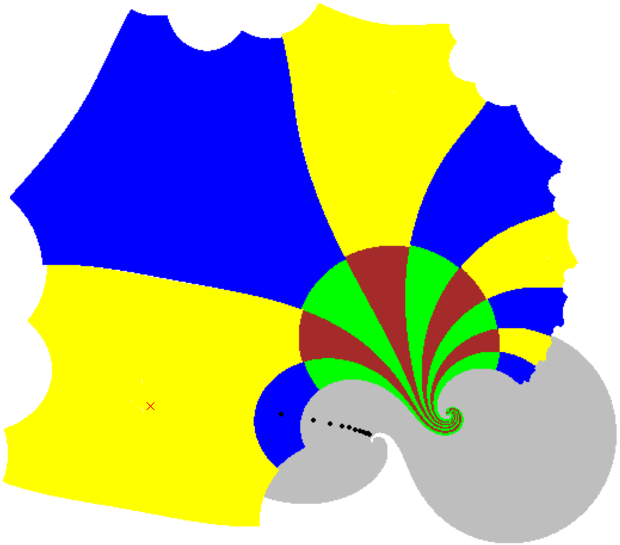
$$A^k \xrightarrow[1-1]{\alpha\ltimes h} A^{k-1} \xrightarrow[1-1]{\alpha\ltimes h} \dots \xrightarrow[1-1]{\alpha\ltimes h} A^1 \xrightarrow[1-1]{\alpha\ltimes h} A$$

and

$$C^k \xrightarrow[1-1]{\alpha\ltimes h} C^{k-1} \xrightarrow[1-1]{\alpha\ltimes h} \dots \xrightarrow[1-1]{\alpha\ltimes h} C^1 \xrightarrow[2-1]{\alpha\ltimes h} C$$

where A^k and C^k are contained in $\mathcal{P}$.

Prop. (Ch.) k is uniformly bounded from above independent of α and h.



Let

$$E = \Phi \circ (\alpha \ltimes h)^{\circ k} \circ \Phi^{-1} : \Phi(A^k \cup C^k) \to \Phi(A \cup C).$$

Let

$$E = \Phi \circ (\alpha \ltimes h)^{\circ k} \circ \Phi^{-1} : \Phi(A^k \cup C^k) \to \Phi(A \cup C).$$

We have $E(\zeta + 1) = E(\zeta) + 1$ on the boundary of $\Phi(A^k \cup C^k)$.

Let

$$E = \Phi \circ (\alpha \ltimes h)^{\circ k} \circ \Phi^{-1} : \Phi(A^k \cup C^k) \to \Phi(A \cup C).$$

We have $E(\zeta + 1) = E(\zeta) + 1$ on the boundary of $\Phi(A^k \cup C^k)$.

E projects under $\mathbb{E}\mathrm{xp}(\zeta) = \frac{-4}{27} e^{2\pi i \zeta}$ to a holomorphic map defined on a punctured neighborhood of 0. That is, there is a map $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)$ with

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) \circ \mathbb{E}\mathrm{xp}(\zeta) = \mathbb{E}\mathrm{xp} \circ E(\zeta)$$

It follows that

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)(z) \simeq e^{-2\pi i \frac{-1}{\alpha}} z + a_2 z^2 + \dots$$

The above map is called the top near-parabolic renormalization of $\alpha \ltimes h$.

It follows that

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)(z) \simeq e^{-2\pi i \frac{-1}{\alpha}} z + a_2 z^2 + \dots$$

The above map is called the top near-parabolic renormalization of $\alpha \ltimes h$.

Q: How does this correspond to a "return map"?

It follows that

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)(z) \simeq e^{-2\pi i \frac{-1}{\alpha}} z + a_2 z^2 + \dots$$

The above map is called the top near-parabolic renormalization of $\alpha \ltimes h$.

Q: How does this correspond to a "return map"?

Key point: while the return map may require large number of iterates, renormalization is defined using the composition of $k+2$ maps?

It follows that

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)(z) \simeq e^{-2\pi i \frac{-1}{\alpha}} z + a_2 z^2 + \dots$$

The above map is called the top near-parabolic renormalization of $\alpha \ltimes h$.

Q: How does this correspond to a "return map"?

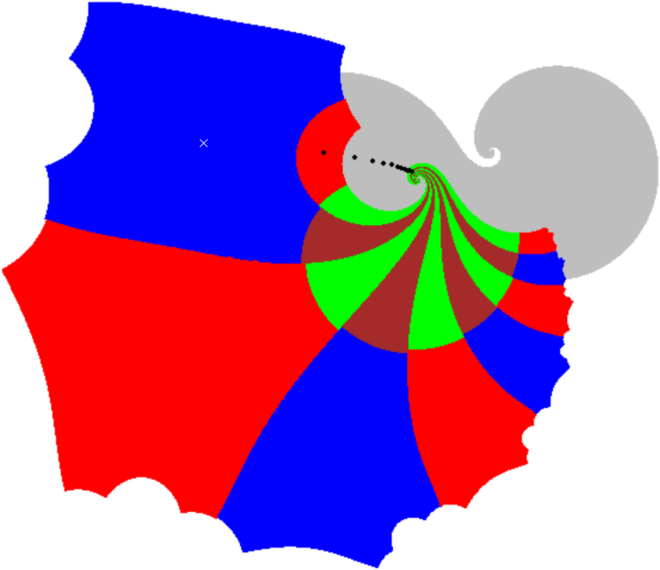
Key point: while the return map may require large number of iterates, renormalization is defined using the composition of $k+2$ maps?

Inou-Shishikura: The above map has the same covering structure as the one of P on U! That is,

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) \in \{\frac{-1}{\alpha} \bmod \mathbb{Z}\} \ltimes \mathcal{F}.$$

There is a similar process to define a "return map" near σ-fixed point: It gives us

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) \in \{\frac{-1}{\beta} \bmod \mathbb{Z}\} \ltimes \mathcal{F}.$$



Let

$$Q_0(z) = z + \frac{27}{16}z^2,$$

so that its critical value lies at $-4/27$.

Let

$$Q_0(z) = z + \frac{27}{16}z^2,$$

so that its critical value lies at $-4/27$.

Then

$$\alpha \ltimes Q_0 = e^{2\pi i\alpha}z + \frac{27}{16}e^{4\pi i\alpha}z^2.$$

Let

$$Q_0(z) = z + \frac{27}{16}z^2,$$

so that its critical value lies at $-4/27$.

Then

$$\alpha \ltimes Q_0 = e^{2\pi i\alpha}z + \frac{27}{16}e^{4\pi i\alpha}z^2.$$

$\alpha \ltimes Q_0$ does not belong to $\alpha \ltimes \mathcal{F}$!

However, $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes Q_0)$ and $\mathcal{R}_{\text{NP-b}}(\alpha \ltimes Q_0)$ are defined in the same fashion, and

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes Q_0) \in \{\frac{-1}{\alpha} \bmod \mathbb{Z}\} \ltimes \mathcal{F},$$

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes Q_0) \in \{\frac{-1}{\beta} \bmod \mathbb{Z}\} \ltimes \mathcal{F}.$$

Lecture II:

Hyperbolicity of the near-parabolic renormalization operators

We are interested in the dynamics of the operators

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) = \hat{\alpha}(\alpha \ltimes h) \ltimes \hat{h}(\alpha \ltimes h)$$

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) = \check{\alpha}(\alpha \ltimes h) \ltimes \check{h}(\alpha \ltimes h)$$

acting on $A(\rho) \ltimes \mathcal{F}$ with values in $\mathbb{C} \ltimes \mathcal{F}$.

We are interested in the dynamics of the operators

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) = \hat{\alpha}(\alpha \ltimes h) \ltimes \hat{h}(\alpha \ltimes h)$$

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) = \check{\alpha}(\alpha \ltimes h) \ltimes \check{h}(\alpha \ltimes h)$$

acting on $A(\rho) \ltimes \mathcal{F}$ with values in $\mathbb{C} \ltimes \mathcal{F}$.

Also recall that

$$\hat{\alpha}(\alpha \ltimes h) = \frac{-1}{\alpha} \bmod \mathbb{Z}, \qquad \check{\alpha}(\alpha \ltimes h) = \frac{-1}{\beta(\alpha \ltimes h)} \bmod \mathbb{Z}.$$

We are interested in the dynamics of the operators

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) = \hat{\alpha}(\alpha \ltimes h) \ltimes \hat{h}(\alpha \ltimes h)$$

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) = \check{\alpha}(\alpha \ltimes h) \ltimes \check{h}(\alpha \ltimes h)$$

acting on $A(\rho) \ltimes \mathcal{F}$ with values in $\mathbb{C} \ltimes \mathcal{F}$.

Also recall that

$$\hat{\alpha}(\alpha \ltimes h) = \frac{-1}{\alpha} \bmod \mathbb{Z}, \qquad \check{\alpha}(\alpha \ltimes h) = \frac{-1}{\beta(\alpha \ltimes h)} \bmod \mathbb{Z}.$$

$\mathcal{R}_{\text{NP-t}}$ preserves vertical fibers, while $\mathcal{R}_{\text{NP-b}}$ does not preserve them.

$\mathcal{F}$ is equipped with a Teichmüller metric:
for $f = P \circ \varphi^{-1}$ and $g = P \circ \psi^{-1}$ in $\mathcal{F}$,

$$\mathrm{d}_{\mathsf{Teich}}(f, g) = \inf \left\{ \log \mathrm{Dil}(\hat{\psi} \circ \hat{\varphi}^{-1}) \right\}$$

where $\inf$ is taken over all quasi-conformal extensions $\hat{\varphi}$ and $\hat{\psi}$ of φ and ψ onto $\mathbb{C}$.
Here,

$$\mathrm{Dil}(\eta) = \sup_{z \in \mathrm{Dom}\,\eta} \frac{|\eta_z| + |\eta_{\overline{z}}|}{|\eta_z| - |\eta_{\overline{z}}|}.$$

is a measure of conformality of h.

$\mathcal{F}$ is equipped with a Teichmüller metric:
for $f = P \circ \varphi^{-1}$ and $g = P \circ \psi^{-1}$ in $\mathcal{F}$,

$$\mathrm{d}_{\mathsf{Teich}}(f, g) = \inf \left\{ \log \mathrm{Dil}(\hat{\psi} \circ \hat{\varphi}^{-1}) \right\}$$

where $\inf$ is taken over all quasi-conformal extensions $\hat{\varphi}$ and $\hat{\psi}$ of φ and ψ onto $\mathbb{C}$.
Here,

$$\mathrm{Dil}(\eta) = \sup_{z \in \mathrm{Dom}\, \eta} \frac{|\eta_z| + |\eta_{\overline{z}}|}{|\eta_z| - |\eta_{\overline{z}}|}.$$

is a measure of conformality of h.
$\mathrm{d}_{\mathsf{Teich}}(f_n, f) \to 0$ implies $f_n \to f$ uniformly on compact sets, but not vice versa.

$\mathcal{F}$ is equipped with a Teichmüller metric:
for $f = P \circ \varphi^{-1}$ and $g = P \circ \psi^{-1}$ in $\mathcal{F}$,

$$\mathrm{d}_{\mathsf{Teich}}(f, g) = \inf \left\{ \log \mathrm{Dil}(\hat{\psi} \circ \hat{\varphi}^{-1}) \right\}$$

where $\inf$ is taken over all quasi-conformal extensions $\hat{\varphi}$ and $\hat{\psi}$ of φ and ψ onto $\mathbb{C}$.
Here,

$$\mathrm{Dil}(\eta) = \sup_{z \in \mathrm{Dom}\,\eta} \frac{|\eta_z| + |\eta_{\overline{z}}|}{|\eta_z| - |\eta_{\overline{z}}|}.$$

is a measure of conformality of h.
$\mathrm{d}_{\mathsf{Teich}}(f_n, f) \to 0$ implies $f_n \to f$ uniformly on compact sets, but not vice versa.

$A(\rho)$ is equipped with the Euclidean metric.

We wish to understand the derivatives of these operators (infinite by infinite matrices!)

$$D\,\mathcal{R}_{\text{NP-t}} = \begin{bmatrix} \dfrac{\partial \hat{\alpha}}{\partial \alpha} & \dfrac{\partial \hat{h}(\alpha \ltimes h)}{\partial \alpha} \\ \dfrac{\partial \hat{\alpha}}{\partial h} & \dfrac{\partial \hat{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix}$$

$$D\,\mathcal{R}_{\text{NP-b}} = \begin{bmatrix} \dfrac{\partial \check{\alpha}}{\partial \alpha} & \dfrac{\partial \check{h}(\alpha \ltimes h)}{\partial \alpha} \\ \dfrac{\partial \check{\alpha}}{\partial h} & \dfrac{\partial \check{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix}$$

The operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are holomorphic.

The operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are holomorphic.
Recall that,

$$h \mapsto \hat{h}(\alpha \ltimes h) : \mathcal{F} \to \mathcal{F}, \quad h \mapsto \check{h}(\alpha \ltimes h) : \mathcal{F} \to \mathcal{F}.$$

By Royden-Gardiner,

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2),$$

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2)$$

The operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are holomorphic.
Recall that,

$$h \mapsto \hat{h}(\alpha \ltimes h) : \mathcal{F} \to \mathcal{F}, \quad h \mapsto \check{h}(\alpha \ltimes h) : \mathcal{F} \to \mathcal{F}.$$

By Royden-Gardiner,

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2),$$

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2)$$

Indeed, Inou-Shishikura showed that these are uniform contractions!

The operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are holomorphic.
Recall that,

$$h \mapsto \hat{h}(\alpha \ltimes h) : \mathcal{F} \to \mathcal{F}, \quad h \mapsto \check{h}(\alpha \ltimes h) : \mathcal{F} \to \mathcal{F}.$$

By Royden-Gardiner,

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2),$$

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2)$$

Indeed, Inou-Shishikura showed that these are uniform contractions!

In my symbolic notations, these mean

$$|\frac{\partial \hat{h}(\alpha \ltimes h)}{\partial h}| \leq 1, \quad |\frac{\partial \check{h}(\alpha \ltimes h)}{\partial h}| \leq 1.$$

Recall that

$$\hat{\alpha}(\alpha \ltimes h) = \frac{-1}{\alpha} \bmod \mathbb{Z}$$

Then,

$$\frac{\partial \hat{\alpha}}{\partial \alpha} = \frac{1}{\alpha^2}$$

and

$$\frac{\partial \hat{\alpha}}{\partial h} = 0.$$

Recall that

$$\check{\alpha}(\alpha \ltimes h) = \frac{-1}{\beta(\alpha \ltimes h)} \bmod \mathbb{Z}, \qquad (\alpha \ltimes h)'(\sigma) = e^{2\pi i\beta}.$$

we need

$$\frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial \alpha}, \quad \frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial h}$$

Proposition $\exists$ a Jordan domain $W \ni 0$, independence of α and h, such that every $\alpha \ltimes h \in A_\rho \ltimes \mathcal{F}$ has only two fixed points 0 and $\sigma(\alpha \ltimes h)$ in W.

Recall that

$$\check{\alpha}(\alpha \ltimes h) = \frac{-1}{\beta(\alpha \ltimes h)} \bmod \mathbb{Z}, \qquad (\alpha \ltimes h)'(\sigma) = e^{2\pi i \beta}.$$

we need

$$\frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial \alpha}, \quad \frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial h}$$

Proposition $\exists$ a Jordan domain $W \ni 0$, independence of α and h, such that every $\alpha \ltimes h \in A_\rho \ltimes \mathcal{F}$ has only two fixed points 0 and $\sigma(\alpha \ltimes h)$ in W.
Then,

$$I(\alpha \ltimes h) := \frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)} \, dz = \frac{1}{1 - e^{2\pi i \alpha}} + \frac{1}{1 - e^{2\pi i \beta}}.$$

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)}\, dz = \frac{1}{1 - e^{2\pi i \alpha}} + \frac{1}{1 - e^{2\pi i \beta}}.$$

Prop. We have

$$|I(\alpha \ltimes h_1)| \leq B_1, \quad |\frac{\partial}{\partial \alpha} I(\alpha \ltimes h_1)| \leq B_2.$$

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i}\int_{\partial W} \frac{1}{z-(\alpha \ltimes h)(z)}\,dz = \frac{1}{1-e^{2\pi i\alpha}} + \frac{1}{1-e^{2\pi i\beta}}.$$

Prop. We have

$$|I(\alpha \ltimes h_1)| \le B_1, \quad |\frac{\partial}{\partial\alpha} I(\alpha \ltimes h_1)| \le B_2.$$

These imply

$$B_3^{-1}|\alpha| \le |\beta(\alpha \ltimes h_1)| \le B_3|\alpha|, \quad B_4^{-1} \le |\frac{\partial\beta}{\partial\alpha}(\alpha \ltimes h_1)| \le B_4.$$

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i}\int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)}\, dz = \frac{1}{1 - e^{2\pi i\alpha}} + \frac{1}{1 - e^{2\pi i\beta}}.$$

Prop. We have

$$|I(\alpha \ltimes h_1)| \leq B_1, \quad |\frac{\partial}{\partial\alpha} I(\alpha \ltimes h_1)| \leq B_2.$$

These imply

$$B_3^{-1}|\alpha| \leq |\beta(\alpha \ltimes h_1)| \leq B_3|\alpha|, \quad B_4^{-1} \leq |\frac{\partial\beta}{\partial\alpha}(\alpha \ltimes h_1)| \leq B_4.$$

and hence

$$\frac{\partial\check{\alpha}(\alpha \ltimes h)}{\partial\alpha} = \frac{1}{\beta^2} \cdot \frac{\partial\beta}{\partial\alpha} \simeq \frac{1}{\alpha^2}.$$

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i}\int_{\partial W} \frac{1}{z-(\alpha \ltimes h)(z)}\,dz = \frac{1}{1-e^{2\pi i\alpha}} + \frac{1}{1-e^{2\pi i\beta}}.$$

$I(\alpha \ltimes h)$ is a holomorphic function of α and h.

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)} \, dz = \frac{1}{1 - e^{2\pi i \alpha}} + \frac{1}{1 - e^{2\pi i \beta}}.$$

$I(\alpha \ltimes h)$ is a holomorphic function of α and h.

Prop. We have

$$|I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2)| \leq B_5 \, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2).$$

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)}\, dz = \frac{1}{1 - e^{2\pi i \alpha}} + \frac{1}{1 - e^{2\pi i \beta}}.$$

$I(\alpha \ltimes h)$ is a holomorphic function of α and h.

Prop. We have

$$|I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2)| \leq B_5 \, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2).$$

$+$ some analysis we get

$$|\beta(\alpha \times h_1) - \beta(\alpha \ltimes h_2)| \leq B_6 |\alpha|^2 \, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2)$$

Recall

$$I(\alpha \ltimes h) := \frac{1}{2\pi i}\int_{\partial W} \frac{1}{z-(\alpha \ltimes h)(z)}\,dz = \frac{1}{1-e^{2\pi i\alpha}} + \frac{1}{1-e^{2\pi i\beta}}.$$

$I(\alpha \ltimes h)$ is a holomorphic function of α and h.

Prop. We have

$$|I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2)| \le B_5\, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2).$$

$+$ some analysis we get

$$|\beta(\alpha \times h_1) - \beta(\alpha \ltimes h_2)| \le B_6|\alpha|^2\, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2)$$

Hence,

$$\begin{aligned}
|\check{\alpha}(\alpha \ltimes h_1) - \check{\alpha}(\alpha \ltimes h_2)| &= \left|\frac{-1}{\beta(\alpha \ltimes h_1)} + \frac{1}{\beta(\alpha \ltimes h_2)}\right| \\
&\le \frac{B_3^2}{|\alpha|^2}|\beta(\alpha \ltimes h_1) - \beta(\alpha \ltimes h_2)| \\
&\le \frac{B_3^2}{|\alpha|^2}B_6|\alpha^2|\, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2) = B_3^2 B_6\, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2).
\end{aligned}$$

In my symbolic notation, the previous bound means

$$|\frac{\partial\check{\alpha}(\alpha \ltimes h)}{\partial h}| \leq B_3^2 B_6$$

For every fixed $h \in \mathcal{F}$,

$$\alpha \mapsto \hat{h}(\alpha, h), \alpha \mapsto \check{h}(\alpha \ltimes h),$$

map A_ρ into $\mathcal{F}$.

For every fixed $h \in \mathcal{F}$,

$$\alpha \mapsto \hat{h}(\alpha, h), \alpha \mapsto \check{h}(\alpha \ltimes h),$$

map A_ρ into $\mathcal{F}$.

These have different nature, mapping a domain in $\mathbb{C}$ into the infinite dimensional Teichmüller space!

For every fixed $h \in \mathcal{F}$,

$$\alpha \mapsto \hat{h}(\alpha, h), \alpha \mapsto \check{h}(\alpha \ltimes h),$$

map A_ρ into $\mathcal{F}$.

These have different nature, mapping a domain in $\mathbb{C}$ into the infinite dimensional Teichmüller space!

To control these maps, we need to understand how the Fatou coordinate $\Phi_{\alpha \ltimes h}$ depends on α, and how the renormalization is constructed.

For every fixed $h \in \mathcal{F}$,

$$\alpha \mapsto \hat{h}(\alpha, h), \alpha \mapsto \check{h}(\alpha \ltimes h),$$

map A_ρ into $\mathcal{F}$.

These have different nature, mapping a domain in $\mathbb{C}$ into the infinite dimensional Teichmüller space!

To control these maps, we need to understand how the Fatou coordinate $\Phi_{\alpha \ltimes h}$ depends on α, and how the renormalization is constructed.

Theorem (Ch. 2015)

There is $L > 0$ such that for every $h \in \mathcal{F}$ the maps

$$\alpha \mapsto \hat{h}(\alpha, h) \text{ and } \alpha \mapsto \check{h}(\alpha, h)$$

are L-Lipschitz with respect to d_{Eucl} on $A(\rho)$ and d_{Teich} on $\mathcal{F}$.

Recall

$$D\,\mathcal{R}_{\text{NP-t}} = \begin{bmatrix} \frac{\partial \hat{\alpha}}{\partial \alpha} & \frac{\partial \hat{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \hat{\alpha}}{\partial h} & \frac{\partial \hat{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix} \qquad D\,\mathcal{R}_{\text{NP-b}} = \begin{bmatrix} \frac{\partial \check{\alpha}}{\partial \alpha} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \check{\alpha}}{\partial h} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix}$$

Recall

$$D\,\mathcal{R}_{\text{NP-t}} = \begin{bmatrix} \frac{\partial\hat{\alpha}}{\partial\alpha} & \frac{\partial\hat{h}(\alpha\ltimes h)}{\partial\alpha} \\ \frac{\partial\hat{\alpha}}{\partial h} & \frac{\partial\hat{h}(\alpha\ltimes h)}{\partial h} \end{bmatrix} \qquad D\,\mathcal{R}_{\text{NP-b}} = \begin{bmatrix} \frac{\partial\check{\alpha}}{\partial\alpha} & \frac{\partial\check{h}(\alpha\ltimes h)}{\partial\alpha} \\ \frac{\partial\check{\alpha}}{\partial h} & \frac{\partial\check{h}(\alpha\ltimes h)}{\partial h} \end{bmatrix}$$

Combining the previous bounds:

$$\left|D\,\mathcal{R}_{\text{NP-t}}\right| \simeq \begin{bmatrix} \frac{1}{\alpha^2} & L \\ 0 & 1 \end{bmatrix} \qquad \left|D\,\mathcal{R}_{\text{NP-b}}\right| \simeq \begin{bmatrix} \frac{1}{\alpha^2} & L \\ C & 1 \end{bmatrix}$$

Recall

$$D\,\mathcal{R}_{\text{NP-t}} = \begin{bmatrix} \frac{\partial \hat{\alpha}}{\partial \alpha} & \frac{\partial \hat{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \hat{\alpha}}{\partial h} & \frac{\partial \hat{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix} \qquad D\,\mathcal{R}_{\text{NP-b}} = \begin{bmatrix} \frac{\partial \check{\alpha}}{\partial \alpha} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \check{\alpha}}{\partial h} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix}$$

Combining the previous bounds:

$$\left| D\,\mathcal{R}_{\text{NP-t}} \right| \simeq \begin{bmatrix} \frac{1}{\alpha^2} & L \\ 0 & 1 \end{bmatrix} \qquad \left| D\,\mathcal{R}_{\text{NP-b}} \right| \simeq \begin{bmatrix} \frac{1}{\alpha^2} & L \\ C & 1 \end{bmatrix}$$

What does this imply?

Let $s \mapsto \Upsilon(s) = (\alpha(s) \ltimes h(s))$, for s in a connected set $\Delta \subseteq \mathbb{C}$, and with values in the set $A_\infty \ltimes \mathcal{F}$.

Let $s \mapsto \Upsilon(s) = (\alpha(s) \ltimes h(s))$, for s in a connected set $\Delta \subseteq \mathbb{C}$, and with values in the set $A_\infty \ltimes \mathcal{F}$.

For $k > 0$, we say that Υ is k-*horizontal*, if Υ is continuous on Δ, and for all $s_1, s_2 \in \Delta$ we have

$$\mathrm{d}_{\mathsf{Teich}}(h(s_1), h(s_2)) \leq k|\alpha(s_1) - \alpha(s_2)|.$$

Let $s \mapsto \Upsilon(s) = (\alpha(s) \ltimes h(s))$, for s in a connected set $\Delta \subseteq \mathbb{C}$, and with values in the set $A_\infty \ltimes \mathcal{F}$.

For $k > 0$, we say that Υ is k-*horizontal*, if Υ is continuous on Δ, and for all $s_1, s_2 \in \Delta$ we have

$$\mathrm{d}_{\mathsf{Teich}}(h(s_1), h(s_2)) \leq k|\alpha(s_1) - \alpha(s_2)|.$$

Theorem (Ch., Shishikura, 2015)

There are $\rho' > 0$ and $k > 0$ such that for every k-horizontal curve Υ in $A_{\rho'} \ltimes \mathcal{F}$, the curves $\mathcal{R}_{\text{NP-t}}(\Upsilon)$ and $\mathcal{R}_{\text{NP-b}}(\Upsilon)$ are k-horizontal in $A_\infty \ltimes \mathcal{F}$.

Let $s \mapsto \Upsilon(s) = (\alpha(s) \ltimes h(s))$, for s in a connected set $\Delta \subseteq \mathbb{C}$, and with values in the set $A_\infty \ltimes \mathcal{F}$.

For $k > 0$, we say that Υ is k-*horizontal*, if Υ is continuous on Δ, and for all $s_1, s_2 \in \Delta$ we have

$$\mathrm{d}_{\mathsf{Teich}}(h(s_1), h(s_2)) \leq k|\alpha(s_1) - \alpha(s_2)|.$$

Theorem (Ch., Shishikura, 2015)

There are $\rho' > 0$ and $k > 0$ such that for every k-horizontal curve Υ in $A_{\rho'} \ltimes \mathcal{F}$, the curves $\mathcal{R}_{\text{NP-t}}(\Upsilon)$ and $\mathcal{R}_{\text{NP-b}}(\Upsilon)$ are k-horizontal in $A_\infty \ltimes \mathcal{F}$.

In other words, $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ map cone-fields of k_1-horizontal curves into themselves.

Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots) \in \{t, b\}^{\mathbb{N}}$. For $n \geq 1$, consider

$$\Lambda(\langle \kappa_i \rangle_{i=1}^{n}) = \left\{ \alpha \ltimes h \;\middle|\; \mathcal{R}_{\mathrm{NP}\text{-}\kappa_{\mathrm{n}}} \circ \cdots \circ \mathcal{R}_{\mathrm{NP}\text{-}\kappa_1}(\alpha \ltimes h) \text{ is defined} \right\}.$$

Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots) \in \{t, b\}^{\mathbb{N}}$. For $n \geq 1$, consider

$$\Lambda(\langle \kappa_i \rangle_{i=1}^{n}) = \left\{ \alpha \ltimes h \;\middle|\; \mathcal{R}_{\mathrm{NP}\text{-}\kappa_{\mathrm{n}}} \circ \cdots \circ \mathcal{R}_{\mathrm{NP}\text{-}\kappa_1}(\alpha \ltimes h) \text{ is defined} \right\}.$$

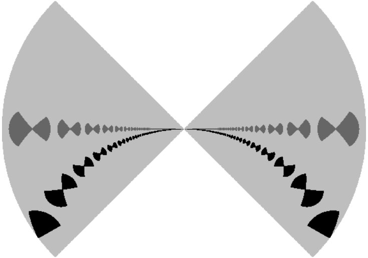
Example, $\Lambda(\kappa_1) = A_\rho \ltimes \mathcal{F}$

Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots) \in \{t, b\}^{\mathbb{N}}$. For $n \geq 1$, consider

$$\Lambda(\langle \kappa_i \rangle_{i=1}^{n}) = \left\{ \alpha \ltimes h \;\middle|\; \mathcal{R}_{\text{NP-}\kappa_n} \circ \cdots \circ \mathcal{R}_{\text{NP-}\kappa_1}(\alpha \ltimes h) \text{ is defined} \right\}.$$

Example, $\Lambda(\kappa_1) = A_\rho \ltimes \mathcal{F}$

$\Lambda(t, \kappa_2)=$ "dark grey region" $\ltimes \mathcal{F}$; $\Lambda(b, \kappa_2) \simeq$ "black region" $\ltimes \mathcal{F}$:



The invariance of k-horizontal curves implies that

Theorem (Ch., Shishikura 2015)

For all k-horizontal family of maps $\Upsilon : A_{\rho'} \to A_{\rho'} \ltimes \mathcal{F}$ and all $\kappa \in \{t, b\}^{\mathbb{N}}$, every connected component of the set $\Lambda(\kappa) \cap \Upsilon(A_{\rho'})$ is a single point.

It follows from the above Theorem and some more work:

Theorem (Ch., Shishikura 2015)

The renormalizations operators $\mathcal{R}_{\text{NP-t}}$ *and* $\mathcal{R}_{\text{NP-b}}$ *are uniformly hyperbolic on* $A_{\rho'} \ltimes \mathcal{F}_0$.
Moreover, $D\,\mathcal{R}_{\text{NP-t}}$ *and* $D\,\mathcal{R}_{\text{NP-b}}$ *at each point in* $A_{\rho'} \ltimes \mathcal{F}_0$ *have an invariant one-dimensional expanding direction and an invariant uniformly contracting co-dimension-one direction.*

It follows from the above Theorem and some more work:

Theorem (Ch., Shishikura 2015)

The renormalizations operators $\mathcal{R}_{\text{NP-t}}$ *and* $\mathcal{R}_{\text{NP-b}}$ *are uniformly hyperbolic on* $A_{\rho'} \ltimes \mathcal{F}_0$*.*
Moreover, $D\,\mathcal{R}_{\text{NP-t}}$ *and* $D\,\mathcal{R}_{\text{NP-b}}$ *at each point in* $A_{\rho'} \ltimes \mathcal{F}_0$ *have an invariant one-dimensional expanding direction and an invariant uniformly contracting co-dimension-one direction.*

The above theorem has applications to

- the Feigenbaum-Coullet-Tresser *universality of the scaling laws*,

It follows from the above Theorem and some more work:

Theorem (Ch., Shishikura 2015)

The renormalizations operators $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ are uniformly hyperbolic on $A_{\rho'} \ltimes \mathcal{F}_0$.
Moreover, $D\,\mathcal{R}_{\text{NP-t}}$ and $D\,\mathcal{R}_{\text{NP-b}}$ at each point in $A_{\rho'} \ltimes \mathcal{F}_0$ have an invariant one-dimensional expanding direction and an invariant uniformly contracting co-dimension-one direction.

The above theorem has applications to

- the Feigenbaum-Coullet-Tresser *universality of the scaling laws*,
- the geometry of the Mandelbrot set (local-connectivity),

It follows from the above Theorem and some more work:

Theorem (Ch., Shishikura 2015)

The renormalizations operators $\mathcal{R}_{\text{NP-t}}$ *and* $\mathcal{R}_{\text{NP-b}}$ *are uniformly hyperbolic on* $A_{\rho'} \ltimes \mathcal{F}_0$.
Moreover, $D\,\mathcal{R}_{\text{NP-t}}$ *and* $D\,\mathcal{R}_{\text{NP-b}}$ *at each point in* $A_{\rho'} \ltimes \mathcal{F}_0$ *have an invariant one-dimensional expanding direction and an invariant uniformly contracting co-dimension-one direction.*

The above theorem has applications to

- the Feigenbaum-Coullet-Tresser *universality of the scaling laws*,
- the geometry of the Mandelbrot set (local-connectivity),
- dynamics of infinitely polynomial-like renormalizable quadratic polynomials with degenerating geometries,