One dimensional real and complex dynamics

Lecture notes for Mathematics Taught Course Centre, Spring 2014

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Lecture 1

Fatou/Julia theory

Normal families of holomorphic maps

A map f from an open set $U \subseteq \mathbb{C}$ to $\mathbb{C}$ is called *holomorphic*, or complex analytic, if the first derivative

$$z \mapsto f'(z) = \lim_{h \to 0}(f(z+h) - f(z)/h)$$

is defined and continuous as a function from U to $\mathbb{C}$. Holomorphic maps from an open set in $\hat{\mathbb{C}}$ into $\hat{\mathbb{C}}$ are defined similarly, using local charts. Recall that a holomorphic map of the Riemann sphere to itself is of the form $P(z)/Q(z)$, for some polynomials P and Q.

A holomorphic map f is called *conformal* if the derivative f' is non-zero everywhere on U. It is called *univalent* if it is conformal and one-to-one on U.

The orbits of a holomorphic map f from $\hat{\mathbb{C}}$ or $\mathbb{C}$ to itself are defined as the sequence of points

$$z_0, z_1, z_2, \ldots, \text{ where } z_{n+1} = f(z_n).$$

To study orbits of a holomorphic map, it is more convenient to study families of nearby orbits at once. This allows to take advantage of the underlying complex structure on the domain and range of the map. For this purpose, we need to introduce a topology on the space of maps defined on a fixed domain.

Let U and V be open subsets of the Riemann sphere $\hat{\mathbb{C}}$, and let $f_n : U \to V$, $n = 1, 2, \ldots$ be a sequence of holomorphic maps. We say that the sequence f_n converges in the *compact-open* topology, or in the topology of *uniform convergence on compact sets*, if there is a holomorphic map $g : U \to V$ such that for every compact subset $K \subseteq U$ the sequence f_n converges uniformly to g on K. On the other hand, we say that the sequence f_n *diverges locally uniformly from* V, if for all compact subsets $K \subseteq U$ and $K' \subseteq V$, we have $f_n(K) \cap K' = \emptyset$, for sufficiently large n. Note that when the target space V is the whole Riemann sphere $\hat{\mathbb{C}}$ no sequence of maps may diverge locally uniformly from V.

A family of maps $\mathcal{F}$ from U to V is called *normal* if every sequence in $\mathcal{F}$ has a subsequence that either converges uniformly on compact subsets of U, or diverges locally uniformly from V. It is convenient (and important) to have a simple criterion for the property of normality for families of maps.

Theorem 1.1 (Montel). *Let U be an open subset of $\hat{\mathbb{C}}$ and $\mathcal{F}$ be a family of maps from U to $\hat{\mathbb{C}}$ which omits three different values. That is, there are distinct points a, b, c in $\hat{\mathbb{C}}$ such that $f(U) \subseteq \hat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathcal{F}$. Then, the family $\mathcal{F}$ is a normal family.*

The proof of the above theorem is based on some foundational theorems in complex analysis and is not proved here. Interested reader may consult Milnor's book "dynamics in one complex variable".

Exercise 1.1. Consider the family of maps $f_n(z) = z + n$ from $\mathbb{C}$ or $\hat{\mathbb{C}}$ to itself. Show that this sequence diverges locally uniformly from $\mathbb{C}$. However, as a family of maps on $\hat{\mathbb{C}}$, this sequence neither converges or diverges locally uniformly, although it converges pointwise to a constant map on $\hat{\mathbb{C}}$. Similarly, show that the family of rational maps $g_n(z) = 1/(n^2 z - n)$ converges pointwise to a constant function, but does not converge locally uniformly.

Exercise 1.2. Show that normality is a local property. That is, let $\mathcal{F}$ is a family of holomorphic maps from U to V. If every point in U has a neighborhood on which the restrictions of maps in $\mathcal{F}$ is normal, then the family $\mathcal{F}$ on U is normal. (Hint: use the diagonal argument.)

Exercise 1.3. Let U be an open subset of $\hat{\mathbb{C}}$ and V be a compact subset of $\hat{\mathbb{C}}$. Show that a family $\mathcal{F}$ of holomorphic maps $f : U \to V$ is normal, if the derivatives $|f'(z)|$ are uniformly bounded as f varies in $\mathcal{F}$ and z varies in a compact subset of U.

Fatou and Julia sets

For a rational map f of $\hat{\mathbb{C}}$ we denote the n times composition of f with itself using $f^{\circ n}$, that is

$$f^{\circ n} = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}.$$

The *Fatou set* of f is the set of points in $\hat{\mathbb{C}}$ that have a neighborhood on which the family of maps $\{f^{\circ n}\}_{n=0}^{\infty}$ is normal. The complement of the Fatou set of f is called the *Julia set* of f. [1] We shall denote these sets by the notations $F(f)$ and $J(f)$, respectively.

By definition, the Fatou set is an open subset of the phase space, and the Julia set is a closed subset of it. Also, it follows that $F(f)$ and $J(f)$ are fully invariant under f, that is, $z \in F(f)$ if and only if $f^{-1}(z) \subseteq F(f)$.

Example 1.2. Let $f(z) = z^2$ on $\hat{\mathbb{C}}$. The iterates are given by the formula $f^{\circ n}(z) = z^{2^n}$. Inside the unit disk $|z| < 1$, the family of iterates $f^{\circ n}$ converges uniformly on compact sets to the constant map 0, and similarly on the set $|z| > 1$ the family uniformly converges to the constant map ∞ on compact sets. (As a family of maps on $\mathbb{C}$, the iterates $f^{\circ n}$ diverges uniformly from $\mathbb{C}$.) Thus, the complement of the unit circle is contained in the Fatou set of f.

[1] The choice of names are after Pierre Fatou (1878–1929) and Gaston Julia (1893–1978), who started the systematic study of the global dynamics of rational maps.

On the other hand, if $|z| = 1$, the family of iterates $f^{\circ n}$ is not normal on any neighborhood of z. That is, in any neighborhood of z there are points whose orbit tends to infinity and there are points whose orbit tends to 0. Therefore, forcing any limiting function of uniformly converegnt subsequence to be discontinuous at z.

The following is a simple consequence of the definition of the Fatou set.

Exercise 1.4. For any $k \in \mathbb{N}$, the Julia set $J(f^{\circ k})$ of the k-fold iterate is identical with $J(f)$.

A rational function of $\hat{\mathbb{C}}$ may be written as the ratio of two polynomials $P(z)/Q(z)$. Then, its *degree* is given as the maximum of the degrees of the polynomials P and Q. The degree of a rational map determines the number of elements in $\{f^{-1}(z)\}$, for any z, counted with multiplicities. It follows that the degree of the n-fold composition of a rational map of degree d is equal to d^n.

Lemma 1.3. *If f is a rational map of the Riemann sphere with* $\deg f \geq 2$*, then the Julia set of f is not empty.*

Proof. If the Julia set is empty, then the family of maps $f^{\circ n}$ must be normal over the whole Riemann sphere. That is, there exits a sequence of iterates $f^{\circ n_j}$, with $n_j \to \infty$, converging to some holomorphic map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. However, g has finite degree, and any map close enough to it must have the same degree. That is because, any two maps of $\hat{\mathbb{C}}$ with distance less than the distance between antipodal points are isotopic. In particular, the iterates $f^{\circ n_j}$ have the same degree for large enough n_j. This contradiction shows that $J(f)$ must be non-empty. □

Exercise 1.5. Any degree one rational map f has either one (indeed it is two if counted with multiplicity) or two fixed points. Show that the derivative of f at the two fixed points are reciprocal numbers λ and $1/\lambda$, for some arbitrary $\lambda \in \mathbb{C}$. Then, classify all possibilities for $J(f)$ and $F(f)$, based on λ.

Periodic points

A *periodic point* of f is a point z with orbit relation

$$z, f(z), f^{\circ 2}(z), \dots, f^{\circ p}(z) = z.$$

A *periodic cycle* is the orbit of a periodic point, that is, the set

$$\{z, f(z), f^{\circ 2}(z), f^{\circ p}(z) = z\}.$$

The *period* of a periodic point z is the smallest positive integer p where $f^{\circ p}(z) = z$. The *multiplier* of a periodic point z of period p is defined as $(f^{\circ p})'(z)$. Note that the multiplier is the same for all points in the orbit of a periodic point. That is, all the points in the orbit of a periodic point have the same multiplier.

In the special case that the point $\infty \in \hat{\mathbb{C}}$ is part of the orbit of a periodic point, special attention should be made when calculating the multiplier of the periodic point! For example, ∞ is a fixed point of $f(z) = z^2$ with multiplier 0!

Periodic points of a rational map are classified into several categories based on the multiplier λ of the cycle. The cycle is called *attracting* if $|\lambda| < 1$, and repelling if $|\lambda| > 1$. The especial case where $\lambda = 0$ is called *super-attracting*. When $|\lambda| = 1$ the cycle or the periodic point is called *indifferent*, or *neutral*. This case breaks down into two cases based on the argument of $\lambda/2\pi$. The cycle is called *rationally indifferent*, or more commonly *parabolic*, if $\lambda = e^{2\pi \frac{p}{q} i}$ for some $p/q \in \mathbb{Q}$, and is called *irrationally indifferent* if $\lambda = e^{2\pi\alpha i}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

For an attracting periodic point $z \in \hat{\mathbb{C}}$ of period p, the basin of attraction of the cycle of z is defined as

$$\mathcal{A} = \{z' \in \hat{\mathbb{C}} \mid d(f^{\circ n}(z'), \{z, f(z), \dots, f^{\circ p-1}(z)\}) \to_{n\to\infty} 0\}$$

where d denotes the spherical distance on $\hat{\mathbb{C}}$ (or any other metric equivalent to it).

Proposition 1.4. *The basin of attraction of every attracting periodic point is contained in the Fatou set. In particular, every attracting periodic point is contained in the Fatou set. On the other hand, every repelling periodic point is contained in the Julia set.*

Proof. First let us assume that z is a fixed point of f. If the multiplier λ at z satisfies $|\lambda| > 1$, then the derivatives $(f^{\circ n})'(z) = \lambda^n$ tends to infinity as $n \to \infty$. Thus, no subsequence of the family may converge to some holomorphic map on any neighborhood of z. That is because, if a sequence of maps converges to a map uniformly on an open set, the sequence of its derivatives must converge to the derivative of the map (recall the Cauchy integral formula).

Now assume that z is an attracting periodic point of f with $|f'(z)| = \lambda < 1$. There exits a neighborhood U of z such that $|f'|$ is bounded by some $c < 1$. If follows from the Taylor's theorem that the sequence of iterates $f^{\circ n}$ converges uniformly on compact sets to the constant map $g = z$. Now, the result on any compact subset of the basin of the attraction of z follows.

The result for periodic points with period bigger than one follows from the one for the fixed point and exercise 1.4 □

Two rational maps f and g are conformally conjugate on $\hat{\mathbb{C}}$, if there exists a Mobius transformation $\theta(z) = (az + b)/(cz + d)$, with $ad - bc \neq 0$, such that $f \circ \theta = \theta \circ g$ on $\hat{\mathbb{C}}$. It follows that $J(f) = \theta(J(g))$.

Proposition 1.5. *If* $\deg f \geq 2$, *then every parabolic periodic point lies in* $J(f)$.

Proof. Composing with a Mobius transformation, we may assume that the parabolic periodic point is at 0. Then there is some iterate $f^{\circ n}$ near 0 that has power series expansion $z \mapsto z + a_m z^m + a_{m+1} z^{m+1} + \dots$, near 0, where a_m is non-zero. The iterates $f^{\circ(jn)}$ have the form $z \mapsto z + j a_m z^m + \dots$. The m-th derivative of $f^{\circ(jn)}$ at 0 is equal to $j a_m m!$, and hence converges to ∞ as $j \to \infty$. This implies that no subsequence of the iterates of f may converge uniformly on any neighborhood of 0. □

Proposition 1.6 (transitivity)**.** *Let z be a point in the Julia set of some rational map f, and B be an arbitrary neighborhood of z. Then, the set $U = \cup_{n=0}^{\infty} f^{\circ n}(B)$ contains all of $\hat{\mathbb{C}}$ except at most two points. In particular, U contains $J(f)$.*

Proof. This immediately follows from Montel normal family theorem, Theorem 1.1 □

The above proposition implies some interesting properties of the Julia set.

Corollary 1.7. *If the Julia set contains an interior point, then it must be the entire Riemann sphere.*

Corollary 1.8. *Let f be a rational map of $\hat{\mathbb{C}}$ with $\deg(f) \geq 2$. If z is an arbitrary point in $J(f)$, then the set of iterated pre-images*

$$\{z' \in \hat{\mathbb{C}} \mid \exists n \in \mathbb{N} \text{ such that } f^{\circ n}(z') = z\}$$

is everywhere dense in $J(f)$.

Proof. First we show that if $z \in J(f)$, then the set of its pre-images is an infinite set. Suppose on the contrary that the set of the pre-images of z is a finite set A. It follows that A consists of a single cycle $z = z_1 \mapsto z_2 \mapsto \ldots \mapsto z_m = z_1$. On the other hand each point in $\hat{\mathbb{C}}$ has d pre-images counted with multiplicity, which implies that each point in the orbit of z_1 must be a critical point. That is, z is a super-attracting periodic point of f and hence, must be in the Fatou set.

Now let z' be an arbitrary point in $J(f)$ and U be an arbitrary neighborhood of z'. By Proposition 1.6, the union of the forward iterates of U contains $\hat{\mathbb{C}}$ except possibly at most two points. Then by the previous argument, some pre-image of z must be contained in U. □

Corollary 1.9. *If f has degree two or more, then $J(f)$ has no isolated point.*

Proof. By the proof of the previous corollary, for any z in $J(f)$, the set of pre-images of z is an infinite set. In particular, it contains an accumulation point z'. Now the set of pre-images of z' is dense in $J(f)$. □

By definition, a property of points is called *generic* in some complete metric space X, if it holds for all points in a countable intersection of open dense subsets of X.

Proposition 1.10. *Let f be a rational map with deg$(f) \geq 2$. For a generic choice of a point $z \in J(f)$, the forward orbit*

$$\{z, f(z), f^{\circ 2}(z), \ldots\},$$

is everywhere dense in $J(f)$.

Proof. For $n \in \mathbb{N}$, $J(f)$ may be covered by a finite number of sets $A_1, A_2, \ldots, A_{m(n)}$ of diameter at most $1/n$. By Corollary 1.8, the set $U_{n,j}$, for $j = 1, 2, \ldots, m(n)$, of pre-images of A_j is an open dense subset of $J(f)$. Let $U_n = \cap_{j=1}^{m(n)} U_{n,j}$, and note that if some z belongs to U_n, then its forward orbit crosses each A_j, for $j = 1, 2, \ldots, m(n)$. The intersection of U_n, for $n \in \mathbb{N}$, consists of points with dense forward orbit. □

Lecture 2

Classification of Fatou components

Parabolic fixed points

In the previous lecture we showed that there is a rather simple local dynamics near an attracting fixed point (and hence near every attracting periodic point). The local dynamics near a repelling fixed point may be understood by looking at the inverse of the map. Also, we showed that every parabolic periodic point belongs to the Julia set. Although this implies that iterates of a map near a parabolic periodic point does not form a normal family, it may still be possible to understand large iterates of a rational map near parabolic fixed points. In this section we look at this local dynamics near a parabolic fixed point.

Let f be a rational map of degree at least two, with expansion $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ near $z = 0$, where the multiplier at 0, λ, is a root of unity. We first assume the simplest case where $\lambda = 1$, and assume a_{n+1} is the first non-zero coefficient of f after λ. That is

$$\begin{aligned} f(z) &= z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \\ &= z(1 + a_{n+1} z^n + \dots), \end{aligned}$$

with $a_{n+1} \neq 0$. The integer $n+1$ is called the *multiplicity* of the fixed point at 0. There are n complex numbers v with $n a_{n+1} v^n = 1$, and n complex numbers v with $n a_{n+1} v^n = -1$. Let us label these $2n$ vectors as v_j, with $0 \leq j \leq 2n-1$, such that

$$v_j = e^{\pi \mathbf{i} j/n} v_0, \qquad \text{and} \qquad n a_{n+1} v_j = (-1)^j.$$

That is, the even indexed numbers are called the *repulsion vectors* for f at 0, while the odd indexed numbers are called the *attraction vectors* for f at 0.

Proposition 2.1. *Let f be a holomorphic map defined on a neighborhood of 0 with expansion $z + az^{n+1} + \dots$, with repulsion and attraction vectors an above. The, if an orbit $z_n = f^{\circ n}(z)$ converges to 0, then either*

- *the sequence is eventually identical to zero, or*
- *$\lim_{k \to \infty} k^{1/n} z_k$ exists and is equal to one of the attraction directions v_j, j odd.*

Similarly, if an inverse orbit $z_1, z_2, z_3, \dots$ of f^{-1} tends to 0 then either

- *the sequence is eventually identical to zero, or*
- $\lim_{k\to\infty} k^{1/n} z_k$ *exists and is equal to one of the repulsion directions* v_j*,* j *even.*

Finally, any of the $2n$ *attraction and repulsion vectors can occur for some (indeed an open set of) not eventually constant sequence.*

Note that the local attraction and repulsion property is preserved under conformal changes of coordinates.

Proof. We work in the coordinate

$$w = \varphi(z) = \frac{c}{z^n}, \text{ where } c = \frac{-1}{na}.$$

The attraction and repulsion vectors are mapped to

$$\varphi(v_j) = \operatorname{Re}\varphi(v_j) = (-1)^{j+1}.$$

First we find appropriate inverse branches for the n-valued function φ^{-1}. To this end, let $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$. Then, consider the half lines $\mathbb{R}_+ v_j$ that cut the plane into sectors of angle π/n. Define,

$$\Delta_j = \{re^{i\theta} v_j \mid r \in (0,\infty), \theta \in (-\pi/n, \pi/n)\}, \text{for } 0 \le j \le 2n-1.$$

Then, φ maps each Δ_j univalently to a slit plane according to

$$\varphi(\Delta_j) = \begin{cases} \mathbb{C}\setminus\mathbb{R}_+ & \text{if } j \text{ even} \\ \mathbb{C}\setminus\mathbb{R}_- & \text{if } j \text{ odd} \end{cases}.$$

Then, there are inverse branches of φ on each Δ_j with

$$\psi_j : \mathbb{C}\setminus\mathbb{R}_{(-1)^j} \to \Delta_j.$$

The intersection of any two consecutive sectors, $\Delta_j \cap \Delta_{j+1}$ (indexes are modulo $2n$) is a sector of angle π/n. Any such intersection is mapped either to the upper half plane or the lower half plane, depending on whether j is odd or even. Recall that

$$f(z) = z(1 + az^n + o(z^n)), \text{ as } z \to 0,$$

where $o(z_n)$ stands for the remainder that satisfies $o(z^n)/z^n \to 0$ as $z \to 0$. Define the maps

$$w \mapsto F_j(w) = \varphi \circ f \circ \psi_j(w), \text{for } 0 \le j \le 2n-1.$$

which are defined on slit w planes. To understand the behavior of f near 0 we study the maps F_j for large values of $|w|$. Note that ψ_j is given as a branch of $w \mapsto (c/w)^{1/n}$. Thus,

$$f \circ \psi_j(w) = (\frac{c}{w})^{\frac{1}{n}}(1 + a\frac{c}{w} + o(\frac{1}{w})), \text{ as } |w| \to \infty.$$

Then,

$$F_j(w) = w(1 + a\frac{c}{w} + o(\frac{1}{w}))^{-n} = w(1 + \frac{-nac}{w} + o(\frac{1}{w})) = w + 1 + o(1), \text{ as } |w| \to \infty,$$

for all j. By definition of little o we can choose $R \in \mathbb{R}$ such that

$$|F(w) - (w+1)| < 1/2, \text{for } |w| > R.$$

In particular, the above equation implies that

$$\operatorname{Re} F(w) > \operatorname{Re} w + 1/2, \text{for } |w| > R. \tag{2.1}$$

This translates to the inequality

$$\operatorname{Re} \varphi(f(z)) > \operatorname{Re} \varphi(z) + 1/2, \text{for } |z| < (\frac{1}{naR})^{\frac{1}{n}}. \tag{2.2}$$

Define

$$\mathbb{H}_R = \{w \in \mathbb{C} \mid \operatorname{Re} w > R\}, \qquad \text{and} \qquad \mathcal{P}_j(R) = \psi_j(\mathbb{H}_R).$$

For odd values of j, F_j is defined on $\mathbb{H}_R$ and by the above property, F_j maps $\mathbb{H}_R$ well into itself. Indeed, for any such j, and $w \in \mathbb{H}_R$ the iterates of w under F_j tend to ∞ in $\mathbb{H}_R$. In terms of f, this implies that f maps $\mathcal{P}_j(R)$ into itself, and the iterates of f on $\mathcal{P}_j(R)$ converge to 0 uniformly on compact subsets of $\mathcal{P}_j(R)$.

Now, let $z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots$ be an orbit of f that tends to 0 with all points in the orbit different from 0. Then, by Equation 2.2, for large enough k, we have $\operatorname{Re} \varphi(z_k) \geq R$. This implies that, there is k such that z_k belongs to one of the attracting petals $\mathcal{P}_j(R) \subseteq \Delta_j$. However, since $f(\mathcal{P}_j(R)) \subseteq \mathcal{P}_j(R)$, for $m \geq k$, we have $z_m \in \mathcal{P}_j(R)$.

Consider the sequence $w_j = \varphi(z_j)$, for $j \in \mathbb{N}$. By definition, $w_k \in \mathbb{H}_R$ and $F_j(w_m) = w_{m+1}$, for all $m \geq k$. By Equation 2.1, we conclude that $\operatorname{Re} w_m \to +\infty$ as $m \to +\infty$. In particular, $|w_m| \to +\infty$, and therefore, $w_{m+1} - w_m \to +1$, as $m \to +\infty$.

By the above paragraph, the average

$$\frac{w_m - w_0}{m} = \frac{1}{m} \sum_{j=0}^{m-1} w_{j+1} - w_j.$$

converges to $+1$, as $m \to +\infty$. That is, w_m/m tends to $+1$ as $m \to +\infty$. Recall that $1/w_m = -naz_m^n$ and $nav_j^n = -1$. Thus, $-namz_m^n \to +1$, and hence, $mz_m^n/v_j^n \to +1$. Since z_m belongs to the petal $\mathcal{P}_j(R)$, it follows that $m^{1/n} z_m/v_j \to +1$, as $m \to +\infty$.

The proof for the repulsion directions is done similarly, by looking at the inverse of the map F_j on some left half plane $-\mathbb{H}_R$. This finishes the proof of the Proposition.

□

Exercise 2.1. Let f be a holomorphic map with a parabolic fixed point at 0. Using the proof of the above proposition show that there is a neighborhood U of 0 such that f has no periodic cycle in U.

Mini-project 1. *There are several mini projects on the dynamics of maps close to a map with a parabolic cycle. When a rational map with a parabolic fixed point is perturbed slightly in an appropriate way, a new periodic point is created (parabolic explosion) near the fixed point. A project may be on the speed of how fast the newly born cycle moves as the map is perturbed away from the original rational map. Another project may be on a technique called parabolic renormalization to describe the local dynamics of the perturbations of a map with a parabolic fixed point.*

If 0 is a parabolic periodic point of f with multiplier $\lambda = 1$, we consider the return map $f^{\circ k}$ where k is the period of the periodic point. Then, 0 is a fixed point of $f^{\circ k}$ with some multiplicity $n + 1 \geq 2$. Using the above proposition, we have n attraction vectors v_j, for j odd, and n repulsion vectors v_j for j even, for the local dynamics of $f^{\circ k}$. Then the conformal map f sends these attraction and repulsion vectors to n attraction and n repulsion vectors at $f(0)$. That is, the sequence of vectors tend to $f(0)$ under $f^{\circ k}$ in one of these attraction vectors, or the inverse orbits of $f^{\circ k}$ tend to $f(0)$ along one of these repulsion vectors. Similarly, this local behavior may be described near any point in the cycle of 0, using one of n attraction or n repulsion vectors.

If 0 is a parabolic fixed point of f with multiplier $\lambda = e^{2\pi i p/q}$, for some rational p/q, the local dynamics may be analyzed similarly.

Lemma 2.2. *Let* 0 *be a parabolic fixed point of a rational map of degree* ≥ 2 *with multiplier* $\lambda = e^{2\pi i p/q}$ *where* $(p, q) = 1$. *Then the number* n *of the attraction vectors at* 0 *must be a multiple of* q.

The proof of the above lemma is similar to the argument in the above paragraph and is left to the reader.

Consider a rational map with a parabolic fixed point at z with multiplier $+1$. For each attraction vector v_j at 0, we may consider the set A_j of all points z_0 in $\hat{\mathbb{C}}$ whose orbit $z_0, z_1, z_2, \ldots$ converges to 0 from the direction v_j. A_j is called the *basin of attraction of* z *in the direction* v_j. By definition, the basins of attractions A_1, A_2, $\ldots$, A_n, are mutually disjoint. The immediate basin of attraction A_j^0 of z in the direction v_j is defined as the unique connected component of A_j that maps into itself.

In general, for a rational map f with a parabolic periodic point z of period k and multiplier $e^{2\pi i p/q}$, the iterate $f^{\circ kq}$ has a parabolic fixed point of multiplier $+1$ at z. Then the parabolic basins of attraction are defined similarly as the attractions and repulsion vectors at other points on the cycle of z.

Lemma 2.3. *For every rational map* f *of degree at least* 2 *with a parabolic periodic point, every parabolic basin of attraction is contained in the Fatou set of* f. *But each basin boundary is contained in the Julia set* $J(f)$.

Proof. This follows from the proof of Proposition 2.1. Further details are left to the reader. □

Dynamics on cycles of Fatou components

A map $f : U \to V$ is called *proper*, if the preimage of every compact set in V is a compact subset of U.

Let f be a rational map of the Riemann sphere with degree $d \geq 2$. Let U be a connected component of the Fatou set of f. It follows from the definition of the Fatou set that $U' = f(U)$ is also a Fatou component of f and $f : U \to U'$ is a proper holomorphic map. Let us first assume that $U' = U$.

Theorem 2.4. *If a rational map f of degree $d \geq 2$ maps a Fatou component U to itself, then there are at most four possibilities,*

1) U is the immediate basin for an attracting fixed point,

2) U is a petal of a parabolic fixed point with multiplier equal to one,

3) U is a simply connected domain and $f : U \to U$ is conformally conjugate to an irrational rotation on the unit disk,

4) U is a doubly connected region and $f : U \to U$ is conformally conjugate to an irrational rotation on an annulus.

We have already seen examples of rational maps with Fatou comments of the first two types. In the third case the component U is called the Siegel disk of f at the unique fixed point inside U. In the fourth case, the component U is called a Herman ring of f. Fatou had anticipated the existence of such connected components of the Siegel disk, without any proof. It is a non-trivial result of C.L. Siegel that there are examples of rational maps with Siegel disks. Indeed, he showed that there are quadratic maps with Siegel disks. The first examples of the fourth type is due to M. Herman. By the maximum principle, a polynomial map of $\mathbb{C}$ can not have a Herman ring. The proofs are rather complicated and out of scope of this course.

An Indifferent periodic point in the Julia set is called a *Cremer* periodic point. This is after H. Cremer who gave the first general arithmetic condition on the rotation of the multiplier at the cycle that implies the cycle is in the Julia set.

One may make a similar statement as in the above theorem for periodic Fatou components. If a Fatou component U is pre-periodic, that is, there is $n \in \mathbb{N}$ such that $f^{\circ n}(U)$ is periodic, then the dynamics on U is also described by the above theorem.

The proof of the above theorem is rather long and requires some language of discrete actions on the Riemann sphere. So, it is not covered in this course.

Mini-project 2. *The proof of the above theorem may be the subject of a mini-project.*

A Fatou component U is called wandering if $f^{\circ j}(U) \cap f^{\circ i}(U) = \emptyset$ for distinct positive integers i and j. A central problem in complex dynamics was whether wandering domains may be realized by a rational map.

Theorem 2.5 (Sullivan)**.** *Every Fatou component of a rational map of the Riemann sphere is pre-periodic.*

Mini-project 3. *Proof of the above theorem.*

Once we know all Fatou components are pre-periodic, the natural question is whether the number of periodic Fatou components is bounded. If the answer is yes, one wants to know an optimal bound on the number of periodic Fatou components.

Theorem 2.6 (Shishikura)**.** *A rational map of degree d has at most $2d-2$ cycles of Fatou components.*

Mini-project 4. *Proof of the above theorem.*

The three theorems listed above, completely describe the nice part of the dynamics of a rational map. In the remaining lectures of this course we shall mainly focus on various ways to describe the dynamics on the Julia sets.

Lecture 3

Measurable dynamics on Julia sets

In this lecture we review some general features of the measurable dynamics of a rational map on its Julia set.

The post critical set

Let f be a rational map of the Riemann sphere. The *post-critical* set of f is defined as the closure of the forward orbits of the critical values of f, that is

$$\mathcal{PC}(f) = \overline{\{f^{\circ n}(c) \mid n \geq 1, c \in \hat{\mathbb{C}}, f'(c) = 0\}}.$$

The post critical set of f is the smallest closed set containing the critical values of all maps $f^{\circ n}$, $n \in \mathbb{N}$. By definition, $f(\mathcal{PC}(f)) \subseteq \mathcal{PC}(f)$, and $\mathcal{PC}(f) = \mathcal{PC}(f^{\circ n})$, for every $n \in \mathbb{N}$.

A rational map is called *post-critically finite*, if $\mathcal{PC}(f)$ is a finite set.

Expansion on the complement of the post-critical set

Let us first take care of the exceptional case.

Exercise 3.1. Let f be a rational map of degree at least two with $|\mathcal{PC}(f)| < 3$. Then, f is conjugate to some $z \mapsto z^d$, $d \in \mathbb{Z}$, by a Mobius transformation. In particular, the Julia set is a round circle, and has zero area.

From now on we assume that the post-critical set consists of at least 3 points.

Recall that the spherical metric on $\hat{\mathbb{C}}$ is given as

$$|ds| = \frac{2|dz|}{1+|z|^2},$$

where $|dz|$ denotes the Euclidean metric on $\mathbb{C}$.

Let $U \subseteq \hat{\mathbb{C}}$ be an open set. A conformal metric on U is a metric of the form $\rho(z)|dz|$ on U, where $\rho : U \to (0, +\infty)$ is a smooth function.

If $|\hat{\mathbb{C}} \setminus U| \geq 3$, there is a complete *conformal metric* of constant negative curvature on U. Here complete means that every Cauchy sequence (w.r.t $\rho(z)|dz|$) of points in U converges to some point in U. This metric is also sometimes called the *Poincaré* *metric* and the domain U is called hyperbolic. In general, by completeness of the metric, $\rho(z) \to \infty$ as z tends to the boundary of U.

For example, on the open unit disk $|z < 1|$ the hyperbolic metric is given by the formula $\rho(z)dz = 2|dz|/(1 - |z|^2)$, and on the upper half plane $\operatorname{Im} z > 0$ it is given by $|dz|/\operatorname{Im} z$. One can see that these metrics are invariant under Mobius transformations of these spaces.

In particular, if $\mathcal{PC}(f)$ has at least three distinct points then $\hat{\mathbb{C}} \setminus \mathcal{PC}(f)$ is hyperbolic. We shall use the following fundamental lemma from complex analysis.

Lemma 3.1 (Schwarz-Pick Lemma)**.** *Let U and V be hyperbolic open subsets of $\hat{\mathbb{C}}$ (or more generally hyperbolic Riemann surfaces) and $f : U \to V$ be a holomorphic map. Then, exactly one of the following three statements is valid:*

Isometry: *f is a conformal isomorphism from U onto V. That is, it maps U with its Poincaré metric isometrically onto V with its Poincaré metric.*

Covering: *f is a covering map from U onto V but is not one-to-one. It is locally but not globally a poincaré isometry, that is, for all z_1, z_2 in U we have*

$$\operatorname{dist}_V(f(z_1), f(z_2)) \leq \operatorname{dist}_U(z_1, z_2),$$

where equality holds if z_1 and z_2 are sufficiently close in U, and strict inequality holds if they are sufficiently apart.

Contracting: *f strictly decreases all non-zero distances. In fact, for any compact set $K \subseteq U$, there is a constant $c_k < 1$ such that for all $z_1, z_2 \in K$,*

$$\operatorname{dist}_V(f(z_1), f(z_2)) \leq c_K \operatorname{dist}_U(z_1, z_2).$$

Note that by the above lemma, if $f : U \to V$ is a holomorphic map such that $f(U)$ is strictly contained in V, then $f : U \to V$ must be contracting.

Proposition 3.2. *Let f be a rational map of degree at least two and $|\mathcal{PC}(f)| \geq 3$. For every $z \in J(f)$ whose forward orbit does not land in $\mathcal{PC}(f)$,*

$$\| (f^{\circ n})'(z) \| \to \infty,$$

with respect to the hyperbolic metric on $\hat{\mathbb{C}} \setminus \mathcal{PC}(f)$.

Proof. Consider the sets $P_n = f^{-n}(\mathcal{PC}(f))$, for $n \in \mathbb{N}$. First see that

$$f^{\circ n} : \hat{\mathbb{C}} \setminus P_n \to \hat{\mathbb{C}} \setminus \mathcal{PC}(f)$$

are proper local homeomorphism. That is, they are covering maps. Then by Schwarz-Pick lemma, they are local isomorphisms with respect to the Poincaré metrics on $\hat{\mathbb{C}} \setminus P_n$ and $\hat{\mathbb{C}} \setminus \mathcal{PC}(f)$.

On the other hand, since $|\mathcal{PC}(f)| \geq 3$, by Montel's normal family theorem, $J(f)$ is contained in the closure of the union of the sets P_n. In particular, the spherical distance $d(P_n, z) \to 0$ as $n \to +\infty$.

Consider the inclusion maps

$$\tau_n : \hat{\mathbb{C}} \setminus P_n \to \hat{\mathbb{C}} \setminus \mathcal{PC}(f).$$

By the above paragraph, $\| \tau_n'(z) \| \to 0$ as $n \to +\infty$, where the norm of the derivative is calculated using the Poincaré metrics on domain and range. Then, it follows that $\| f^{\circ n} \circ \tau_n^{-1}(z) \| \to +\infty$, as $n \to +\infty$, with respect to the Poincare metric on $\hat{\mathbb{C}} \setminus \mathcal{PC}(f)$. □

The above proposition is only useful when the post-critical set of f is not too large. For example, if $\mathcal{PC}(f) = J(f)$, then it has no content. There are examples of rational maps with $J(f) = \mathcal{PC}(f)$, and there are even quadratic polynomials with this property. This suggests that it is important to control (the geometry of) the post critical set, in many interesting circumstances.

Also, the above proposition suggests that the post critical set is related to attraction in the dynamics of f. We make this more precise in the next two statements.

Corollary 3.3. *Let f be a rational map of degree at least two. The post-critical set of f contains attracting cycles of f and the indifferent cycles of f that lie in the Julia set.*

Proof. For the attracting cycles, since the cycle is not contained in the Julia set one can not directly use the above proposition. However, one applies the proof of the proposition to one iterate of the map f and concludes that $\| f'(z) \| \geq 1$. □

Proposition 3.4. *The boundary of every Siegel disk and Herman ring is contained in the post critical set.*

Proof. Let Δ be a Siegel disk or Herman ring of period p for some map f. Let $h = f^{\circ p}$. If we have $z_0 \in \partial\Delta$ such that $z_0 \notin \mathcal{PC}(f)$, choose $\varepsilon > 0$ with $B_\varepsilon(z_0) \cap \mathcal{PC}(f) = \emptyset$.

For integers $n \geq 0$, let z_{-n} be the pre-image of z_0 under $h^{\circ n}$ contained in Δ. Then define g_n as the inverse branch of $h^{\circ n}$ which maps z_0 to z_{-n}. By making ε small enough, we may assume that all of $g_n(B_\varepsilon(z_0))$, for $n \geq 1$, omit a fixed neighborhood in Δ. In particular, the family $\{g_n\}_{n=0}^{\infty}$ is a normal family.

Define the set $V = \cup_{n=0}^{\infty} g_n(B_\varepsilon(z_0))$. We claim that $h(V) \subseteq V$. Since $h(V) \subseteq V \cup h(B_\varepsilon(z_0))$, we only need to show that $h(B_\varepsilon(z_0)) \subseteq V$. The family g_n has a subsequence that converges to the identity on $B_\varepsilon(z_0) \cap \Delta$. Thus, every subsequence of that subsequence, say g_{n_k}, converges to the same limit on $B_\varepsilon(z_0)$. By the argument principle this implies that every point in $B_\varepsilon(z_0)$ is contained in the image of g_{n_k}, for sufficiently large n_k. That is, the union of $g_{n_k-1}(B_\varepsilon(z_0))$ covers $h(B_\varepsilon(z_0))$.

By the above paragraph, the family of iterates $h^{\circ n} : V \to V$ are defined and normal, and $B_\varepsilon(z_0) \subseteq V$ is contained in the Fatou set. This contradicts the choice of z in the Julia set. □

Indeed, one can see that the boundary of every Siegel disk and Herman ring is contained in the closure of the orbits of some critical points in the Julia set.

Hyperbolic rational maps

A rational map f of the Riemann sphere is called hyperbolic if all critical points of f lie in the basin of attraction of attracting periodic points.

Theorem 3.5. *Let f be a rational map of degree at least two. Then the following statements are equivalent.*

(1) The post critical set of f is disjoint from the Julia set of f.

(2) There are no critical points or parabolic cycles in the Julia set.

(3) Every critical point of f tends to an attracting cycle under iterates of f.

(4) There is a constant $C > 1$ and a smooth conformal metric ρ defined on a neighborhood of $J(f)$ such that $\|f'\|_\rho > C$, for all $z \in J(f)$.

(5) There is an integer $n \geq 1$ such that $f^{\circ n}$ strictly expands the spherical metric on $J(f)$.

Proof. By Exercise 3.1, if $|\mathcal{PC}(f)| = 2$ then, f is conjugate to some $z \mapsto z^d$, and the equivalence of the above statements is clear. So below, we assume that $|\mathcal{PC}(f)| \geq 3$.

(1) ⇒ (2) If $\mathcal{PC}(f) \cap J(f) = \emptyset$, then there is no critical point in $J(f)$. Also, since every parabolic cycle is contained in both $J(f)$ and $\mathcal{PC}(f)$, by Proposition 1.5 and Corollary 3.3, there is no parabolic cycle in the Julia set.

(2) ⇒ (3) Since the boundary of every Siegel disk and Herman ring are contained in the closure of the orbit of some critical points in the Julia set. Since there is no critical point in the Julia set, there can not be Siegel disks and Herman rings. Also, since there is no parabolic cycle in the Julia set, there is no basin of parabolic cycles. Thus, by the classification of Fatou components, the only components of the Fatou set are basins of attracting cycles.

(3) ⇒ (1)
Since attracting cycles are contained in the Fatou set, this is clear.

(3) ⇒ (4)
We have that $\mathcal{PC}(f) \cap J(f) = \emptyset$. We will use the idea of Poincaré expansion outside of $\mathcal{PC}(f)$. Since all critical points are attracted to attracting cycles, $\mathcal{PC}(f)$ is a countable set with a finite number of accumulation points. Thus, the complements of $\mathcal{PC}(f)$ is connected. Similarly, $Q(f) = \hat{\mathbb{C}} \setminus \mathcal{PC}(f)$ is also connected. The covering map

$$f : \hat{\mathbb{C}} \setminus Q(f) \to \hat{\mathbb{C}} \setminus \mathcal{PC}(f)$$

is a covering map, and hence an isometry of respective Poincaré metrics.

On the other hand, since $|\mathcal{PC}(f)| \geq 3$, $Q(f) \setminus \mathcal{PC}(f)$ is a non-empty set, and thus

$$\tau : \hat{\mathbb{C}} \setminus Q(f) \to \hat{\mathbb{C}} \setminus \mathcal{PC}(f)$$

is strictly contracting on $\hat{\mathbb{C}} \setminus Q(f)$, with respect to the hyperbolic metric domain and range. This implies that f expands the Poincaré metric on $\hat{\mathbb{C}} \setminus \mathcal{PC}(f)$. Since, $J(f)$ is a compact subset of $\hat{\mathbb{C}} \setminus \mathcal{PC}(f)$, the expansion is uniform.

(4) $\Rightarrow$ (5)

Choose $K \in \mathbb{R}$ such that $1/K \leq \rho \leq K$ on $J(f)$, and choose $n \in \mathbb{N}$ such that $C^n/K^2 > 2$. Then, $(|f^{\circ n})'| \geq \|(f^{\circ n})'\|/K^2 \geq C^n/K^2 \geq 2$, where $|f'|$ denotes the absolute value of derivative with respect to the spherical metric.

Note that here the spherical metric may be replaced by any other C^1 conformal metric on $\hat{\mathbb{C}}$.

(5) $\Rightarrow$ (2)

Obviously, if there is expansion on $J(f)$, there can not be any critical point on parabolic cycles in $J(f)$. □

Ergodic or attracting

We shall use two classical results from analysis.

Theorem 3.6 (Lebesgue's density theorem). *Let $A \subseteq \mathbb{R}^n$ be a Lebesgue measurable subset. Then, for almost every $x \in A$ we have*

$$\lim_{\varepsilon \to 0} \frac{\text{area}(A \cap B_\varepsilon(x))}{\text{area}\, B_\varepsilon(x)} = 1.$$

Theorem 3.7 (Koebe distortion Theorem). *Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent map with $f(0) = 0$ and $f'(0) = 1$. Then,*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

In particular, $f(\mathbb{D})$ contains the disk of radius $1/4$.

The main result of this section is the following general statement.

Theorem 3.8. *Let f be a rational map of degree at least two and $J(f) \neq \hat{\mathbb{C}}$. Then, the spherical distance $d(f^{\circ n}(z), \mathcal{PC}(f)) \to 0$ as $n \to \infty$, for almost every $z \in J(f)$.*

Proof. Let V be an arbitrary neighborhood of $\mathcal{PC}(f)$. We need to show that for almost every $z \in J(f)$ the orbit of z eventually remains in V.

Consider the set

$$\Gamma := \{z \in J \mid \text{for infinitely many integers } k > 0, f^{\circ k}(z) \notin V\}.$$

If area of Γ is not zero, let z be a Lebesgue density point of Γ. Let n_k be an increasing sequence of positive integers with $f^{\circ n_k}(z) \to y$, for some $y \in \hat{\mathbb{C}} \setminus V$. As $y \notin V$, it has a definite distance δ from $\mathcal{PC}(f)$. For sufficiently large values of n_k, let E_{n_k} denote the component of $f^{\circ(-n_k)}(B(y, \delta/2))$ containing z. As $B(y, \delta/2)$ does not intersect $\mathcal{PC}(f)$, $f^{\circ n_k} : E_{n_k} \to B(y, \delta/2)$ is univalent. Moreover, its inverse has a univalent extension over the larger domain $B(y, \delta)$. Hence, by the Koebe distortion theorem, the domains E_{n_k} have uniformly bounded eccentricities, and the maps $f^{\circ n_k} : E_{n_k} \to B(y, \delta/2)$ have uniformly bounded distortions.

If the domains E_{n_k} do not shrink to z as $n_k \to \infty$, their uniformly bounded eccentricities implies that they contain a ball $B(z, r)$ for some constant $r > 0$. Thus, every $f^{\circ n_k}$ maps $B(z, r)$ into $B(y, \delta)$. This implies that $\{f^{\circ n_k}\}$ is a normal family, by Montel's Theorem, contradicting z being in $J(f)$. Therefore, $\text{diam}\,(E_{n_k}) \to 0$. Furthermore, the family E_{n_k} *shrinks regularly* to z, i.e. there exists a constant $c > 0$ such that for each E_{n_k} there exists a round ball $B \supseteq E_{n_k}$, with

$$\text{area}(E_{n_k}) \geq c \cdot \text{area}\, B.$$

Hence, the Lebesgue density theorem implies that

$$\lim_{n_k \to \infty} \frac{area(E_{n_k} \bigcap \Gamma)}{area(E_{n_k})} = 1.$$

As the maps $f^{\circ n_k}$ have uniformly bounded distortions, and that Γ is f invariant, we obtain

$$\lim_{n_k \to \infty} \frac{area(B(y, \delta/2) \bigcap \Gamma)}{area(B(y, \delta/2))} = \lim_{n_k \to \infty} \frac{area(f^{\circ n_k}(E_{n_k} \bigcap \Gamma))}{area(f^{\circ n_k}(E_{n_k}))} = 1.$$

One concludes from this equality and that $\Gamma \subseteq J$ to get $B(y, \delta/2) \subseteq J$. This implies that $J = \hat{\mathbb{C}}$, contradicting the assumption in the proposition. □

Let f be a rational map of $\hat{\mathbb{C}}$ and E be a measurable subset of $\hat{\mathbb{C}}$ which is fully invariant under f, that is, $f^{-1}(E) = E$. The map $f : E \to E$ is called ergodic, if for every fully invariant measurable set $F \subseteq E$ we have $area(F) = 0$ or $\text{area}(E \setminus F) = 0$.

Theorem 3.9. *Let f be a rational map of degree at least two and $J(f) = \hat{\mathbb{C}}$. If for z in a set of positive area the spherical distance $d(f^{\circ n}(z), \mathcal{PC}(f)) \nrightarrow 0$, as $n \to \infty$, then, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is ergodic.*

Proof. Let E be the set of points z with $d(f^{\circ n}(z), \mathcal{PC}(f)) \nrightarrow 0$, as $n \to \infty$. Clearly, $f^{-1}(E) = E$.

We break the proof into several steps.

Step 1. For every fully invariant measurable measurable set $F \subseteq \hat{\mathbb{C}}$ with positive area and every $\delta > 0$, we have

$$\inf_{z \in \hat{\mathbb{C}}} \text{area}(B_\delta(z) \cap F) > 0.$$

Recall that for every $z \in J(f) = \hat{\mathbb{C}}$, the set $U = \cup_{i=0}^{\infty} f^{\circ i}(B_\delta(z))$ covers the whole Riemann sphere except at most two points. Thus, there is $n \in \mathbb{N}$ such that $f^{\circ n}(B_\delta(z))$ covers at least half area of F. However, since F is fully invariant, this implies that $\text{area}(B_\delta(z) \cap F) > 0$. On the other hand, the map $z \mapsto \text{area}(B_\delta(z) \cap F)$ is a continuous function of $z \in \hat{\mathbb{C}}$. This implies the first step of the proof.

Step 2. The set E has full Lebesgue measure.

If this is not the case, then $E^c = \hat{\mathbb{C}} \setminus E$ is fully invariant and has positive area. In particular, for every $\delta > 0$ we have the above inequality for the set E^c.

On the other hand, since E has positive area, it has Lebesgue density points. Let z be a Lebesgue density point of E, and let n_k be the moments with $d(f^{\circ n_k}(z), \mathcal{PC}(f) \geq$

δ, for some $\delta > 0$. By the proof of the above theorem, there is $z' \in \hat{\mathbb{C}}$, an accumulation point of the sequence $f^{\circ n_k}(z)$, such that $\text{area}(B(z', \delta/2) \cap E) = \text{area}\, B(z', \delta/2)$. This contradicts the above inequality for the set E^c with $\delta/2$.

Step 3. $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is ergodic.

Let F be a fully invariant subset of $\hat{\mathbb{C}}$ that has positive area. We need to prove that $\hat{\mathbb{C}} \setminus F$ has measure zero.

By step 2, almost every point in F is contained in E. Let z be a Lebesgue density point of F that is also contained in E. Repeating the argument in the proof of the above theorem, we conclude that $\hat{\mathbb{C}} \setminus F$ has measure zero. □

Theorem 3.10. *The Julia set of a hyperbolic rational map has zero area.*

Proof. Since the Julia set is disjoint from the post critical set, then $J(f) \neq \hat{\mathbb{C}}$ and we may apply Theorem 3.8. If the Julia set has positive area, then the orbit of almost every point in it must tend to the post critical set. Since both $J(f)$ and $\mathcal{PC}(f)$ are closed subsets of $\hat{\mathbb{C}}$, this implies that $\mathcal{PC}(f) \cap J(f) \neq \emptyset$. This contradiction implies that $J(f)$ may not have positive area. □

Lecture 4

Polynomial dynamics

Filled Julia sets

Let

$$P(z) = a_0 + a_1 z + \cdots + a_{d-1} z^{d-1} + a_d z^d,$$

be a polynomial of degree $d \geq 2$. Without loss of generality, we may assume that the polynomial is monic, that is $a_d = 1$. This may be achieved by conjugating P with an appropriate linear map $z \mapsto cz$, where $c^{d-1} = a_d$.

The *filled Julia* set of P is defined as

$$K(P) = \{z \in \mathbb{C} \mid \text{ the orbit of } z \text{ under } P \text{ remians uniformly bounded}\}.$$

This is a non empty set as it contains many periodic points of P.

Recall that infinity is a supper attracting fixed point of P. the basin of attraction of infinity is contained in the Fatou set of P.

Lemma 4.1. *For every polynomial P of degree at least two, we have*

- *$K(P)$ is compact,*
- *$\mathbb{C} \setminus K(P)$ is connected,*
- *the topological boundary of $K(P)$, $\partial K(P)$, is equal to $J(P)$,*
- *the interior of $K(P)$ is equal to the set of all bounded Fatou components of P.*

Proof. First note that there is $R > 0$ such that for every $z \in \mathbb{C}$ with $|z| \geq R$ the orbit of z under P tends to infinity. For every $z \in \mathbb{C} \setminus K(P)$, there is $n \in \mathbb{N}$ such that $|P^{\circ n}(z)| > R$. It follows that there is a neighborhood U of z such that for all $z' \in U$ we have $|P^{\circ n}(z')| > R$. That is U is contained in $\mathbb{C} \setminus K(P)$. This implies that $\mathbb{C} \setminus K(P)$ is an open set, proving the first part of the lemma.

If $\mathbb{C} \setminus K(P)$ is not connected, it has a connected component U different from the component containing infinity. However, the iterates of every point on $\partial U \subset K(P)$ remain uniformly bounded. By the maximum principle, all iterates of P on U must be uniformly bounded. That is, $U \subseteq K(P)$.

By definition, every point on $\partial K(P)$ is contained in $J(P)$. On the other hand, by Montel's theorem, every point in the interior of $K(P)$ is contained in the Fatou set of P.

The last part is also clear using the Montel's normal family theorem. □

The Green's function of $K(P)$ is a function $G : \mathbb{C} \to [0, \infty)$ which is identically zero on $K(P)$ and for every z in the complement of $K(P)$

$$G_P(z) = \lim_{k\to\infty} \frac{1}{d^k} \log |P^{\circ k}(z)| > 0.$$

By definition, $G(P(z)) = dG(z)$.

Exercise 4.1. Show that G_P is a continuous function on $\mathbb{C}$ that is harmonic outside of $K(P)$. Moreover, it satisfies $G(z) = \log |z| + o(1)$ as $|z| \to \infty$.

The curves $G(z) = c$, for constants $c \in (0, +\infty)$, are called equipotentials of $K(P)$. It follows that each equipotential is mapped to an equipotential by P.

Theorem 4.2. *Let P be a polynomial of degree at least 2. Then, $K(P)$ and $J(P)$ are connected iff the orbit of all finite critical points of P remain uniformly bounded. In other word, $K(P)$ is connected iff it contains all finite critical points of P.*

Proof. For every $c > 0$, let V_c be the set of $z \in \mathbb{C}$ with $G_P(z) < c$. For each $c > 0$, V_c is a bounded open set, and by the maximum principle, every connected component of V_c is simply connected. Moreover, each component of V_c intersects $K(P)$. On the other hand, $K(P) = \cap_{c>0} V_c$.

For each $c > 0$, let $\chi(V_c)$ denote the number of connected components of V_c. Note that $P : V_c \to V_{cd}$ is a proper branched covering. It follows that $d \cdot \chi(V_{cd}) - \chi(V_c)$ is equal to the number of critical points of P in V_c counted with multiplicity.

For c sufficiently large, V_c is a connected set (It looks like a large disk). If all $d-1$ critical points of P are contained in $K(P)$, and hence all V_c, then we conclude from the above formula that each V_c has only one connected component. That is, each V_c is connected. Therefore, $K(P_c)$ is connected. On the other hand, if there is at least one critical point of P outside of $K(P)$, some V_c has more than one connected component. Since each component of V_c intersects $K(P)$, it follows that $K(P)$ is disconnected. (Indeed one can use this argument to show that when $K(P)$ is not connected, it has infinitely many connected components.) $\square$

Bötcher coordinates, external rays, and equipotentials

From now on we assume that P is a polynomial of degree at least two with a connected Julia set. By the above results $\mathbb{C} \setminus K(P)$ is a doubly connected region. Then, there is a conformal isomorphism

$$\Phi_P : \mathbb{C} \setminus K(P) \to \mathbb{C} \setminus \overline{\mathbb{D}},$$

where $\mathbb{D}$ is the unit disk centered at 0 in $\mathbb{C}$. The map Φ_P is unique upto rotations of the circle. The holomorphic map $\Phi_P \circ P \circ \Phi_P^{-1}$ is a proper covering of degree d on $\mathbb{C} \setminus \overline{\mathbb{D}}$. This implies that $\Phi_P \circ P \circ \Phi_P^{-1}$ is equal to $z \mapsto az^d$ for some constant a with $|a| = 1$ (just lift to the universal cover of $\mathbb{C} \setminus \overline{\mathbb{D}}$). By composing Φ_P with an appropriate rotation, we may assume that $a = 1$. That is, P on the complement of $K(P)$ is conformally conjugate to the power map $z \mapsto z^d$ on the complement of $\overline{\mathbb{D}}$. The conformal map Φ_P is called the Bẗocher coordinate of P at infinity. It follows that the Green's function G_p defined above is equal to $\log |\Phi_P|$.

The complement of $\overline{\mathbb{D}}$ is foliated by circles and straight rays passing through 0. The covering $z \mapsto z^d$ preserves these foliations. That is, it maps leaves of these foliations to corresponding leaves of these foliations. We may pull-back these foliations to obtain two foliations on the complement of $K(P)$ with some nice properties that we explore below.

Define the *equipotential* of radius $h \in (0, \infty)$ as

$$E_P^h = \{\Phi_P^{-1}(he^{2\pi it}) \mid \forall t \in [0,1]\},$$

and *external ray* of angle $2\pi\theta \in [0, 2\pi]$ as

$$R_P^\theta = \{\Phi_P^{-1}(he^{2\pi i\theta}) \mid \forall h \in (0, \infty)\}.$$

By definition, each E_P^h is mapped to E_P^{dh} and each R_P^θ is mapped to $R_P^{d\theta}$, under P respectively. In particular, if some angle θ is periodic under the map $t \mapsto dt$, mod $\mathbb{Z}$, then the ray R_P^θ is mapped to itself under some iterate of P.

Exercise 4.2. Let P be a polynomial of degree $d \geq 2$ and connected filled Julia set. Then, a ray R_P^θ is fixed under P, iff θ is a rational number of the form $j/d-1$, for $j = 1, 2, d-1$. In particular, P has $d-1$ fixed angles.

A ray R_P^θ is called *pre-periodic*, if it is mapped to some periodic ray under some iterate of P.

Exercise 4.3. Every ray R_P^θ with $\theta \in \mathbb{Q} \cap [0, 1]$ is either periodic or pre-periodic.

Landing of rays and local connectivity

An external ray R_P^θ is said to *land* at a point $z \in \partial K(P)$, if

$$\lim_{h \to 0^+} \Phi^{-1}(e^h e^{2\pi i\theta}) = z.$$

An important relation between landing property of rays and local connectivity of $J(P)$ is given by the following theorem.

Theorem 4.3 (Caratheodory). *For any polynomial P with degree $d \geq 2$ and connected Julia set $J(P)$, the following conditions are equivalent:*

- *Every external ray R_P^θ lands at a point $\gamma(\theta)$ which depends continuously on θ,*
- *the Julia set $J(P)$ (similarly $K(P)$) is locally connected.*

If all rays of P land, the map $\gamma : \mathbb{R}/\mathbb{Z} \mapsto J(P)$ semi-conjugates the multiplication by d to the action of P on $J(P)$, that is, $\gamma(d\theta) = P(\gamma(\theta))$, for all $\theta \in [0, 1]$.

However, not all polynomials have locally connected Julia sets. For example, if P has an irrationally indifferent periodic point in $J(P)$, then $J(P)$ may not be locally connected (D. Sullivan). Recall that an irrationally indifferent cycle in the Julia set is called a Cremer cycle. There are other examples of quadratic polynomials with non locally connected Julia sets due to A. Douady.

We are interested in breaking the Julia set into several pieces using external rays landing at the same point. Evidently, if a finite number of rays land at the same point in $z \in K(P)$, then $K(P) \setminus \{z\}$ has a finite number of connected components.

Theorem 4.4. *Every periodic ray lands at a periodic point which is either repelling or parabolic.*

Since all rational angles are either periodic or pre-periodic under iterates of $z \mapsto z^d$, we conclude the following corollary from the above theorem.

Corollary 4.5. *All external rays with rational angle land at some periodic or pre-periodic point in the Julia set.*

On the other hand, it is possible to prove the following.

Theorem 4.6. *All repelling and parabolic periodic points are landing points of at least one, but at most finitely many, external rays.*

The proof of the above theorems are lengthy and are not treated here.

Puzzle pieces

Let P be a polynomial of degree $d \geq 2$ with all its critical points either repelling or parabolic. Then, it has d fixed point points in the complex plane. While the map $\theta \to d\theta$ on the unit circle has $d-1$ fixed points. These $d-1$ fixed angles land at at most $d-1$ of the fixed points of P. In other words, at least one of the fixed points of P is not a landing point of a fixed ray. Since all repelling and parabolic periodic points are landing points, there must be a periodic ray of period strictly bigger than one landing at that fixed points. Then, all iterates of the ray landing at the fixed point must also land at the same fixed point.

The union of all the rays landing at this fixed point divide the complex plane into a finite number of connected components, say $Y_1, Y_2, \dots, Y_n$. These are unbounded open subsets of $\mathbb{C}$. Let us cut off all these domains by some equipotential in $\mathbb{C} \setminus K(P)$ to obtain bounded open sets $Y_1^0, Y_2^0, \dots, Y_n^0$. That is, these are bounded components of the complex plane minus the external rays landing at a particular fixed point and an external rays. All these pieces are taken as open sets, and are called *puzzle pieces* of level 0.

The pre-image of each Y_i^0, $1 \leq i \leq n$, under P has at most d connected components. Let us call these pieces puzzle pieces of level 1 and denote them by $Y_1^1, Y_2^1, \dots, Y_m^1$. In general, puzzle pieces of level j are defined as the connected components of the pre-images of the puzzle pieces of level 0 under $P^{\circ j}$. Let us denote them by Y_l^j, for l in an index set A_l. The collection of all puzzle pieces of all levels of P has the following Markov properties:

- the union of all puzzle pieces of any level j forms a neighborhood of the Julia set minus a finite number of pre-images of the fixed point;

- any two puzzle pieces are either disjoint or nested;

- for any puzzle piece Y_m^j of level $j \geq 1$, $P : Y_m^j \to P(Y_m^j)$ is a proper holomorphic map, which is either univalent or branched covering of finite degree, depending on whether there is any critical points in Y_m^j or not.

Note that the intersection of each puzzle piece with the Julia set is a connected subset of the Julia set.

The interior of the closure of all puzzle pieces of level j forms a neighborhood of the Filled Julia set (and Julia set) bounded by equipotential of level $1/d^j$. Clearly, as $j \to +\infty$, these neighborhoods shrink to the $K(P)$. This may be used to describe the dynamics of the polynomial P on its Julia set. However, the complicated patterns that arise are far from trivial.

In general, there is always a periodic point that is the landing point of at least two external rays. One may use these rays to define the puzzle pieces of level 0 and then continue as above to build the puzzle pieces are arbitrary level. They will enjoy the Markov properties listed above.

For each z in the Julia set that is not mapped to the dividing fixed point under iterates of P, and every $j \in \mathbb{N}$, there is a unique puzzle pieces of level j containing z. The natural question is whether any nest of puzzle pieces shrink to a point. Also, it is important to understand the behavior of the orbits of the critical points in the Julia set. For example the recurrence of each critical point to itself, and also its accumulation on other critical points. To this end, one may arrange the puzzle pieces intersecting $\mathcal{PC}(P)$ in a table with arrows according to how they map one into another. We shall analyze this in the quadratic case more carefully in the following lectures.

Modulus of an annulus

Let U be a doubly connected subset of the complex plane. There are r and R in $[0, \infty]$ and a univalent map $\varphi : U \to B(0, R) \setminus D(0, r)$, where $B(0, R)$ denotes the open ball of radius R and $D(0, r)$ denotes the closed disk of radius r. Then, if $R = \infty$ or $r = 0$ the *modulus* of U, denoted by $\mod(U)$, is defined as infinity. Otherwise, we define

$$\mod(U) = \frac{1}{2\pi} \log(\frac{R}{r}).$$

Since $B(0, R) \setminus D(0, 1)$ is conformally isomorphic to $B(0, R') \setminus D(0, 1)$ if and only if $R = R'$, the above value is well-defined and independent of the choice of R and r. Intuitively, $\mod(U)$ measures how "fat" the annulus U is relative to its size. We need the following two properties of the modulus of annuli.

We say that a doubly connected domain U is non-trivially contained in a doubly connected region V if the bounded components of $\mathbb{C} \setminus V$ is contained in the bounded components of $\mathbb{C} \setminus U$.

Lemma 4.7. *Let U_1 and U_2 be disjoint doubly connected regions non-trivially contained in a doubly connected region U. Then* $\mod(U) \geq \mod(U_1) + \mod(U_2)$.

Lemma 4.8. *Let U be a doubly connected region that is a bounded subset of $\mathbb{C}$. If* $\mod(U) = \infty$, *then the bounded component of $\mathbb{C} \setminus U$ is a single point.*

Lecture 5

Quadratic polynomials and renormalization

The Mandelbrot set

In this section we focus on the family of maps

$$P_c(z) = z^2 + c, c \in \mathbb{C},$$

on the complex plane.

Recall that when the unique finite critical point of P_c at 0 remains bounded under iterates of P_c, the Julia set of P_c is connected. The set of all such parameters c where $J(P_c)$ is connected is the well-known Mandelbrot set. That is,

$$M = \{c \in \mathbb{C} \mid J(P_c) \text{ is connected}\}.$$

Exercise 5.1. Show that M is compact.

Mini-project 5. *The set M is a connected subset of $\mathbb{C}$, and $\mathbb{C} \setminus M$ is a doubly connected domain.*

Exercise 5.2. Prove that

1) if $c \notin M$, then $J(P_c)$ is a Cantor set;

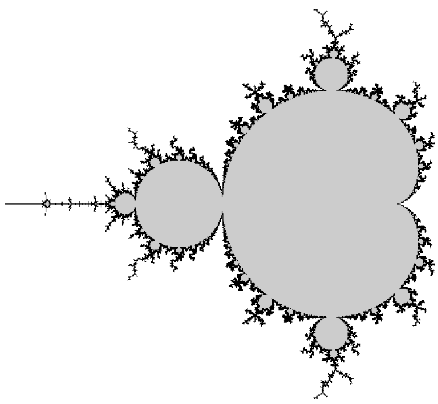


Figure 5.1: The Mandelbrot set

2) if c_1 and c_2 belong to $\mathbb{C} \setminus M$, then P_{c_1} and P_{c_2} are topologically conjugate on a neighborhood of their Julia sets.

Recall the general definition from topological dynamics that the orbit of a point z is *topologically recurrent*, if z is an accumulation point of its orbit.

Lemma 5.1. *Let $c \in M$ such that the critical point of P_c is not topologically recurrent. Then, the Julia set of P_c is locally connected and has zero area.*

Proof. The parameters satisfying the property in the lemma are called semi-hyperbolic. By carefully analyzing the proof in the hyperbolic case, one can give a proof of the above lemma. Further details may be found in Orsay notes of Douady-Hubbard. $\square$

From now on we assume that c belongs to the Mandelbrot set. Recall from the previous lecture that the *Bötcher* coordinate of P_c is the conformal isomorphism

$$\varphi_c : \mathbb{C} \setminus K(P_c) \to \mathbb{C} \setminus \overline{\mathbb{D}}, \text{ where } \mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\},$$

that is tangent to the identity near infinity. It conjugates P_c on $\mathbb{C} \setminus K(P_c)$ to $w \mapsto w^2$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$. By means of this isomorphism, the external ray of angle $2\pi\theta \in [0, 2\pi]$, denoted by R_c^θ, and equipotential of radius $r \in (1, +\infty)$, denoted by E_c^r, are defined. We have $P_c(E_c^r) = E_c^{r^2}$ and $P_c(R_c^\theta) = R_c^{2\theta}$.

The unique fixed ray of P_c, R_c^0, lands at a fixed point that is either repelling or parabolic with multiplier $+1$ (why?). We denote this fixed point by β.

At $c = 1/4$, P_c has a parabolic fixed point of multiplicity two. The Julia set has a rather simple structure; it is the boundary of the basin of attraction of the parabolic fixed point.

For $c \neq 1/4$ there are two distinct fixed points, $(1 \pm \sqrt{1-4c})/2$, with multipliers $1 \pm \sqrt{1-4c}$. One of them is already denoted by β, and we denote the other one by α.

The curve $|P_c'(\alpha)| = 1$ is a closed curve dividing the complex plane into two components. For c inside this component, which is called the *main hyperbolic component* of M, α is attracting. On the boundary of the main hyperbolic component, α is neutral, while on the complement of the closure of the main hyperbolic component α becomes repelling. In Figure 5, the main hyperbolic component of the Mandelbrot set is the region bounded by the cardioid at the center of the Mandelbrot set. In particular, the boundary of the main hyperbolic component of M is parametrized by $\theta \in [0, 1]$, where $P_c'(\alpha) = e^{2\pi\theta i}$ at parameter θ.

Exercise 5.3. Prove that for every c in the main hyperbolic component, P_c on a neighborhood of $J(P_c)$ is homeomorphic to P_0 on a neighborhood of the unit circle.

For c on the boundary of the main hyperbolic component, P_c has a neutral fixed point which is either parabolic, or is an irrationally indifferent fixed point that lies in $F(P_c)$ or $J(P_c)$. Quadratics of the latter type have very complicated dynamics, which we do not discuss them here.

Nests of puzzle pieces

When α is *repelling*, by Theorem 4.6 there are at least two, but a finite number of, external rays landing on α.

Lemma 5.2. *Prove that the set of rays landing at α consists of the orbit of a single periodic ray.*

Proof. Using the rays landing at α one cuts the complex plane into unbounded sectors mapped one into another. Each such sector has an opening angle at infinity that is multiplied by 2 by P_c. Then, one can see that any cycle of sectors must contain at least one critical point, using the argument principle. □

Let $2\pi\theta_j \in [0, 2\pi)$, for $1 \leq j \leq q$ and $q \geq 2$, denote the angles of the external rays landing at α, labeled in increasing order. There is a non-zero integer $p \in (0, q)$, with $(p, q) = 1$, such that $P_c(R_c^{\theta_j}) = R_c^{\theta_{j'}}$ where $j' = j + p \pmod q$. The rational number p/q is called the *combinatorial rotation* of P_c at α. The fixed point α is sometimes referred to as *the dividing fixed point* of P_c. That is because $J(P_c) \setminus \{\alpha\}$ has at least two connected components. It follows that for any rational $p/q \in (0, 1)$, there are parameters c in the Mandelbrot set where α has combinatorial rotation p/q at α. For example, $c \in M$ with $P_c'(\alpha) = e^{2\pi i p/q}$.

The closure of the q rays landing at α cut the complex plane into q (open) connected components which we denote by Y_j, for $0 \leq j \leq q-1$. The map P_c on these pieces has a simple covering property. Let Y_0 denote the one containing the critical point 0. Then, P_c on Y_0 is a double cover, while it is univalent on all other pieces. In particular it follows that Y_0 contains the other preimage of the fixed point, $-\alpha$. The image of Y_0 under P_c covers all Y_j, for $0 \leq j \leq q-1$. We may relabel the these components so that $P_c(Y_j) = Y_{j+1}$, for $1 \leq j \leq q-2$, and $P_c(Y_{q-1}) = Y_0$. Thus, the critical point is mapped into Y_1 in one iterate of P_c, and is mapped back into Y_0 under q iterates of P_c.

Fix $r > 1$. The equipotential E_c^r divides each piece Y_j, for $0 \leq j \leq q-1$, into two connected components. We denote by Y_j^1, for $0 \leq j \leq q-1$, the interior of the bounded connected component of $Y_j \setminus E_c^r$. These are puzzle pieces of level 1. The connected components of $P_c^{-i}(Y_j^1)$, for $i \in \mathbb{N}$ and $0 \leq j \leq q-1$, are puzzle pieces of level i. They form nests of pieces breaking the Julia set into components.

If the orbit of 0 under iterates of P_c lands on α, P_c is critically non-recurrent and we have Lemma 5.1. Thus we may assume from now on that the orbit of 0 never lands at α. This guarantees that at every level there is a unique puzzle piece containing 0.

Since $P_c(Y_0)$ covers $\cup_{j=0}^{q-1} \operatorname{int} Y_j^1$, the connected components of $P_c^{-1}(Y_j^1)$, for $0 \leq j \leq q-1$, which are contained in Y_0 divide Y_0 into q pieces. We denote the one containing 0 by Z_0^2 and the remaining ones by Z_j^2, for $1 \leq j \leq q-1$. Recall that $P_c^{\circ q}(0) \in Y_0^1$, and therefore belongs to one of Z_j^2, for $0 \leq j \leq q-1$.

We must have one of the following two possibilities:

A: $P_c^{\circ(nq)}(0) \in Z_0^2$, for all $n \in \mathbb{N}$;

B: there exists $n \in \mathbb{N}$ such that $P_c^{\circ(nq)}(0) \in Z_j^2$, for some $1 \leq j \leq q-1$.

Lemma 5.3. *If A occurs for some $c \in M$, the map $P_c^{\circ q} : Z_0^2 \to Z_0^1$ is a proper branched covering of degree two. The nest of puzzle pieces containing* 0 *do not shrink to the critical point.*

Proof. There are two different ways to prove this. One way is to who that all puzzle pieces containing zero also contain α and $-\alpha$ on their boundary. In particular, they may not shrink to a point. The other approach is use the double covering property and the argument principle to first show that $P_c^{\circ(nq)}(0) \in Z_0^2$ must have a fixed point in Z_0^2, and since it is a double cover the is also the other pre-image of this fixed point Z_0^2 distinct from the fixed point. Then all pre-images of Z_0^2 containing 0 also contain the fixed point and its pre-image. In particular, since all puzzle pieces are either nested or disjoint, one concludes that the diameters of puzzle pieces of P_c containing 0 may not shrink to 0. $\square$

For a parameter $c \in M$ such that all periodic points of P_c are repelling and property A holds, the map P_c is called *satellite renormalizable*, and $P_c^{\circ q} : Z_0^2 \to Z_0^1$ is called the first renormalization of P_c. More precisely, it may be called *satellite renormalizable of type* p/q, where p/q is the combinatorial rotation of α.

If B occurs for some $c \in M$, there is the smallest $n \in \mathbb{N}$ such that $P_c^{\circ(nq)}(0) \in Z_j^2$, for some $1 \leq j \leq q-1$. Let V^1 denote the connected component of $P_c^{-nq}(Z_j^2)$ containing 0 that is obtained by pulling back along the orbit $0, P_c(0), \dots, P_c^{\circ nq}(0)$. If the orbit of 0 under P_c never enters V^1, then the orbit of the critical point of P_c is not recurrent, and the dynamics of P_c is described in Lemma 5.1. Thus, we may assume that there is the smallest $j_1 \in \mathbb{N}$ such that $P_c^{\circ j_1}(0) \in V^1$. We must have $j_1 > nq$.

Denote by V^2 the component of $P_c^{-j_1}(V^1)$ that is obtained by pulling back along the orbit $0, P_c(0), \dots, P_c^{\circ j_1}(0)$. Here, $P_c^{\circ j_1} : V^2 \to V^1$ is a proper branched covering of degree two. If P_c is topologically recurrent, we may inductively define domains $V^1 \supset V^2 \supset V^3 \supset \dots$, and positive integers $j_1, j_2, j_3, \dots$ such that each j_m is the smallest positive integer with $P_c^{\circ j_m}(0) \in V^m$, and V^{m+1} is the pull-back of V^m along the orbit $0, P_c(0), \dots, P_c^{\circ j_m}(0)$. For example, it is possible that $P_c^{\circ j_1}(0) \in V^2$, and then $j_2 = j_1$.

Lemma 5.4. *For c as above and every integer $m \geq 2$, V^m is compactly contained in V^{m-1} and $P_c^{\circ j_m} : V^m \to V^{m-1}$ is a proper branched covering of degree two.*

Proof. The first part is because, the closure of every two puzzle piece only intersect at equipotentials and closures of rays. Puzzle pieces of different levels may not intersect at equipotentials. On the other hand if the rays bounding V^m and V^{m-1} intersect, there must be a periodic point on the boundary of V^m (and also V^{m-1}). But this is not possible since the rays bounding V^m are mapped to the rays landing at α. $\square$

Now we consider two possibilities.

$\mathbf{B_1}$: $\exists m \geq 2$, such that the orbit of 0 under $P_c^{\circ j_m} : V^m \to V^{m-1}$ remains in V^m.

$\mathbf{B_2}$: $\forall m \geq 2$, the orbit of 0 under $P_c^{\circ j_m} : V^m \to V^{m-1}$ leaves V^m.

Note that when B_1 occurs, the sequence j_k is eventually constant, but when B_2 occurs, the sequence j_k tends to infinity.

Lemma 5.5. *If B_1 occurs for some $c \in M$, then the nest of puzzle pieces containing the critical point does not shrink to the critical point.*

Proof. The proof is similar to the one for Lemma 5.4. □

For a parameter $c \in M$ such that all periodic points of P_c are repelling and property B_1 holds, the map P_c is called *primitive renormalizable*, and the map $P_c^{\circ j_m} : V^m \to V^{m-1}$ is called the first renormalization of P_c.

For a parameter $c \in M$ such that all periodic points of P_c are repelling and property B_2 holds, the map P_c is called *non-renormalizable.*

In general, P_c is called *quadratic-like renormalizable*, or sometimes *Douady-Hubbard renormalizable*, if it is either satellite or primitive renormalizable. The satellite renormalizable is sometimes called *immediately renormalizable*, since it happens immediately.

Theorem 5.6 (Yoccoz)**.** *If $c \in M$ such that all periodic points of P_c are repelling and P_c is not quadratic-like renormalizable, then every nest of puzzle pieces shrinks to a single point. In particular, $J(P_c)$ is locally connected.*

Mini-project 6. *Detail a proof of the above theorem.*

The above theorem has some important consequences. For example,

Theorem 5.7 (Lyubich, Shishikura)**.** *If $c \in M$ such that all periodic points of P_c are repelling and P_c is not renormalizable, then $J(P_c)$ has zero area.*

Mini-project 7. *Detail a proof of the above theorem.*

Quadratic-like mappings and straightening

Now we study what happens to the intersection of the nest of puzzle pieces containing the critical point for parameters satisfying properties A and B_1 above. Before we do that we need some definitions from quasi-conformal mappings.

For a differentiable function $\varphi : U \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a coordinate $z = x + iy$ on U, with x and y real, let us denote the complex partial derivatives of φ as follows

$$\varphi_z = \frac{1}{2}(\varphi_x - i\varphi_y), \varphi_{\overline{z}} = \frac{1}{2}(\varphi_x + i\varphi_y),$$

where φ_x and φ_y denote the first partial derivatives of φ with respect to x and y, respectively. In particular, if φ is a holomorphic function, then $\varphi_z = \varphi'$ is the usual derivative, and $\varphi_{\overline{z}} \equiv 0$ on the domain of φ.

A homeomorphism $\varphi : U \to V$ between open subsets of $\hat{\mathbb{C}}$ is called *quasi-conformal* if

- φ has distributional partial derivatives φ_z and $\varphi_{\overline{z}}$ of class L^1_{loc}
- there exists $k \in [0, 1)$ such that $|\varphi_{\overline{z}}(z)| \leq k|\varphi_z(z)|$ for almost every $z \in U$.

We shall abbreviate quasi-conformal as qc from now on.

The second condition implies that a qc map is orientation preserving. A homeomorphism is called K-qc if it is qc with constant $k = \frac{K-1}{K+1}$, $K \geq 1$. By Weyl's lemma, a qc homeomorphism such that $\varphi_{\overline{z}} = 0$ almost everywhere is conformal.

A *quadratic-like mapping* is a proper branched covering holomorphic map f from U to V of degree 2, where U and V are simply connected domains with U compactly contained in V. For example, the restriction of any quadratic polynomial P to $P^{-1}(B(0, R))$, for sufficiently large R, is a quadratic-like map.

One may define the *filled Julia* set and the *Julia* set of a quadratic-like map in the same fashion;

$$K(f) := \{z \in U \mid f^{\circ j}(z) \in U, \forall j \in \mathbb{N}\}, J(f) := \partial K(f).$$

By a similar proof one can show that $K(f)$ and $J(f)$ are connected iff the orbit of the critical point of f remains in U.

Two quadratic-like mappings $f : U \to V$ and $g : U' \to V'$ are *qc* conjugate if there is a qc map $h : V \to V'$ such that $g \circ h = h \circ f$ on U. They are called *hybrid* conjugate if they are qc conjugate and the qc conjugacy h between them may be chosen so that $\overline{\partial} h = 0$ on $K(f)$. A remarkable result of Douady and Hubbard is that the dynamics of a quadratic-like map is similar to the dynamics of some quadratic polynomial.

Theorem 5.8 (Douady-Hubbard)**.** *Every quadratic-like map is hybrid conjugate to an appropriate restriction of some quadratic polynomial. Moreover, if the Julia set of the quadratic-like map is connected, then the corresponding quadratic polynomial is unique.*

Although the hybrid conjugacy h in the above theorem is not unique, h is uniquely determined on $J(f)$. For example, it is uniquely determined on the α fixed point and all its pre-images.

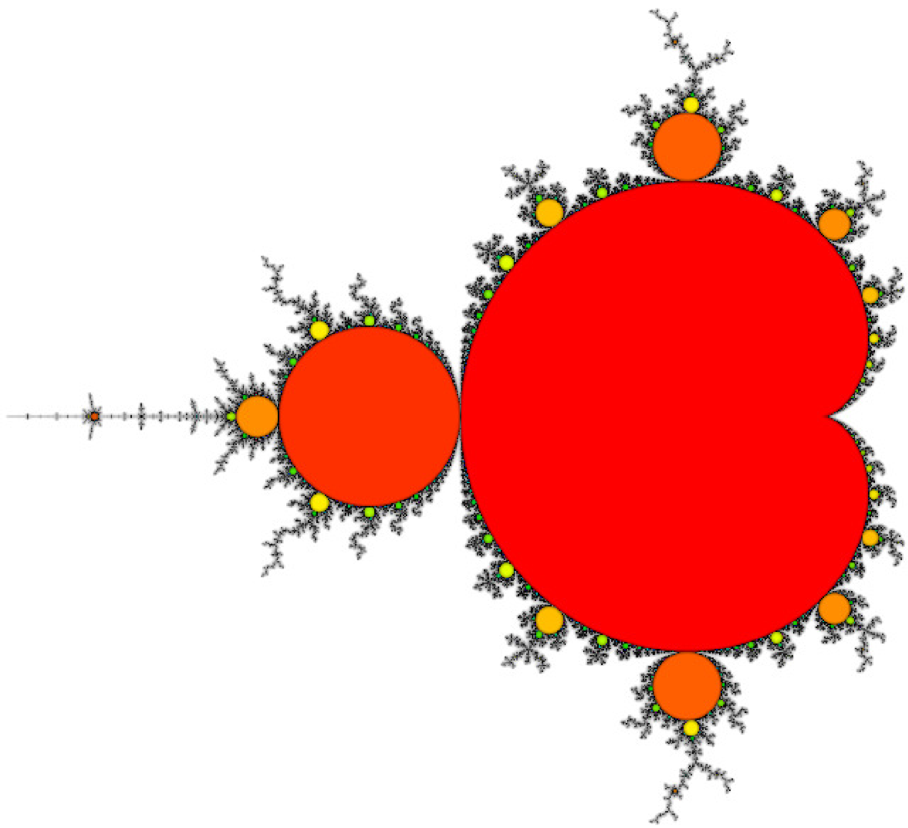
quadratic-like renormalization

When P_c is primitive renormalizable, the map $P_c^{\circ j_m} : V^m \to V^{m-1}$ is quadratic-like with connected Julia set. In particular, by the above theorem, there is a unique quadratic polynomial $P_{c'}$, $c' \in M$, such that $P_c^{\circ j_m} : V^m \to V^{m-1}$ is conjugate to $P_{c'}$ restricted to a neighborhood of its filled Julia set. Moreover, as the conjugacy is hybrid, fine scale geometric properties of the dynamics of the two maps are preserved to some extent. It turns out (requires a lot of work) that when P_c varies among all primitive renormalizable maps with the same combinatorial behavior as in P_c upto level m, the map $P_{c'}$ covers M. This correspondence is one to one and continuous. Douady and Hubbard used this technique to explain the appearance of little Mandelbrot copies with in the Mandelbrot set.

When P_c is satellite renormalizable, the map $P_c^{\circ q} : Z_0^2 \to Z_0^1$ is almost quadratic-like with connected Julia set, except that the domain is not compactly contained in the range. To fix this problem we slightly "thicken" the domain and range of this map. That is, using $|P_c'(\alpha)| > 1$ and that P_c is expanding on the complement of $K(P_c)$ in the Botcher coordinate, there is a simply connected domain

$\tilde{Z}_0^2$ containing the closure of Z_0^2 such that $P_c^{\circ q}(\tilde{Z}_0^2)$ contains the closure of $\tilde{Z}_0^2$ and $P_c^{\circ q} : \tilde{Z}_0^q \to P_c^{\circ q}(\tilde{Z}_0^2)$ is a proper branched covering of degree two. Note that the orbit of the critical point under iterates of $P_c^{\circ q}$ remains in $\tilde{Z}_0^2$. Then, as in the above paragraph, the dynamics of $P_c^{\circ q}$ is hybrid conjugate to the dynamics of some quadratic polynomial on a neighborhood of its filled Julia set.

Similar to the primitive case, each satellite renormalizable P_c gives rise to a homeomorphic copy of the Mandelbrot set. These copies are attached to the boundary of the main hyperbolic copy of the Mandelbrot set.



Lecture 6

Periodic points in the Julia set

Density of hyperbolicity

Recall that a rational map is called hyperbolic if the orbit of all its critical points tend to attracting cycles of the map. We showed that a hyperbolic rational map has zero area Julia set and is expanding on a neighborhood of its Julia set. Thus, the orbit of almost every point on the Riemann sphere under iterates of the map is asymptotic to one of a finite number of periodic cycle. A rational map of degree d may have at most $2d-2$ attracting cycles.

The space of all rational maps of degree d may be considered as a subset of $\mathbb{C}^{2d+2}$, and the space of polynomials of degree d naturally embeds into $\mathbb{C}^{d+1}$. In particular, it is meaningful to talk about convergence of sequences in these spaces. In any continuous family of rational maps, the hyperbolic ones form an open subset of the family. A central problem in complex dynamics, which goes back to Fatou, is that the hyperbolic maps are also dense in any non-trivial family of rational maps.

Conjecture 6.1. *The set of hyperbolic maps is dense in the space of rational maps of any given degree. Also, the set of hyperbolic maps is dense in the space of all polynomials of any given degree.*

The above conjecture has been extensively studied for many families of maps, but it has not been confirmed even in the quadratic family.

Conjecture 6.2. *The set of $c \in \mathbb{C}$ such that the map $z \mapsto z^2 + c$ is hyperbolic forms a dense (and open) subset of $\mathbb{C}$.*

We shall study the Fatou's efforts in studying Conjecture 6.1, and study some recent advances on this conjecture using quasi-conformal mappings.

Global linearizations in the basins of attraction

Theorem 6.3. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\deg f \geq 2$ and $p \in \hat{\mathbb{C}}$ with $f(p) = p$ and $|f'(p)| \in (0, 1)$). Then, there exists a holomorphic map $\varphi : \mathcal{A} \to \mathbb{C}$, where $\mathcal{A}$ is the immediate basin of attraction of p, such that $\varphi(f(z)) = f'(p)\varphi(z)$, for all $z \in \mathcal{A}$. Moreover, f has at least one critical point in $\mathcal{A}$.*

Proof. By making a Mobius change of coordinate we may assume that $p = 0$. Let $\lambda = f'(0)$ and choose $c \in \mathbb{R}$ with $c^2 < \lambda < c$.

Choose $r > 0$ such that for all $z \in B(0, r)$ we have $|f(z)| \leq c|z|$. By Taylor's expansion theorem, there is $C > 0$ such that $|f(z) - \lambda z| \leq C|z|^2$ for all $z \in B(0, r)$. Define the sequence of maps $\varphi_k : B(0, r) \to \mathbb{C}$ as $\varphi_k(z) = f^{\circ k}(z)/\lambda^k$. We have

$$\begin{aligned}|\varphi_{k+1}(z) - \varphi_k(z)| &\leq |\frac{1}{\lambda^{k+1}} f^{\circ k+1}(z) - \frac{1}{\lambda^k} f^{\circ k}(z)| \\ &\leq \frac{1}{\lambda^{k+1}} C|f^{\circ k}(z)|^2 \leq \frac{1}{\lambda^{k+1}} Cr^2 c^{2k} \leq Cr^2 c^{k-1}.\end{aligned}$$

Then, φ_k form a Cauchy sequence on $B(0, r)$. In particular, the sequence of univalent maps φ_k converges to a holomorphic map $\varphi : B(0, r) \to \mathbb{C}$ that is either univalent or constant. However, since $\varphi_k'(0) = 1$, for all k, $\varphi'(0) = 1$ and can not be a constant map. By definition, we have $\varphi(f(z)) = \lambda\varphi(z)$, for all $z \in B(0, r)$.

Now extend φ onto the immediate basin of attraction of 0, $\mathcal{A}$. Let $z \in \mathcal{A}$ and choose $n \in \mathbb{N}$ such that $f^{\circ n}(z) \in B(0, r)$. Define, $\varphi(z) = \varphi(f^{\circ n}(z))/\lambda^n$. To see that this gives a well-defined holomorphic map on $\mathcal{A}$, assume $n_1 > n_2$ be two integers with $f^{\circ n_1}(z) \in B(0, r)$ and $f^{\circ n_2}(z) \in B(0, r)$. Then,

$$\frac{\varphi(f^{\circ n_1}(z))}{\lambda^{n_1}} = \frac{\varphi(f^{\circ(n_1-n_2)} \circ f^{\circ n_2}(z))}{\lambda^{n_1}} = \frac{\lambda^{n_1-n_2}\varphi(f^{\circ n_2}(z))}{\lambda^{n_1}} = \frac{\varphi(f^{\circ n_2}(z))}{\lambda^{n_2}},$$

by applying the functional equation $\varphi(f(z)) = \lambda\varphi(z)$, $n_1 - n_2$ times on $B(0, r)$. By definition, the extended map satisfies the desired functional equation.

Assume in the contrary that f has no critical point in $\mathcal{A}$. We define a non-constant holomorphic map $\psi : \mathbb{C} \to \mathcal{A}$, which is not possible by Liouville's theorem.

For small enough $\varepsilon > 0$, there is an inverse branch ψ of φ defined on $B(0, \varepsilon)$, with $\psi(0) = 0$. Let r be the supremum of all ε such that ψ extends as a univalent holomorphic map on $B(0, \varepsilon)$. If $r = \infty$, then we are done. Otherwise, choose $r' > 0$ with $|\lambda|r < r' < r$, and let $U = \psi(B(0, r')) \subseteq \mathcal{A}$. Since ψ is univalent on $B(0, r)$, U is simply connected. On the other hand f has no critical value on U. This implies that there is a univalent inverse branch g of f defined on U. We extend ψ univalently onto $B(0, r'/|\lambda|$ as $\psi(z) = g\psi(z\lambda)$. This contradicts r being finite. □

Theorem 6.4. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\deg f \geq 2$ and assume that f has a parabolic fixed point p of multiplier $+1$. Then, there exists a holomorphic map $\psi : \mathcal{B} \to BBC$, where $\mathcal{B}$ is the immediate basin of attraction of p, such that $f(\psi(w)) = \psi(w + 1)$, for all $w \in \mathbb{C}$. Moreover, f has at least one critical point in $\mathcal{B}$.*

Proof. Let us put the parabolic fixed point of f at 0 using a a Mobius change of coordinate. Then, f near 0 has power series expansion

$$f(z) = z + az^{k+1} + \cdots,$$

with $a \neq 0$. Let v denote the attraction direction in the attracting basin $\mathcal{B}$. Recall that the map $\varphi(z) = -1/(naz^k)$ maps the immediate basin of attraction $\mathcal{B}$ to an open set containing some right-half plane

$$\mathbb{H}_R = \{w \in \mathbb{C} \mid \operatorname{Re} w > R\}.$$

Moreover, f on a neighborhood of 0 lifts under φ to a univalent map F defined on $\mathbb{H}_R$, where $F(w) - (w+1)| \leq 1/2$. A careful calculation gives

$$|F(w) - (w+1)| \leq \frac{C}{|W|^{1/k}},$$

on $\mathbb{H}_R$. Define the sequence of maps β_k on $\mathbb{H}_R$ as $\beta_k(w) = F^{\circ k}(w) - k$. It follows that β_k forms a Cauchy sequence converging to some univalent map β on $\mathbb{H}_R$. Further calculations are left to the reader.

Let U denote the connected component of $\varphi^{-1}(\mathbb{H}_R)$ containing 0 on its boundary. We may define $\psi = \beta \circ \varphi$ on U. Since φ conjugates the iterates of f to iterates of F, and β conjugates iterates of F to the iterates of the translation by $+1$, ψ conjugates the iterates of f to the iterates of the translation by $+1$ on U.

One uses the functional equation $\psi(f(z)) = \psi(z) + 1$ on U to extend ψ onto all of $\mathcal{B}$. That is, for $z \in \mathcal{B}$, by Proposition 2.1, there is $n \in \mathbb{N}$ such that $f^{\circ n}(z) \in U$. Then, define $\psi(z) = \psi(f^{\circ n}(z)) - n$. Let $n_1 > n_2$ be integers with $f^{\circ n_1}(z) \in U$ and $f^{\circ n_2}(z) \in U$. Then

$$\begin{aligned}\psi(f^{\circ n_2}(z)) - n_2 = \psi(f^{\circ n_2+1}) - n_2 - 1 = \psi(f^{\circ n_2+2}) - n_2 - 2 = \dots \\ = \psi(f^{\circ n_1}) - n - 1.\end{aligned}$$

That is, ψ is a well-defined map on $\mathcal{B}$.

First note that there is $\varepsilon > 0$ such that there is a univalent inverse branch of ψ, say η defined on $\mathbb{H}_\varepsilon$. Then let R be the infimum of all R such that there is a univalent extension of η defined on $\mathbb{H}_R$. If $R = -\infty$, then we are done, other wise let $R' = R - 1/2$ and $V = \eta(\mathbb{H}_{R'})$. Note that $V \subseteq \mathcal{B}$ is simply connected and contains no critical value of f. Therefore there is an inverse branch of f, say g, defined on V such that $g(z) \to 0$ as $z \to 0$ in V. On $\mathbb{H}_{R'}$ we may extend η as $\eta(w) = g \circ \eta(w+1)$. □

The above results naturally extend to an statement for periodic points as in the following corollary.

Corollary 6.5. *Let f be a rational map of degree $d \geq 2$. Then, the basin of attraction of every attracting and parabolic periodic point contains at least one critical point. In particular, a rational map of degree $d \geq 2$ has at most $2d-2$ cycles which are either attracting or parabolic.*

Proof. Let z be an attracting or parabolic fixed point of f of period p. Then, z is an attracting or parabolic fixed point of $f^{\circ p}$, and therefore by the above theorems, the immediate basin of attraction of z contains a critical point of $f^{\circ p}$. By the chain rule, f must have a critical point of f in the basin of attraction p.

Clearly the basins of attractions of distinct periodic cycles are disjoint. As a rational map of degree d has at most $2d-2$ critical points, the number of attracting and parabolic cycles of f is bounded by $2d-2$. □

Persistent cycles

Let M be an n dimensional complex manifold and $F : M \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. For each $m \in M$, the map $z \mapsto F(m, z)$ is a rational map of $\hat{\mathbb{C}}$, which we denote by f_m. Assume for some $m_0 \in M$ the map f_{m_0} has a fixed point at $z(m_0)$ with multiplier $\lambda_0 \neq 1$. By implicit function theorem, there is a neighborhood $U \subseteq M$ of m_0 and a holomorphic map $z : U \to \hat{\mathbb{C}}$ such that for all $m \in U$ we have $f_m(z(m)) = z(m)$. That is, fixed points of multiplier different from one persist under small perturbations of the map. The same argument applies to see that every periodic point of multiplier different from 1 persists under perturbation. In particular, in any holomorphic family of rational maps, the repelling periodic point of a map in the family persists as long as it remains repelling.

Lemma 6.6. *For a rational map of degree $d \geq 2$, the number of indifferent cycles with multiplier $\lambda \neq 1$ is at most $4d - 4$.*

Shishikura gives the optimal upper bound of $2d-2$ in the above lemma. However, the bound $4d - 4$ (or any bound depending on degree) is enough for our purposes. The proof we present here is due to Fatou. The idea is that one may perturb the map in such a way that at least half of its indifferent cycles with multiplier $\neq 1$ become attracting.

Proof. If $f(z) = z^d$, then the map has no indifferent cycle and there is nothing to prove. So below we assume that $f(z)$ is not identically equal to z^d.

Let $f(z) = p(z)/q(z)$, where p and q are polynomials with no common divisor, and where at least one of them has degree d. We consider the one parameter family of maps

$$f_t(z) = \frac{p(z) - tz^d}{q(z) - t}$$

for $t \in \hat{\mathbb{C}}$. We have $f_0(z) = f(z)$ and $f_\infty(z) = z^d$.

If $\hat{t} \in \hat{\mathbb{C}}$ is a parameter such that $f_{\hat{t}}$ has degree strictly less than d, then for this value of parameter the numerator and the denominator of $f_{\hat{t}}$ must have common roots, say $\hat{z}$. It follows that $\hat{z}$ must be a solution of the equation $f_{\hat{t}}(\hat{z}) = \hat{z}^d$. In particular, these points are independent of $\hat{t}$ and there are a finite number of them. If $\hat{z} \neq \infty$, then the value of $\hat{t}$ is determined by $q(\hat{z}) - \hat{t} = 0$. If $\hat{z} = \infty$, then $\hat{t}$ is the unique solution of $\hat{t} = p(\hat{z})/\hat{z}^d$, interpreting this quotient as the leading coefficient of the polynomial $p(z)$ when $\hat{z} = \infty$. Removing these finite values of t from the parameter space, the maps f_t have degree equal to d and depend smoothly on t.

Assume $f = f_0$ has k distinct indifferent cycles with multipliers $\lambda_j \neq 1$, for $1 \leq j \leq k$. We need to show that $k \leq 4d - 4$. In each of these cycles choose a representative z_j. By implicit function theorem, we may follow all these cycles in a neighborhood of f_0. That is, there is T such that for $|t| \leq T$ there are holomorphic maps $z_j(t)$ with multiplier $\lambda_j(t)$ which depend holomorphiccally on t. We shall show that there is a parameter t in this neighborhood such that at least half of $|\lambda_j(t)|$ are strictly less than 1.

Sublemma 6.7. *For all j, the map $t \mapsto \lambda_j(t)$ is a non-constant function on $|t| \leq T$.*

Proof. Choose an angle θ such that the ray

$$r \mapsto \gamma(r) = re^{i\theta}, r \in [0, \infty]$$

avoids all the finite exceptional parameters t where the family fails to be a smooth degree d map.

Now assume some map $t \mapsto \lambda_j(t)$ is constant on some small neighborhood of 0. Let A be the set of $r \in [0, \infty]$ such that $f_{\gamma(r)}$ has a periodic point of multiplier $\lambda_j(0)$ and period the same as periodic of $z_j(0)$. The set A is closed, and contains a neighborhood of 0. On the other hand, by Implicit Function Theorem, the set A is also open. Hence, $A = [0, \infty]$. This implies that $f_{\gamma(\infty)}(z) = z^d$ must have an indifferent periodic cycle. However, the only multipliers of the map $z \mapsto z^d$ are 0 and 2^k, for $k \geq 1$. This is a contradiction. □

Let us expand the multipliers λ_j, for $1 \leq j \leq k$, for $|t| \leq T$, as

$$\lambda_j(t)/\lambda_0(t) = 1 + a_j t^{n_j} + o(t^{n_j}),$$

for $a_j \neq 0$ and $n_j \geq 1$. Hence,

$$|\lambda_j(t)| = 1 + \mathrm{Re}(a_j t^{n_j}) + o(|t|^{n_j}).$$

Let us define the functions $\sigma_j(\theta)$ as the sign of $\mathrm{Re}(a_j e^{i\theta n_j})$. Evidently, if $\sigma_j(\theta) = +1$ then, $|\lambda_j(re^{i\theta})| > 1$ for small enough r, and if $\sigma_j(\theta) = -1$ then, $|\lambda_j(re^{i\theta})| < 1$ for small enough r. There are n_j sectors in $\mathbb{C}$ where σ_j is $+1$ and n_j complimentary sectors where σ_j becomes -1. There are $2n_j$ points where σ_j has jump discontinuity. Moreover,

$$\frac{1}{2\pi} \int_0^{2\pi} \sigma_j(\theta) d\theta = 0.$$

The function $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_k$ is a step function with a finite number of jump discontinuities. Moreover, it has zero average on the interval $[0, 2\pi]$.

If k is even, then σ takes even values at almost every point. If $\sigma(\theta) = 0$ for almost every $\theta \in [0, 2\pi]$, then half of the $\sigma_j(\theta)$ must be -1 at almost every point. Otherwise there is $\theta \in [0, 2\pi]$ such that $\sigma(\theta) < 0$. It follows that at least $k/2$ of the functions $\sigma_j(\theta)$ are equal to -1.

If k is odd, then σ takes odd values at all points except at the finite jump discontinuities. Then, there is $\theta \in [0, 2\pi]$ such that at least $(k+1)/2$ of $\sigma_j(\theta)$, for $1 \leq j \leq k$, are -1.

The above two paragraphs imply that for small enough r, the map f_t with $t = re^{i\theta}$ has at least $k/2$, when k is even, or $(k+1)/2$, when k is odd, attracting periodic cycles. By the previous corollary, we have $k \leq 4d - 4$. □

Repelling cycles are dense

Finally, we prove the main statement we need.

Theorem 6.8. *Let f be a rational map of $\hat{\mathbb{C}}$ of degree at least 2. Then, the set of repelling periodic points of f is dense in $J(f)$.*

The property in the above theorem may be taken as an alternative definition of the Julia set.

Proof. Let z_0 be a point in the Julia set which is not a fixed point or a critical value of f. Then, $f^{-1}(z_0)$ has d distinct elements z_j, for $j = 1, 2, \dots, d$ that are disjoint from z_0. As f is univalent on some neighborhood of each z_j, there is a neighborood U of z_0 and holomorphic maps φ_j defined on U such that $f \circ \varphi_j(z) = z$ and $\varphi(z_0) = z_j$.

We claim that for some $z \in U$ and some positive integer n the map $f^{\circ n}(z)$ takes one of the values z, $\varphi_1(z)$, or $\varphi_2(z)$. If this does not happen, then the family of maps

$$g_n(z) = \frac{(f^{\circ n}(z) - \varphi_1(z))(z - \varphi_2(z))}{(f^{\circ n}(z) - \varphi_2(z))(z - \varphi_1(z))}$$

defined on U does not take the three values 0, 1, ∞. Thas is, $\{g_n\}_n$ forms a normal family. This implies that the family $f^{\circ n}$ is a normal family on U, which is a contradiction since $z_0 \in J(f)$. Therefore, we can find $z \in U$ such that $f^{\circ n}(z)$ is equal to z, $\varphi_1(z)$, or $\varphi_2(z)$. Providing us with a period point of period n or $n + 1$.

Recall that $J(f)$ has no isolated point. Thus, for every $z \in J(f)$ there is a sequence of points in $J(f)$ that tends to z. Not all points in the sequence may be fixed or critical value. Hence, by the above argument, there is a sequence of periodic points tending to z. Since, by Theorems 6.5 and 6.6, a rational map of degree d has at most $6d - 6$ attracting or neutral cycles, most of these periodic points tending to z must be repelling. This finishes the proof of the theorem. $\square$

Lecture 7

Holomorphic motions and invariant line fields

Holomorphic motions

Let E be a subset of $\hat{\mathbb{C}}$, and Λ be a connected complex manifold with a base point λ_0. A holomorphic motion of E parametrized by (Λ, λ_0) is a family of maps $\varphi_\lambda : E \to \hat{\mathbb{C}}$, for $\lambda \in \Lambda$, that satisfies

- for every $\lambda \in \Lambda$, the map $\varphi_\lambda : E \to \hat{\mathbb{C}}$ is injective,
- for every fixed $z \in E$, the map $\lambda \to \varphi_\lambda(z)$ is holomorphic on Λ,
- φ_{λ_0} is the identity map on E.

A rather surprising property of the holomorphic motions is given by the following lemma.

Lemma 7.1 (λ-lemma of Mañe-Sad-Sullivan). *Let Λ be a complex manifold and $E \subseteq \hat{\mathbb{C}}$. Let $\varphi : \Lambda \times E \to \hat{\mathbb{C}}$ be a holomorphic motion of E parametrized by (Λ, λ_0). Then, $\varphi : \Lambda \times E \to \hat{\mathbb{C}}$ is continuous and has continuous extension to a holomorphic motion $\psi : \Lambda \times \overline{E} \to \hat{\mathbb{C}}$. Moreover, for each $\lambda \in \Lambda$, the map $z \mapsto \psi(\lambda, z)$ is a qc mapping on $\overline{E}$.*

Although the notion of quasi-conformality we gave in the previous section does not make sense if the map is not defined on an open set, there is an equivalent definition of quasi-conformality that makes sense for arbitrary subsets of the Riemann sphere. The following extension of the λ-lemma is meaningful for the definition we have presented. However, one has to impose an strong condition on the parameter space Λ.

Theorem 7.2 (Slodkowski). *Let E be a subset of $\hat{\mathbb{C}}$. Any holomorphic motion $\varphi : \mathbb{D} \times E \to \hat{\mathbb{C}}$ extends to a holomorphic motion $\psi : \mathbb{D} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

In particular the fiber maps $z \mapsto \psi(\lambda, z)$, for each fixed $\lambda \in \mathbb{D}$, of the extended motion obtained in the above theorem, are qc mappings of the Riemann sphere.

Beltrami coefficients

Let f be an orientation preserving homeomorphism of $\mathbb{C}$ or $\hat{\mathbb{C}}$ that has first order partial derivatives $\partial_x f$ and $\partial_y f$ defined almost everywhere and belong to L^1_{loc}. The complex partial derivatives

$$\partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f), \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f),$$

are defined almost everywhere. Then, for every $\alpha \in [0, 2\pi]$, the derivative of f in the direction $e^{i\alpha}$, $Df \cdot e^{i\alpha}$, is equal to $\partial_z f + \partial_{\bar{z}} f e^{-2i\alpha}$. In particular, if f is conformal at some point, then $\partial_{\bar{z}} f = 0$ and all directional derivatives of f at z are equal to $\partial_z f(z)$. Moreover,

$$\max_\alpha |Df(z) \cdot e^{i\alpha}| = |\partial_z f(z)| + |\partial_{\bar{z}} f(z)|,$$
$$\min_\alpha |Df(z) \cdot e^{i\alpha}| = |\partial_z f(z)| - |\partial_{\bar{z}} f(z)|.$$

Since the Jacobian of the orientation preserving homeomorphism f, given by $J_f(z) = |\partial_z f(z)|^2 - |\partial_{\bar{z}} f(z)|^2$, is positive, the difference $|\partial_z f(z)| - |\partial_{\bar{z}} f(z)|$ must be positive almost everywhere. In particular the *dilatation quotient*

$$K_f(z) = \frac{\max_\alpha |Df(z) \cdot e^{i\alpha}|}{\min_\alpha |Df(z) \cdot e^{i\alpha}|} = \frac{|\partial_z f(z)| + |\partial_{\bar{z}} f(z)}{|\partial_z f(z)| - |\partial_{\bar{z}} f(z)|}$$

is a finite number. Clearly, the dilatation quotient is conformally invariant. That is, if g and h are conformal mappings where $s = g \circ f \circ h$ is defined then, $K_f(h(z)) = K_s(z)$.

One can see that the condition

$$\max_\alpha |Df(z) \cdot e^{i\alpha}| \leq K \min_\alpha |Df(z) \cdot e^{i\alpha}|,$$

for some constant K, is equivalent to the condition

$$|\partial_{\bar{z}} f(z)| \leq \frac{K-1}{K+1} |\partial_z f(z)|.$$

The function

$$\mu(z) = \frac{\partial_{\bar{z}} f(z)}{\partial_z f(z)}$$

is called the *complex dilatation* of f at z. On the other hand, if f is a qc mapping, $K_f(z) \leq K$, almost everywhere.

By a direct calculation, when $\mu(z) \neq 0$, the maximal stretching of f at z, i.e. $\max_\alpha |Df(z) \cdot e^{i\alpha}|$, occurs when $\alpha = \arg \mu(z)/2$.

It turns out that the correspondence $f \mapsto \mu$ is a "good" way to represent qc mappings. This is confirmed by the following classical theorem.

Theorem 7.3 (Measurable Riemann Mapping Theorem)**.** *Let $\mu : \mathbb{C} \to \mathbb{D}$ be a measurable function with $\|\mu\|_\infty < 1$. There exists a qc mapping $f : \mathbb{C} \to \mathbb{C}$ such that $\partial_{\bar{z}} f(z) = \mu(z) \partial_z f(z)$, for almost every $z \in \mathbb{C}$.*

Moreover,

- *the normalized solution leaving* 0*,* 1*, and* ∞ *fixed is unique.*
- *Let* Λ *be a complex manifold and* $\mu_\lambda : \mathbb{C} \to \mathbb{D}$ *be an analytic family of measurable functions with* $\|\mu_\lambda\|_\infty < 1$*, for all* $\lambda \in \Lambda$*. Then, the normalized solutions* f_λ*, for* $\lambda \in \Lambda$*, depends analytically on* λ *(for each fixed* $z \in \mathbb{C}$*).*

The equation $\partial_{\bar{z}} f = \mu \partial_z f$ is called the *Beltrami equation* and the function μ is called the *Beltrami coefficient* of the equation. Intuitively, the Beltrami coefficient μ determines a measurable ellipse field defined on $\mathbb{C}$. The ratio of the major axis to the small axis of the ellipse at each z is given by $|\mu(z)|$, and the angle of the major axis is given by $\arg \mu(z)/2$. The measurable Riemann mapping theorem states that given any measurable ellipse field on $\mathbb{C}$ with uniformly bounded eccentricities, then there is a qc mapping that straightens the field. When μ is zero, the solutions are holomorphic, and if μ is constant almost everywhere, the solution is an affine mapping on $\mathbb{C}$. The study of the above equation has a long history going back to Gauss. The above version is proved in 1960 by Ahlfors and Bers.

As in the holomorphic case, it is useful to have a simple criterion for normality of a family of qc mappings.

Proposition 7.4. *Let* $\{f_a\}_a$ *be a family of quasi-conformal mappings of* $\hat{\mathbb{C}}$ *with* $|\partial_{\bar{z}} f_a / \partial_z f_a| \leq k < 1$ *and normalized by mapping three distinct points on* $\hat{\mathbb{C}}$ *to three distinct points on* $\hat{\mathbb{C}}$*. Then, the family* $\{f_a\}_a$ *is normal.*

Invariant line fields

A *line field* on a measurable set $E \subseteq \hat{\mathbb{C}}$ is a measurable function $\Theta : E \to [0, 2\pi]/\sim$, where θ_1 and θ_2 in $[0, 2\pi]$ are equivalent if $\theta_1 - \theta_2 \in \pi\mathbb{Z}$. In other words, a line field on E assigns a line in the tangent space at each point in E that depends in a measurable fashion on the point in E. Note that a measurable function may be defined only almost everywhere on E. For example, if f is a quasi-conformal mapping defined on a neighborhood of a measurable set E, the map $z \mapsto \arg \mu(z)/2$ defines a line field on E that determines the directions of maximal stretching of f on E.

Let $E \subseteq \hat{\mathbb{C}}$ be a measurable set and $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational map such that $g^{-1}(E) = E$. A line field Θ on E is called invariant under g, if for almost every $z \in E$ we have $\Theta(g(z)) \sim \Theta(z) + \arg g'(z)$. In other words, the action of g' on the tangent space maps the line field Θ to itself.

The quadratic family

Let M denote the Mandelbrot set.

Theorem 7.5. *The following two conditions are equivalent.*

(1) the set of $c \in \mathbb{C}$ *such that* $P_c(z) = z^2 + c$ *is hyperbolic is dense in* $\mathbb{C}$*;*

(2) there is no $c \in \mathbb{C}$ *such that* $J(P_c)$ *has positive area and supports an invariant line field.*

We shall prove the above theorem in a several steps.

Proposition 7.6. *The boundary of the Mandelbrot set is equal to the set of $c \in \mathbb{C}$ such that the family of maps $\{c \mapsto P_c^{\circ n}(0)\}_n$ is not normal.*

Proof. For c_0 in $\mathbb{C} \setminus M$, as M is closed, there is a neighborhood of c_0 on which the family of maps $P_c^{\circ n}(0)$ converges uniformly to the constant map ∞. For c_0 in the interior of M, there is a neighborhood of c_0 on which the iterates $P_c^{\circ n}(0)$ remain uniformly bounded. Hence by Montel's normal family theorem, the family is normal on that neighborhood. If c_0 belongs to the boundary of M, then on any neighborhood of c_0 there are points generating an orbit that tends to infinity, and there are points generating uniformly bounded values. Hence, no subsequence of the family may converge to a holomorphic map. □

Proposition 7.7. *Let $c \in \mathbb{C}$ such that P_c has an indifferent cycle. Then, $c \in \partial M$.*

Proof. Let z_c be the indifferent periodic point of P_c with period q and multiplier λ. First note that c must be in the Mandelbrot set. Otherwise, the orbit of the critical point tends to the supper attracting fixed point at infinity and the map is hyperbolic. In particular, it can not have a neutral cycle.

Assume to the contrary that c is in the interior of the Mandelbrot set. Let W be the connected component of M containing c. By the previous proposition, there is a sequence $n_j \in \mathbb{N}$ such that $b \mapsto P_b^{\circ n_j}(0)$ converges to some holomorphic map g uniformly on compact subsets of W. We consider two cases A and B belo

(A): $\lambda \neq 1$.

By implicit Function theorem, there is a connected neighborhood $V \subseteq W$ of c and a holomorphic map $\eta : V \to \mathbb{C}$ such that $\eta(c) = z_c$ and for all $c' \in V$, $\eta(c')$ is a period point of $P_{c'}$ with period q.

We consider two sub-cases.

(A_1): $(P_b^{\circ q})'(\eta(b))$ is constant on V.

As b tends to a point on the boundary of V, $\eta(b)$ remains uniformly bounded in $\mathbb{C}$. Then, every convergent subsequence of thus a sequence tends to a point that is periodic of period q and multiplier λ. In other words, the connected component containing c of $b \in \mathbb{C}$ such that P_b has a periodic cycle of period q and multiplier λ is both open and closed in $\mathbb{C}$. Therefore, this component must be the whole $\mathbb{C}$. But P_0 has no indifferent cycle. (Note that this argument does not work if $\lambda = 1$.)

(A_2): $(P_b^{\circ q})'(\eta(b))$ is not identically constant on V.

By open mapping property of holomorphic maps, there are a and b in V such that $\eta(a)$ is an attracting periodic point of P_a and $\eta(b)$ is a repelling periodic point of P_b. Since the attracting periodic point must attract the whole orbit of 0, $g(a') = \eta(a')$ on a neighborhood of a. Hence, $\eta \equiv g$ on V, and this holds for the limiting map g of any convergent subsequence of $t \mapsto P_t^{\circ n_j}(0)$. It follows that, the orbit of 0 under P_b must tend to the repelling cycle $\eta(b)$, which is impossible.

(B): $\lambda = 1$.

Let X_j denote the set of $a \in W$ such that P_a has a periodic point of period j with multiplier $+1$. We consider two sub-cases again.

(B_1): For all $i \in \mathbb{N}$, X_i has no accumulation point in W.

Fix $i \in \mathbb{N}$ such that X_i is not empty (we know that at least X_q is not empty), and fix y in X_i. First assume that there is $x \in W$ such that P_x has an attracting periodic point of period i. Choose a smooth closed curve γ in $W \setminus X_i$ separating y from the rest of X_i and passing through x. By Part A above, P_b has no indifferent cycle of period i on γ. Thus, there is a holomorphic motion w_b of the attracting cycle of P_b, for $b \in \gamma$. We have $g(b) = w_b$ on a neighborhood of γ. The holomorphic map $b \mapsto (P_b^{\circ i})'(g(b))$ is defined on W and is strictly contained in the unit disk on a neighborhood of γ. By the maximum principle, $|(P_y^{\circ i})'(g(y))| < 1$. This is a contradiction, since $g(y)$ is the parabolic cycle of P_y.

By the above paragraph, all periodic points of P_b of period i are repelling for $b \in W \setminus X_i$. Thus, there is a holomorphic motion of every repelling cycle of period i on an omitted neighborhood of y. It follows that at least one of these repelling cycles, say w_b, must become neutral at y. Then, y is a removable singularity of w_b. By the open mapping property of holomorphic maps, P_b must have an attracting cycle near y. This is a contradiction.

(B_2): There is $i \in \mathbb{N}$ such that X_i has an accumulation point in W.

At every $b \in X_i$, P_b has a parabolic periodic point that must attract the orbit of 0. Hence, $g(b)$ is equal to a point in the parabolic cycle of multiplier $+1$. The holomorphic maps $b \mapsto (P_b^{\circ i})'(g(b))$ and $P_b^{\circ i}(g(b)) - g(b)$ are defined on W and are equal to $+1$ and 0 on X_i, respectively. Hence, they must be identically constant maps. That is, $g(b)$ is a parabolic cycle of P_b of multiplier $+1$ and period i, for all $b \in W$. By Hartog's extension theorem, this implies that for all $b \in \mathbb{C}$, P_b must have a parabolic point of multiplier $+1$. But, for large values of $|b|$, P_b has no such cycle. □

Let M^o denote the interior of M. If W is a connected component of M^o and there is $c \in W$ such that P_c is hyperbolic, by the above proposition, for all $c' \in W$, $P_{c'}$ is hyperbolic. Any component W of M^o consisting of hyperbolic parameters is called a *hyperbolic component* of the Mandelbrot set. Let M^{hyp} denote the union of all hyperbolic components of the Mandelbrot set, and M^{queer} denote the union of the remaining components. Clearly, the density of hyperbolicity conjecture in the quadratic family is equivalent to the statement that M^{queer} is empty.

Proof of Theorem 7.5.

$\neg(1) \Rightarrow \neg(2)$: Assume that there is a non-empty component W of M^{queer}. We want to show that there is $c \in W$ such that $J(P_c)$ has positive area and carries an invariant line field.

By Proposition 7.7, for all $c \in W$ all cycles of P_c are repelling. Then, for all $c \in W$, P_c has no parabolic or basin of attraction or a Siegel disk. Also, P_c may not have any basin of attraction of an attracting cycle, as W is a queer component. Polynomials have no Herman ring by the maximum principle. Thus, by the classification of Fatou components, the only Fatou component of P_c, for $c \in W$, is the basin of attraction of ∞. This implies that the interior of the filled Julia set is empty, for all $c \in W$.

Fix $c \in W$. Recall that Botcher coordinate $\varphi_b : \mathbb{C} \setminus J(P_b) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ conformally conjugates P_b to $z \mapsto z^2$ on the complement of the Julia set. Therefore, for all c' in

W, the map

$$\psi_{c'} = \varphi_{c'}^{-1} \circ \varphi_c : \mathbb{C} \setminus J(P_c) \to \mathbb{C} \setminus J(P_{c'}),$$

is a univalent map conjugating $P_{c'}$ to P_c on the complement of their Julia sets. It easily follows that the above map depends holomorphically on $c' \in W$. The above family is a holomorphic motion of the the complement of $J(P_c)$, parametrized by (W, c). As $J(P_c)$ has no interior point, by the λ-lemma, the above holomorphic motion extends to a holomorphic motion of the whole complex plane. We denote the extended motion by the same notion $\psi_{c'}$.

By definition, $\psi_{c'} \circ P_c = P_{c'} \circ \psi_{c'}$ on $\mathbb{C} \setminus J(P_c)$. By the continuity of $\psi_{c'}$ on $\hat{\mathbb{C}}$, this relation must hold on $J(P_c)$ as well.

Fix $b \in W$ distinct from c. As ψ_b is qc, its complex dilatation μ is defined almost everywhere on $J(P_c)$, and then defines a line field on $J(P_c)$. The conjugacy relation on $J(P_c)$ implies that μ is invariant under P_c.

We claim that $J(P_c)$ has positive area. The qc mapping ψ_b is conformal outside $J(P_c)$ and conjugates the two maps on their Julia sets. If $J(P_c)$ has zero area, by Weyl's Lemma ψ_b is a conformal map. On the other hand, the only conformal maps of $\mathbb{C}$ are linear maps. It follows that $c = b$.

$\neg(2) \Rightarrow \neg(1)$: Assume that $J(P_c)$ has positive area and supports an invariant line field. We shall show that M has a non-empty queer component.

For $t \in \mathbb{D}$ define the Beltrami coefficient μ_t on $J(P_c)$ as $te^{i\Theta}$, where Θ is the argument of the invariant line field on the Julia set. Extend μ_t onto $\mathbb{C} \setminus J(P_c)$ by 0. The Beltrami coefficient μ_t is invariant under P_c.

By the measurable Riemann Mapping theorem, there are qc mappings φ_t solving the Beltarmi equation with coefficient μ_t. We normalize this solution by requiring that 0 is mapped to 0, ∞ is mapped to ∞, and φ_t is tangent to the identity at ∞.

Consider the map $Q_t = \varphi_t \circ P_c \circ \varphi_t^{-1}$, for $t \in \mathbb{D}$. Each Q_t has zero complex dilatation almost everywhere on $\mathbb{C}$. Therefore, it must be holomorphic on $\mathbb{C}$. On the other hand, since it has degree two, and has a supper attracting fixed point at infinity, it must be a quadratic polynomial. That is, for all $t \in \mathbb{D}$, $Q_t(z) = c_t + b_t z + a_t z^2$. As Q_t is tangent to the identity at infinity, we must have $a_t = 1$. The, since it is locally a double cover at 0, it must have a critical point at 0. This implies that $b_t = 0$. Hence, $Q_t(z) = z^2 + c_t$, for some $c_t \in \mathbb{C}$.

We claim that $t \mapsto c_t$ is an injective function on $\mathbb{D}$. If not, let t and s be distinct points in $\mathbb{D}$ with $c_t = c_s$. Then $J(P_{c_s}) = J(P_{c_t})$ and $\varphi_s \circ \varphi_t^{-1}$ conjugates these two maps on their Julia sets. However, since this map is identity on the complement of the Julia set, it must be identity on all of $\mathbb{C}$. Therefore, $t\mu = s\mu$ on the Julia set. As the Julia set has positive area and μ is non-zero, this can only happen if $t = s$.

The map Q_t depends holomorphically on $t \in \mathbb{D}$ (by λ-lemma). This implies that $t \mapsto c_t = Q_t(0)$ is a holomorphic and injective map on $\mathbb{D}$. Moreover, the values of this function lie in M. Therefore, $c_0 = c$ lies in the interior of the Mandelbrot set. On the other hand, since $J(P_c)$ has positive area, P_c is not hyperbolic. Therefore, c must lie in a queer component of M^o. □

Lecture 8

Density of hyperbolicity

In this lecture we study the problem of invariant line fields for a class of infinitely renormalizable quadratic polynomials.

Finitely renormalizable quadratic polynomials

Recall that by Theorem 7.5, the density of hyperbolicity in the quadratic family is equivalent to the absence of quadratic polynomials with positive area Julia set that support an invariant line field. The latter statement has been understood for several classes of maps, listed below.

- If P_c has an attracting or a parabolic cycle, then it has zero area Julia set.
- If P_c has a non-recurrent critical point then its Julia set has zero area.
- If P_c has an irrationally indifferent cycle, by Proposition 7.6 and the proof of Theorem 7.5, either $J(P_c)$ has zero area or it can not support an invariant line field.
- If P_c is not renormalizable, then its Julia set has zero area; Theorem 5.7.

Let P_c be renormalizable of either satellite or primitive type. That means there are simply connected domains U_1 and V_1, $U_1 \Subset V_1$, and a positive integer n_1 such that $P_c^{\circ n_1} : U_1 \to V_1$ is a proper branched covering of degree two with $P_c^{\circ (in_1)}(0) \in U_1$, for all $i \in \mathbb{N}$. By the straightening theorem there is a qc mapping $S_1 : V_1 \to \mathbb{C}$ and a parameter c_1 in the Mandelbrot set M such that $P_{c_1} \circ S_1 = S_1 \circ P_c^{\circ n_1}$ holds on U_1. Let $\mathcal{R}(P_c) = P_c^{\circ n_1} : U_1 \to V_1$.

If P_{c_1} is also renormalizable, we say P_c is twice renormalizable. By definition, there are data U_2', V_2', and $n_2' \in \mathbb{N}$ for the renormalization of P_{c_1}. There is a qc mapping S_2 and a parameter $c_2 \in M$ straightening the renormalization of P_{c_1}, that is, $P_{c_2} \circ S_2 = S_2 \circ P_{c_1}^{\circ n_2'}$ on U_2'. We may assume that V_2' is contained in the image of S_1. Otherwise, we may replace V_2' by the connected component of $P_{c_1}^{\circ -n_2 i}(V_2')$, for some $i \in \mathbb{N}$, that contains 0 in its interior such that it is contained in the image of S_1. Then, U_2' is the connected component of $P_{c_1}^{\circ -n_2}(V_2')$ containing 0 and compactly contained in V_2'. Define, $U_2 = S_1^{-1}(U_2')$, $V_2 = S_1^{-1}(V_2')$, and $n_2 = n_1 n_2'$. It follows that $S_2 \circ S_1$ conjugates $P_c^{\circ n_2}$ on U_2 to P_{c_2} on a neighborhood of its Julia set. That is,

$S_2 \circ S_1 \circ P_c^{\circ n_2} = P_{c_1} \circ S_2 \circ S_1$. Note that $S_2 \circ S_1$ is also a qc mapping, with dilatation quotient bounded by the product of the dilatation quotients of S_1 and S_2. It follows that $\mathcal{R}^{\circ 2}(P_c) = P_c^{\circ n_2} : U_2 \to V_2$ is a proper branched covering of degree 2 such that the orbit of 0 under $\mathcal{R}^{\circ 2}(P_c)$ remains in U_2. The map $\mathcal{R}^{\circ 2}(P_c)$ is called the second renormalization of P_c.

In general, one may continue the above process of renormalization for at most finite or infinitely many times. In the former case P_c is called *finitely renormalizable*, or k times renormalizable if k is the largest integer for which the renormalization process may be carried out k times. In the latter case, P_c is called *infinitely renormalizable.*

Let P_c be k times renormalizable, where k is a finite number or infinity. By the above construction, for each j with $1 \leq j < k+1$, there are simply connected domains U_j and V_j, $U_j \Subset V_j$, as well as $n_j \in \mathbb{N}$ such that

$$\mathcal{R}^{\circ j}(P_c) = P_c^{\circ n_j} : U_j \to V_j$$

is a proper branched covering of degree two and for all $m \in \mathbb{N}$, $\mathcal{R}^{\circ (jm)}(0) \in U_j$. Fix a j with $1 \leq j < k+1$, and consider the orbit $0, P_c(0), P_c^{\circ 2}(0), \ldots, P_c^{\circ n_j}(0)$. Note that by the definition of renormalization, 0 and $P_c^{\circ n_j}(0)$ belong to U_j, while for all i with $1 \leq i \leq n_j - 1$, $P_c^{\circ i}(0)$ does not belong to U_j. That is, $P_c^{\circ n_j}(0)$ is the first return of 0 back to U after leaving U. Define $U_{j,0} = U_j$ and $V_{j,0} = V_j$. Then, inductively we define the simply connected domains $U_{j,i}$ and $V_{j,i}$, for $1 \leq i \leq n_j - 1$, as follows. Since, $P_c(0)$ does not belong to $V_{j,0}$, there is a univalent P_c-preimage of $V_{j,0}$ and $U_{j,0}$ containing $P_c^{\circ n_j - 1}(0)$. We denote these components by U_{j,n_j-1} and V_{j,n_j-1}, respectively. It follows that $P_c(0)$ does not belong to U_{j,n_j-1} and V_{j,n_j-1}. Then, there is univalent preimage of these domains under P_c containing $P_c^{\circ n_j - 2}(0)$, which are denotes by U_{j,n_j-2} and V_{j,n_j-2}, respectively, and so on. For each i, $P_c^{\circ n_j} : U_{j,i} \to V_{j,i}$ is conjugate to $P_c^{\circ n_j} : U_{j,0} \to V_{j,0}$ via the univalent map $P_c^{\circ n_j - i}$ on $U_{j,i}$. Thus, for each i, $P_c^{\circ n_j} : U_{j,i} \to V_{j,i}$ is a proper branched covering of degree 2 with a son-escaping critical point.

For every i and j with $1 \leq j < k+1$ and $0 \leq i \leq n_j - 1$ define the Julia sets

$$J_{j,i} = J(P_c^{\circ n_j} : U_{j,i} \to V_{j,i}).$$

We have, $P_c(J_{j,i}) = J_{j,i+1}$, where the second subscript is considered module n_j. These are called the little Julia sets of the j-th renormalization of P_c. Let us define

$$\mathcal{J}_j = \cup_{i=0}^{n_j-1} J_{j,i}.$$

Proposition 8.1. *If P_c is k times renormalizable, where $k \in \mathbb{N} \cup \{\infty\}$, for every $1 \leq j < k+1$, we have*

$$\text{area}\left(J(P_c) \setminus \cup_{i=0}^{\infty} P_c^{-i}(\mathcal{J}_j)\right) = 0.$$

Proof. Fix j and let $U = \cup_{i=0}^{n_j-1} U_{j,i}$. By the definition of renormalization, the post-critical set of P_c is contained in U. Then, by Theorem 3.8, the orbit of almost every point in $J(P_c)$ eventually remains in U. However, the only points of U whose orbits remain in U under iterates of P_c are contained in $\mathcal{J}_j$ (look at two possibilities of satellite and primitive renormalizations). In other words, $J(P_c)$, up to a set of measure zero, is equal to the union of the pre-images of $\mathcal{J}_j$. □

Corollary 8.2. *If P_c is finitely renormalizable, then $J(P_c)$ has zero area.*

Proof. Let k be the largest integer such that P_c is k times renormalizable with the last renormalization given by $\mathcal{R}^{\circ k}(P_c) = P_c^{\circ n_k} : U_{k,0} \to V_{k,0}$. By the straightening theorem, $\mathcal{R}^{\circ k}(P_c)$ is conjugate to a non-renormalizable quadratic polynomial $P_{c'}$ by a qc mapping. As $J(P_{c'})$ has zero area, Theorem 5.7, and qc mappings map sets of positive area to sets of positive area, $J_{k,0}$ must have zero area. It follows that for all i, $J_{k,i}$ has zero area. The statement of the proposition follows from Proposition 8.1. □

Compactness in the family of quadratic-like mappings

Proposition 8.3. *For every $M > 0$ there is $K > 1$ such that if $f : U \to V$ is a quadratic-like mapping with* mod $(V \backslash U) \geq M$*, the qc mapping S in the straightening theorem may be chosen such that the dilatation quotient of S is bounded from above by K.*

Moreover, for any $M > 0$, the set of quadratic-like mappings $f : U \to V$, upto affine conjugacy, with connected Julia set and mod $(V \setminus U) \geq M$ *is a compact class.*

Proof. This follows from the Koebe distortion theorem, the compactness of the Mandelbrot set, and Theorem 7.4. We leave further details to the reader. □

Let P_c be infinitely renormalizable. For each $k \in \mathbb{N}$, we may define

$$\text{mod }(\mathcal{R}^{\circ k}(P_c)) = \sup_{U,V} \text{ mod }(V \setminus U),$$

where the supremum is taken over all simply connected domains U and V with $U \Subset V$ and $\mathcal{R}^{\circ k}(P_c)$ is a proper branched covering from U to V with a non-escaping critical point. An infinitely renormalizable P_c is said to have *a priori* bounds, if there is $\varepsilon > 0$ such that mod $(\mathcal{R}^{\circ k}(P_c)) \geq \varepsilon$, for all $k \in \mathbb{N}$. It follows from the above proposition that for every infinitely renormalizable P_c, the renormalizations $\{\mathcal{R}^{\circ k}(P_c)\}_k$, up to affine conjugacy, is contained in a compact class of maps.

For some classes of maps the *a priori* bounds have been established. For example, when c is real and also, when P_c is infinitely renormalizable with all its renormalizations of certain primitive type. However, it is also known that there are examples of infinitely renormalizable maps that do not have a priori bounds. This problem is widely open at this stage.

An infinitely renormalizable P_c is said to have *unbranched a priori bounds*, if there is $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, the domains U and V for $\mathcal{R}^{\circ k}(P_c)$ may be chosen in such a way that mod $(V \setminus U) \geq \varepsilon$ and $\mathcal{PC}(P_c) \cap (V \setminus U) = \emptyset$.

Proposition 8.4. *For every infinitely renormalizable P_c with unbranched a priori bounds, $\cap_{j=0}^{\infty} \mathcal{J}_j$ has zero area.*

Proof. We want to show that $\cap_{j=0}^{\infty} \mathcal{J}_j$ has no Lebesgue density point.

By the definition of a priori bounds, there is $\varepsilon > 0$ such that for each $j \in \mathbb{N}$ and $0 \leq i \leq n_j$ there are simply connected domains $U_{j,i} \Subset V_{j,i}$ such that $J(P_c^{\circ n_j} : U_{j,i} \to V_{j,i}) = J_{j,i}$ and mod $V_{j,i} \setminus U_{j,i} \geq \varepsilon$. We claim that there is $M < 1$ depending only on ε such that area $U_{j,i} \leq M$ area $V_{j,i}$.

Let $\varphi : B(0, e^{2\pi\varepsilon}) \setminus B(0, 1)$ be an injective homorphic map into $V_{j,i} \setminus U_{j,i}$. Let γ be the circle of radius $e^{2\pi\varepsilon}/2$. Then, there must be z on γ such that $|\varphi'(z)|\pi e^{2\pi\varepsilon} \geq \operatorname{diam} U_{j,i}$. Applying Koebe distortion theorem finite number of times (depending on ε) on circles of radius $e^{2\pi\varepsilon} - 1$ we conclude that there is $m > 0$ such that for all $z \in \gamma$, $|\varphi'(z)| \geq m \operatorname{diam}(U_{j,i})$. Invoking Koebe distortion theorem once more time on circles of radius $e^{2\pi\varepsilon} - 1$, we conclude that there is $m' > 0$, depending only on ε such that $|\varphi'(z)| \geq m' \operatorname{diam} U_{j,i}$, for all $z \in B(0, 3e^{2\pi\varepsilon}/4) \setminus B(0, e^{2\pi\varepsilon/4}$. Integrating the Jacobian of φ on this annulus, one has $\operatorname{area}(V_{j,i} \setminus U_{j,i}) \geq m' \operatorname{diam} U_{j,i}^2$. On the other hand by the isoperimetric inequality, the area enclosed by a curve of length L is at most $L^2/(4\pi)$. This finishes the proof of the lemma. $\square$

Corollary 8.5. *Let P_c be infinitely renormalizable with a priori bounds. Then, for almost every $z \in J(P_c)$ we have*

- *the forward orbit of z does not meet $\cap_{j=0}^{\infty} \mathcal{J}_j$,*
- *$\|(P_c^{\circ n})'(z)\| \to \infty$, with respect to the Poincare metric on $\hat{\mathbb{C}} \setminus \mathcal{PC}(P_c)$.*
- *for any $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $P_c^{\circ k}(z) \in \mathcal{J}_n$.*
- *for any $k \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $P_c^{\circ k}(z) \notin \mathcal{J}_n$.*

Robust infinitely renormalizable quadratics

Let P_c be a quadratic polynomial, and let $\Theta : J(P_c) \to [0, 2\pi]$ be a measurable invariant line field on $J(P_c)$. Define, $\mu(z) = e^{i\Theta}$. On may extend μ as 0 onto $\hat{\mathbb{C}} \setminus J(P_c)$ to obtain an invariant measurable function under the action of the derivative of P_c on the tangent space.

We need the following classical result on the regularity of measurable functions.

Theorem 8.6. *Let $\mu : \hat{\mathbb{C}} \to \mathbb{C}$ belong to $L^1(\hat{\mathbb{C}})$ (L^1_{loc} is enough). Then, for all $\varepsilon > 0$, and almost every $z \in \hat{\mathbb{C}}$,*

$$\lim_{r \to 0} \frac{\operatorname{area}\{w \in B(z, r) \mid |\mu(z) - \mu(w)| \leq \varepsilon\}}{\operatorname{area} B(z, r)} = 1.$$

For a measurable function $\mu : \hat{\mathbb{C}} \to \mathbb{C}$, we say that μ is *almost continuous* at some point z in $\hat{\mathbb{C}}$ if the limit in the above theorem is $+1$ for all $\varepsilon > 0$. It follows that μ is almost continuous at almost every point in $\hat{\mathbb{C}}$.

Theorem 8.7 (McMullen)**.** *Let P_c be an infinitely renormalizable map with unbranched a priori bounds. Then, there is no invariant line field supported on the Julia set of P_c.*

A line field μ on an open subset $U \subseteq \mathbb{C}$ is called *univalent line field*, if there is a holomorphic map $h : U \to \mathbb{C}$ such that μ is the pull back of the horizontal line field on $\mathbb{C}$ under h. That is, $\mu(z) = (h^{-1})'(h(z))/|(h^{-1})'(h(z))|$.

Lemma 8.8. *Let $f : U \to V$ be a quadratic like map. Then, there is no univalent invariant line field on V.*

Proof. The critical point in the domain of f implies that the holomorphic map inducing the univalent line field must have a critical point in its domain of definition. Hence, the map may not be univalent. $\square$

Proof of Theorem 8.7. We shall show that if P_c has a measurable invariant line field μ on $J(P_c)$, then some quadratic-like map must have a univalent invariant line filed, which is not possible by the above lemma.

Fix $z \in J(P_c)$ such that μ is almost continuous at z and we have the properties in Corollary 8.5. By multiplying μ by a constant on $\mathbb{C}$, we may assume that $\mu(z) = 1$.

For every $j \in N$ let $k(j)$ be the smallest positive integer with $P_c^{\circ k(j)}(z) \in \mathcal{J}_j$. Then, as $j \to \infty$, $k(j) \to \infty$. As $k(j)$ is the smallest integer with this property, $P_c^{\circ k(j)}(z)$ must belong to one of little Julia sets $J_{j,i(j)}$, for some $i(j) \neq 1$ between 0 and n_j. Let E_j denote the connected component of $P_c^{-k(j)}(U_{j,i(j)})$ containing z. Then, $P_c^{\circ k(j)} : E_j \to U_{j,i(j)}$ is univalent. Also, $P_c^{\circ n_j - i(j)} : U_{j,i(j)} \to U_{j,0}$ is a univalent map. Thus, we have a univalent onto map $P_c^{\circ k(j)+n_j-i(j)} : E_j \to U_{j,0}$. Let us denote this map by h_j, that is $h_j : E_j \to U_{j,0}$, is an iterate of P_c that is univalent. We have, $|h_j'(z)| \to \infty$, since z belongs to the Julia set. By Koebe distortion theorem, this implies that diam $E_j \to 0$.

Define the rescaling maps $A_j(z) = z/\operatorname{diam}(J_{j,0})$, and consider the quadratic-like maps $g_j = A_j \circ P_c^{\circ n_j} \circ A_j^{-1}$. Then, each g_j is a quadratic-like map with a connected Julia set, and modulus at least ε, where ε is the bound guaranteed by the a priori bounds.

Let us define the line fields η_j as the push forward of μ under A_j. As μ is invariant under $P_c^{\circ n_j}$, and A_j is a conjugacy, η_j must be invariant under g_j. By Proposition 8.3, there is a subsequence of g_j converging to some quadratic-like map $g : U' \to V'$. On the other hand, since μ on E_j tends to the constant line field 1 in measure, the corresponding subsequence of η_j converges to a univalent line field η invariant under g. This finishes the proof of the theorem. $\square$

References:

Below is a rather wide list of relevant literature on one dimensional real and complex dynamics. We plan to touch upon appropriate parts of the materials 5, 6, 7, 8, 9, 10, 11 from the list below.

Books and relevant papers on real and complex dynamics:

- A. Beardon, The Iteration of Rational Functions: Complex Analytic Dynamical Systems, Springer-Verlag, 1991.
- P. Blanchard, Complex analytic dynamics on the Riemann sphere, Bull. Amer. Math. Soc. (N.S.) 11 (1984) 85-141.
- L. Carleson and T. Gamelin, Complex Dynamics, Springer-Verlag, 1993.
- W. de Melo, S. van Strien One dimensional dynamics, Springer-Verlag, 1993.
- A. Douady, J. H. Hubbard Orsay notes, 1982.
- A. Douady, J. H. Hubbard On the dynamics of polynomial-like mappings, Ann. Sci. Ecole Norm. Sup. (4) 18 (1985), no. 2, 287-343.

- M. Lyubich, On the Lebesgue measure of the Julia set of a quadratic polynomial, Preprint 1991.
- C. McMullen, Complex Dynamics and Renormalization, Annals of Math Studies 135, Princeton University Press, 1996.
- J. Milnor, Dynamics in One Complex Variable, Third edition. Annals of Mathematics Studies, 160, Princeton University Press, Princeton, NJ, 2006.
- J. Milnor, Periodic orbits, externals rays and the Mandelbrot set: an expository account, Asterisque No. 261 (2000).
- S. Morosawa, Y. Nishimura, M. Taniguchi, Holomorphic dynamics, Cambridge Studies in Adv. Math., 66. Cambridge University Press, Cambridge, 2000.
- M. Shishikura, Bifurcation of parabolic fixed points, (chapter in the book) The Mandelbrot set, theme and variations, 325-363, Cambridge Univ. Press, 2000.
- N. Steinmetz, Rational Iteration : Complex Analytic Dynamical Systems, Walter de Gruyter, 1993.

Some survey articles on real and complex dynamics:

- A. Eremenko, M. Lyubich, The dynamics of analytic transformations, Leningrad Math. J. 1 (1990), no. 3, 563-634
- M. Lyubich, Renormalization Ideas in Conformal Dynamics,
- C. T. McMullen, Frontiers in complex dynamics , Bull. Amer. Math. Soc. (N.S.) 31 (1994) 155-172.
- C. T. McMullen, The Classification of Conformal Dynamical Systems,
- S. van Strien, One-dimensional dynamics in the new millennium, Discrete Contin. Dyn. Syst. 27 (2010), no. 2, 557-588.
- M. Urbanski, Measures and Dimensions in Conformal Dynamics,

Background in complex analysis and quasi-conformal mappings:

- L. V. Ahlfors, Complex analysis, Int. Series in Pure and Applied Math., McGraw-Hill Book Co., 1978.
- L. V. Ahlfors, Conformal invariants: topics in geometric function theory, McGraw-Hill, 1973.
- L. V. Ahlfors, Lectures on quasiconformal mappings, Van Nostrand, 1966.
- J. B. Conway, Functions of one complex variable. Second edition. Graduate Texts in Mathematics, 11. Springer-Verlag, New York-Berlin, 1978.

- O. Lehto and K. Virtanen, Quasiconformal mappings in the plane, 2nd edition, Springer-Verlag, 1973.
- C. Pommerenke, Univalent functions, Studia Mathematica/Mathematische Lehrbucher, Vandenhoeck and Ruprecht, Gottingen, 1975.