The Complex Hénon Family

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joint with Eric Bedford

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The Hénon "map" was introduced by the astronomer and applied mathematician Michel Hénon in the 1960's. This is the diffeomorphism of $\mathbf{R}^2$ given by the following formula.

Definition (Hénon Family)

$$f_{c,\delta}(x, y) = (c + \delta y - x^2, -x).$$

The parameter δ is the Jacobian of the map and the map is invertible when $\delta \neq 0$.

The Hénon family of diffeomorphisms can be written in the following form:

$$f_{a,b}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-x^2+a-by\\x\end{pmatrix}$$

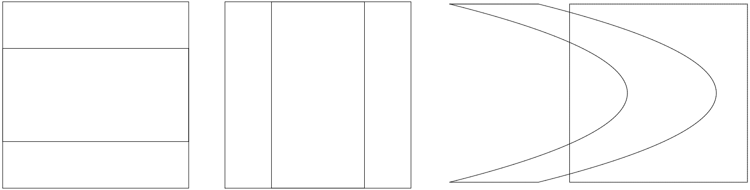
This diffeomorphism is the composition of three simpler maps which squeeze, rotate and shear.

$$f_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\by\end{pmatrix}$$

$$f_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}$$

$$f_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+(-y^3+a)\\y\end{pmatrix}$$

Squeeze, rotate and shear

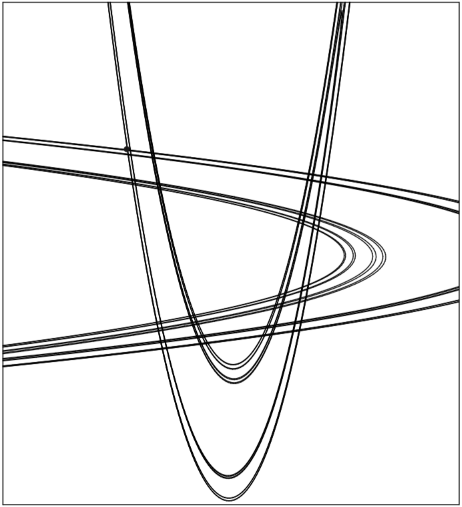


This diffeomorphism expands some directions, contracts some directions and folds.
The dynamics is easier to understand and better behaved when there is no recurrent folding.

Definition

We say that $f_{c,\delta}$ is Axiom A (or *hyperbolic*) if f the tangent bundle splits into a uniformly expanding and uniformly contracting subbundles over the nonwandering set.

A nice hyperbolic example is the real horseshoe.

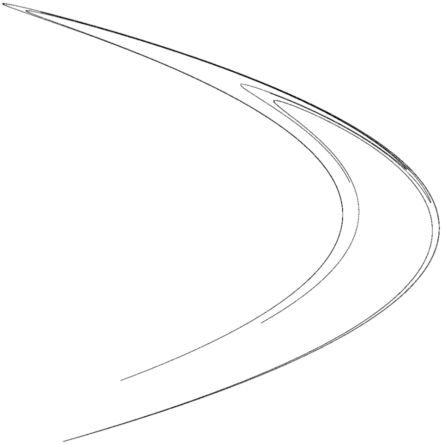


Hyperbolic diffeomorphisms are *structurally stable* meaning that a small change in the parameters produces a topologically conjugate diffeomorphism.

Hénon parameters

For the particular values of the parameters suggested by Hénon the Hénon diffeomorphism seems to exhibit a strange attractor. In particular it demonstrates expansion, contraction and folding on many scales.

This behaviour contrasts with that shown by the horseshoe.



In the 1980's Hubbard suggested that it would be profitable to study the extensions of these polynomial diffeomorphisms to $\mathbf{C}^2$. This is the complex Hénon family:

$$f_{c,\delta} : \mathbf{C}^2 \to \mathbf{C}^2.$$

We allow the coefficients to be real or complex. Thus the parameter space is also $\mathbf{C}^2$.

When the parameters are real then $\mathbf{R}^2$ is an invariant submanifold and we can think of the real dynamical system as contained in the complex dynamical system.

When the Jacobian is zero then the map is not invertible. In this case the dynamics reduce to that of a one dimensional complex quadratic polynomial.

Hubbard was motivated in part by the successful theory of the dynamics of the family $z \mapsto z^2 + c$ and in part by the prominence of the (real) Hénon family in the field of dynamical systems.

We can also think of following the model of algebraic geometry which might suggest that dynamics over $\mathbf{C}$ is more regular and should be studied first while dynamics over $\mathbf{R}$ might be more idiosyncratic and should be studied after the dynamics over $\mathbf{C}$ is understood.

A simpler analogy is the study of roots polynomials in one variable. The complex case is simpler and illuminates the real case.

Definition

Let $K = \{z \in \mathbf{C} : f^n(z) \not\to \infty \text{ as } n \to \infty\}$.

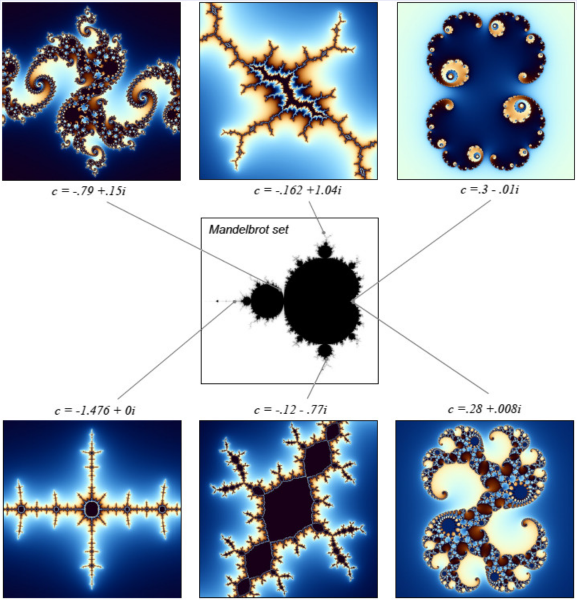
Definition

Let $J = \partial K$.

The chaotic dynamics (expanding recurrent behaviour) is contained in J.

Definition

The Mandelbrot is the subset of parameter space for which the Julia set is connected.


c = -.79 +.15i
c = -.162 +1.04i
c =.3 - .01i
Mandelbrot set
c = -1.476 + 0i
c = -.12 - .77i
c =.28 +.008i

Some themes in one variable complex dynamics

- Understanding combinatorics of Julias sets and relating that to the combinatorics of the Mandelbrot set
- Understanding structural stability and the relation with hyperbolicity (Structurally stable maps are those not in the boundary of the Mandelbrot set.)
- Understanding renormalization phenomena (for example small copies of the Mandelbrot set contained in the Mandelbrot set)

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Hoped for connections with computation

In one complex dimension there is a nice interaction between the theoretical analysis of the dynamics and computer pictures. The pictures help one discover phenomena that can be rigorously proved and proofs are often well illustrated by computer pictures.

One feature which helps in drawing pictures is that the dynamics is taking place in a (real) two dimensional space. Another feature is that the dynamical properties are captured by the behaviour of the iterates of critical points and this makes certain to computer pictures easy to draw.

Hubbard put in a great deal of effort developing a computer tool for studying complex Hénon maps.

Connection between dynamics and critical points

Connectivity of the Julia set: The Juia set is connected if the critical point (0 in the case of the map $z^2 + c$) has a bounded orbit.

Hyperbolicity of the map (expansion on the Julia set): The map is hyperbolic if the critical point is attracted to a sink or to ∞.

Are there analogues of critical points for two dimensional diffeomorphisms? Perhaps critical points in one variable are analogous to tangencies of stable and unstable manifolds in two variables?

With the family $z \mapsto z^2 + c$ in mind Hubbard defined analogs of Julia sets and filled Julia sets for the complex Hénon family.

Definition

$$K^{\pm} = \{p \in \mathbf{C}^2 : f^n(p) \nrightarrow \infty \text{ as } n \to \pm\infty\}$$

Definition

$$J^{\pm} = \partial K^{\pm} \text{ and } J = J^+ \cap J^-$$

The set J contains all hyperbolic periodic points. The set J^+ contains stable manifolds of points in J. The set J^- contains unstable manifolds of points in J.

The real horseshoe illustrates some of these sets.
In the case of the horseshoe the set J is actually contained in $\mathbf{R}^2$ so we are seeing all of J. We are seeing parts of J^+ and J^-.

Often in complex dynamics there are nice functions associated with the sets we define. This is somewhat analogous to algebraic geometry in which we study varieties defined by polynomial equations.

The sets we study are often limits of sets defined by polynomial equations of increasing degree and the functions are built from the polynomials that define them.

Rate of escape functions

Corresponding to the set $K \subset \mathbf{C}$ there is a "rate of escape" function

$$G(z) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |f^n(z)|.$$

The simplest thing to do with these rate of escape functions is use them to draw elegant color pictures of the sets we are interested in. The Julia set pictures in that we saw used these functions.

A deeper relation is connected to potential theory and the idea that G is the Green function of K.

There are corresponding “rate of escape functions” for complex Hénon maps.

Definition

Let

$$G^{\pm}(p) = \lim_{n\to\infty} \frac{1}{d^n} \log^{+} ||f^{\pm n}(p)||.$$

These functions are pluri-subharmonic which means that they are subharmonic when restricted to complex one dimensional submanifolds such as coordinate slices or unstable manifolds of saddle points.

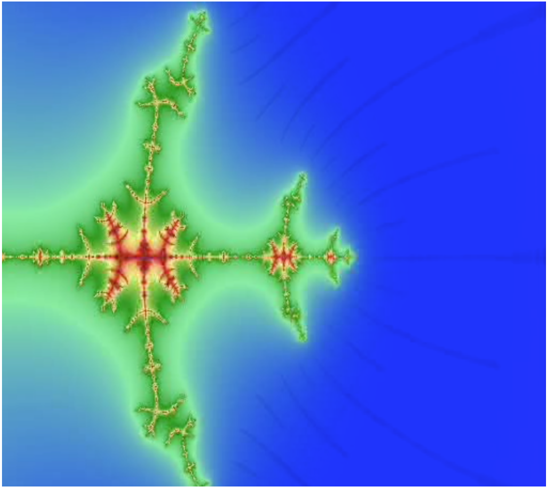
How do we draw pictures capturing the dynamics of Hénon maps in $\mathbf{C}^2$?

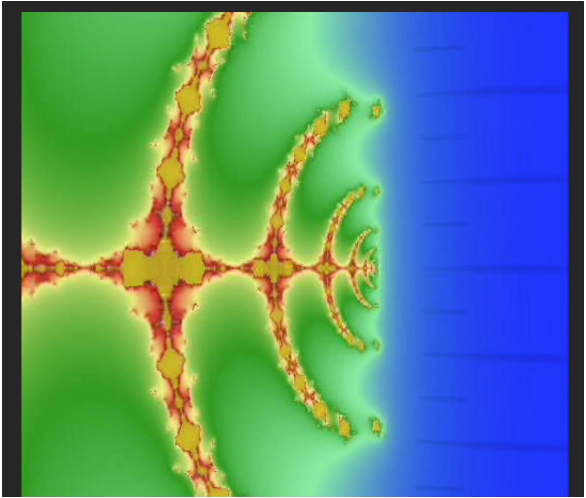
Is it reasonable to draw one dimensional slices of the sets we are interested in?

Do these slices capture all of the important information?

This is not such an unreasonable idea if the slices are (generic) complex submanifolds. (If we draw K^+ and we slice by a complex submanifold we should also draw G^+.)

An unstable manifold of a fixed saddle point





One way to describe the harmonic measure or equilibrium measure (or balanced measure) on a Julia set is as follows where d is the exterior derivative and d^c is the "twisted" exterior derivative rotated using the complex structure.

Definition

$$\mu = dd^c G$$

In one complex variable we can interpret dd^c as a holomorphically invariant version of the Laplacian. The Laplacian takes real functions to real functions but is not holomorphically invariant. dd^c takes the smooth real functions h to the two form $\triangle h \, dx \wedge dy$. It can be extended to on operator taking subharmonic functions to measures.

The potential theory related to $G^{\pm}$ was first investigated not by Hubbard but by Fornaess-Sibony and Bedford-S.
The operator dd^c is defined in any complex manifold so it makes sense in $\mathbf{C}^2$.

Definition

$$\mu^{\pm} = dd^c G^{\pm}$$

$\mu^{\pm}$ are currents. We can think of them as transverse measures which assign a measure to holomorphic transversals. They are analogous to the Margulis transverse invariant measure in hyperbolic dynamics.

Cohomological interpretation

In the case of a variety V defined by a polynomial P the "current" $dd^c \log|P|$ represents the Poincaré dual class to the variety V. In other words when we evaluate this current on a complex disk it counts the number of intersection points between the disk and the variety.

From this point of view we can interpret $\mu^{\pm}$ as dual classes to $J^{\pm}$ even though these sets are fractal objects and are not manifolds.

Definition

$$\mu = \mu^+ \wedge \mu^-$$

In the hyperbolic horseshoe case μ is the Bowen measure which is the unique measure of maximal entropy.

Theorem (Bedford-Lyubich-S)

The measure μ *is the unique measure of maximal entropy. It describes the distribution of periodic points. The support of* μ *is the closure of the set of periodic saddle points.*

Definition Hubbard forgot to make

Definition
J^* is the closure of hyperbolic periodic points. Alternatively J^* is the support of μ or the Shilov boundary of K.

J^* is contained in J. In all examples we know $J^* = J$ but it is not known if $J^* = J$ in general.

What else can you learn by slicing by (say) an unstable manifolds?

- You can detect periodic sink basins. ([BS3])
- You can tell whether the Julia set is connected. ([BS6])

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Why do holomorphic slices reveal so much about dynamical properties?

What happens to holomorphic slices when we iterate?

Note that the situation for the real slice when the parameters are real is very different.

Can you tell whether the diffeomorphism is hyperbolic by looking at an unstable manifold?

Is it related to the *John condition*?

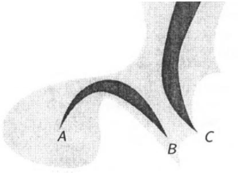
The John condition

Let $E \subset \mathbf{C}$ denote a path from p to ∞. For $c > 0$ we let

$$car(E, c) = \{z \in \mathbf{C} : |z - x| < c|x - p| \text{ for some } x \in E\}$$

An unstable manifold satisfies the *John condition* if every point outside of K is connected to ∞ by a carrot lying outside of K.

Carrots and Cigars



An unstable manifold that satisfies the John condition?

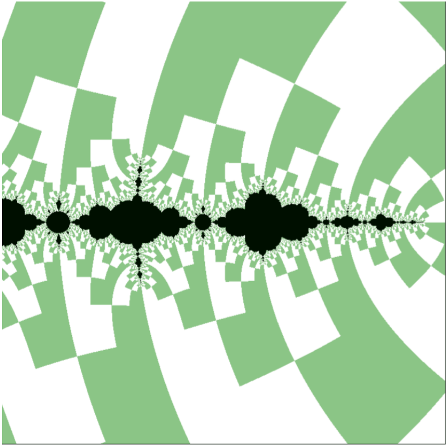


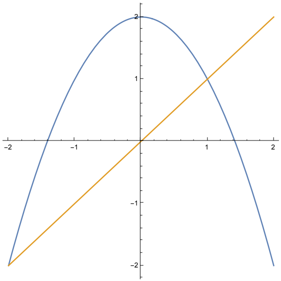
Figure 3.10: $W^u_{a,c}$ picture for $a = 0.3,\ c = -1.17$

This question needs to be refined since it is not actually true in one dimension that the John condition implies hyperbolicity. The polynomial maps which satisfy the John condition include hyperbolic maps but include other maps as well such as the map $z \mapsto z^2 - 2$.

Example

The Ulam-Von Neuman map $z \mapsto z^2 - 2$ can be thought of either as a real or complex dynamical system. It demonstrates Misiurewicz behavior in 1 dimension. It is *not* expanding.

The critical point 0 is pre-periodic but not periodic, it maps to the fixed point 2. This map is expanding but not uniformly expanding and not structurally stable.



When viewed as a complex dynamical system the Julia set of this map is the interval $[-2, 2] \subset \mathbf{C}$. The complement of this interval does satisfy the John condition.

What is the class of diffeomorphisms defined by the John condition and how do we recognize the hyperbolic maps within this class?

The sad fact is that we cannot answer this question. In the course of investigating the question we came up with an analogous property which we conjecture to be equivalent to the John condition. It is this second property that we will study.

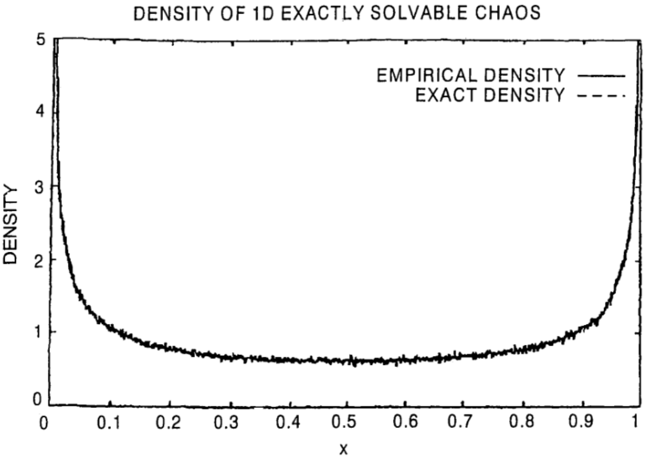
Quasi-hyperbolicity is defined in terms of a locally bounded area condition which is related to the amount of folding for stable and unstable manifolds.

Let W_p^u be the unstable manifold through the point p. We say f is quasi-expanding if there are constants $C > 0$ and $R > 0$ so that the component of $W_p^u \cap B(p, R)$ containing p has area bounded by C.

Quasi-contraction is defined similarly. We say that f is quasi-hyperbolic if it is quasi-expanding and quasi-contracting.

It can also be defined in terms of the existence of an expanding (but not always finite) metrics.

FIG. 7

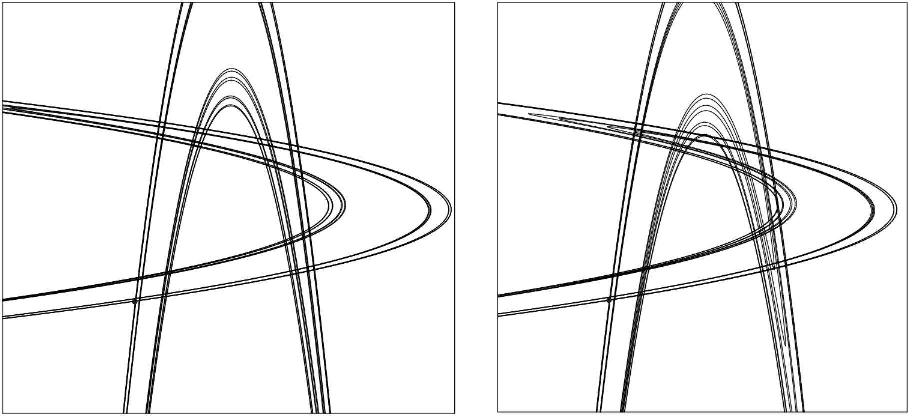

DENSITY OF 1D EXACTLY SOLVABLE CHAOS
EMPIRICAL DENSITY
EXACT DENSITY
DENSITY
x
0
1
2
3
4
5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

In general a quasi-hyperbolic map will have a finite invariant filtration and a uniformly expanding metric on each piece of this filtration which blows up on the next lower piece of the filtration.

Theorem (Bedford-S)

Suppose that f is quasi-hyperbolic. Then f is uniformly hyperbolic on J^* if and only if there is no tangency between $\mathcal{W}^s$ and $\mathcal{W}^u$.

Misiurewicz type behavior for Hénon diffeomorphisms



Our proof makes use of the following result.

Theorem

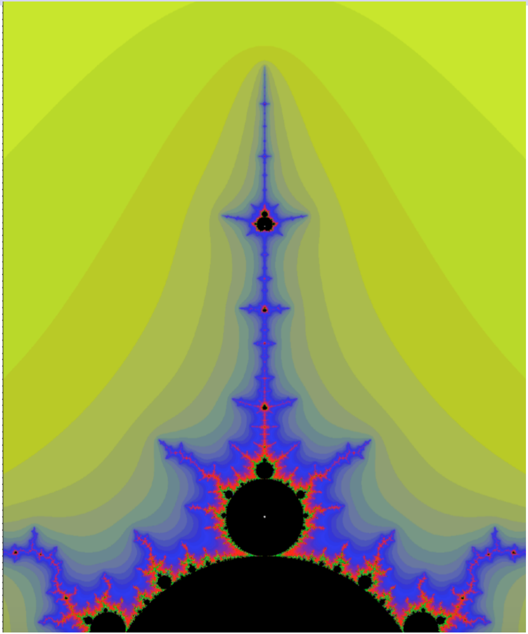
(Lyubich-Peters). Let $\psi_n : \mathbf{C} \to \mathbf{C}^2$ *be a sequence of injective holomorphic mappings of the plane which converge to a non-constant map* ψ. *The set* $\psi(\mathbf{C}) \subset \mathbf{C}^2$ *is a complex manifold without singular points.*

Question

Is there a natural plurisubharmonic function on parameter space which is analogous to the rate of escape functions on dynamical space?

The answer is "yes". In one variable this is obtained by evaluating the function G_c on the critical value. This function can also be interpreted as the Lyapunov exponent with respect to μ.

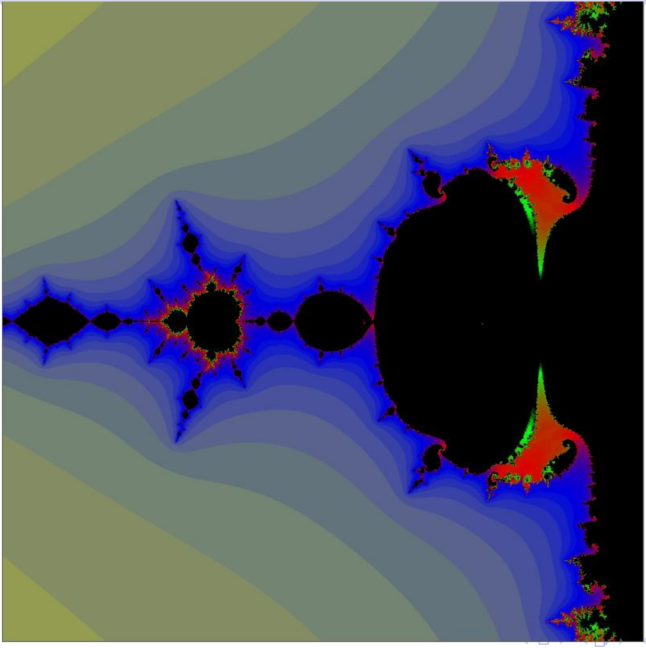
One variable parameter space



In two variables the larger Lyapunov exponent with respect to μ also provides a plurisubharmonic function on parameter space which takes its minimum value on the connectivity locus.

Applying dd^c to this function gives a current which conjecturally should be interpreted as the "bifurcation current".

Slice of Hénon parameter space



As I mentioned earlier Hubbard seems more interested in discovering phenomena than in proving phenomena. So if we leave proofs aside for the moment what has he discovered?

Hubbard was very interested in the combinatorial properties of Hénon parameter space. In particular in the monodromy of the horseshoe locus and the natural map into the group of automorphisms of the two-sided shift on two symbols.

His student Chris Lipa has made some very interesting conjectures in this direction.