Equidistribution in arithmetic geometry and dynamics

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The probabilistic viewpoint in arithmetic geometry

1. Roots of LITTLEWOOD and EINSENSTEIN polynomials;
2. Discrepancy;
3. Equidistribution.

Roots of LITTLEWOOD polynomials

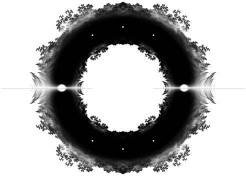


Figure : Roots of polynomials with coefficients $+1$ and -1, by TIOZZO.

CHRISTENSEN, DERBYSHIRE, BAEZ, ...

Roots of EINSENSTEIN polynomials

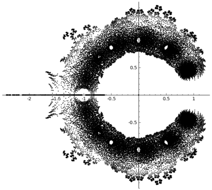
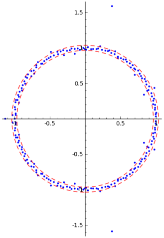


Figure : Roots of monic polynomials of degree 12 with coefficients 0 or 2.

Roots of single Einsenstein polynomial



Discrepancy

Theorem (Radial discrepancy)

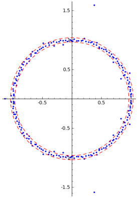
$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0 \in \mathbb{C}[z]$, $a_d a_0 \neq 0$.

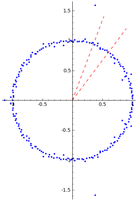
For every ε in $(0, 1)$, we have

$$\frac{1}{d}\#\left\{\alpha \text{ root of } P : |\alpha| < 1 - \varepsilon \quad \text{or} \quad |\alpha| > \frac{1}{1-\varepsilon}\right\} \leq \frac{2}{\varepsilon}\left(\frac{1}{d}\log\left(\frac{\sum_{j=0}^{d}|a_j|}{\sqrt{|a_0 a_d|}}\right)\right).$$

Hughes–Nikeghbali, 2008;
Applications: Roots of Littlewood and Einsenstein polynomials $\left(\varepsilon \sim \frac{1}{\sqrt{d}}\right)$.

Discrepancy





Radial discrepancy: Hughes–Nikeghbali, 2008; Angular discrepancy: Erdös–Turán, 1950;
Higher dimension: D'Andrea–Galligo–Sombra, 2014.

Discrepancy ⇒ Equidistribution

Corollary (Equidistribution)

λ: *Uniform probability on S^1 (Haar measure);*
$(P_n)_{n=1}^{+\infty}$: Littlewood *polynomials such that*

$$d_n := \deg(P_n) \xrightarrow[n\to+\infty]{} +\infty.$$

For every continuous function $\varphi : \mathbb{C} \to \mathbb{R}$ with compact support,

$$\frac{1}{d_n}\sum_{\alpha \text{ root of } P_n} \varphi(\alpha) \xrightarrow[n\to+\infty]{} \int \varphi(z)\, \mathrm{d}\lambda(z).$$

Equivalently:

$$\boxed{\frac{1}{d_n}\sum_{\alpha \text{ root of } P_n} \delta_\alpha \xrightarrow[n\to+\infty]{} \lambda.}$$

Heights and equidistribution

1. MAHLER measure and naive height;
2. Arithmetic equidistribution;
3. Dynamical heights;
4. Proving equidistribution;
5. Adelic heights.

Discrepancy

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HUGHES–NIKEGHBALI, 2008;
Applications: Roots of LITTLEWOOD and EINSENSTEIN polynomials $\left(\varepsilon \sim \frac{1}{\sqrt{d}}\right)$.

MAHLER measure

$\frac{1}{d}\log\left(\frac{\sum_{j=0}^{d}|a_j|}{\sqrt{|a_0 a_d|}}\right)$ can by replaced by:

$$T_{ET}(P) := \frac{1}{d}\log\left(\frac{\sup_{z\in S^1}|P(z)|}{\sqrt{|a_0 a_d|}}\right);$$

GANELIUS, 1954;
ERDÖS–TURÁN size of P.

$$p > 0,\ M_p(P) := \left(\int |P(z)|^p \,\mathrm{d}\lambda(z)\right)^{\frac{1}{p}};$$

L^p measure of P.

$$M(P) := \exp\left(\int \log|P(z)| \,\mathrm{d}\lambda(z)\right);$$
$$= \lim_{p\to 0^+} M_p(P).$$

MAHLER measure of P (“geometric mean”).

Naive height

α: Algebraic number;

$P_\alpha :=$ Minimal polynomial of α (with integer coefficients);

$d_\alpha := \deg(P_\alpha)$;

$h_W(\alpha) := \frac{1}{d_\alpha}\log M(P_\alpha)$.

Naive or WEIL height of α;
Comparison with $T_{ET} = \frac{1}{d_\alpha}\log\left(\frac{\sup_{z\in S^1}|P_\alpha(z)|}{\sqrt{|a_0 a_d|}}\right)$.

- $h_W(\alpha)$ measures the arithmetic complexity of α, *e.g.*,

$$h_W\left(\frac{p}{q}\right) = \log\max\{|p|,|q|\};$$

Exercise!

- $h_W(\alpha) \geq 0$ with equality if and only if α is a root of unity.

Naive height and equidistribution

Theorem (Bilu, 1997)

$(\alpha_n)_{n=1}^{+\infty}$: *Algebraic numbers such that*

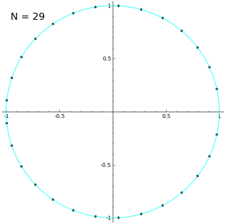
$$h_W(\alpha_n) \xrightarrow[n\to+\infty]{} 0 \quad \text{and} \quad d_{\alpha_n} \xrightarrow[n\to+\infty]{} +\infty.$$

We have,

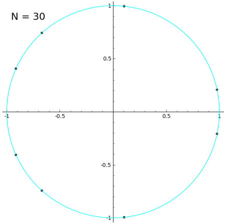
$$\frac{1}{d_{\alpha_n}} \sum_{\alpha \text{ root of } P_{\alpha_n}} \delta_\alpha \xrightarrow[n\to+\infty]{} \lambda.$$

Similar to the Néron–Tate height of an Abelian variety
Szpiro–Ullmo–Zhang, 1997.
Application: α_n primitive root of unity of order n;

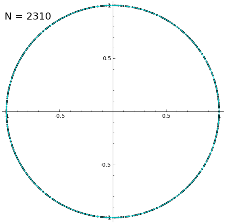
Equidistribution of roots of unity



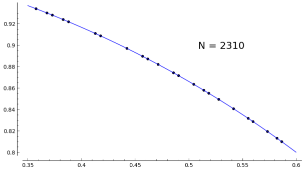
Equidistribution of roots of unity



Equidistribution of roots of unity



Equidistribution of roots of unity



Dynamical heights

$R(z) \in \overline{\mathbb{Q}}(z)$, rational function of degree $D \geq 2$,

$$R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}};$$

Discrete time dynamical system on the Riemann sphere $\widehat{\mathbb{C}}$.

ρ_R: Maximal entropy measure of R.

$$h_R := \lim_{n \to +\infty} \frac{1}{D^n} h_W \circ R^n.$$

Canonical height of R.

- Unique "adelic" height such that $h_R \circ R = D \cdot h_R$;
- $h_R(\alpha) \geq 0$ with equality if and only if α is a periodic point of R (a solution of $R^n(z) - z = 0$, for some $n \geq 1$).

Comparison: Naive height / Néron–Tate height;
Roots of unity / torsion points.

Dynamical heights

Theorem

$(\alpha_n)_{n=1}^{+\infty}$: *Algebraic numbers such that*

$$h_R(\alpha_n) \xrightarrow[n \to +\infty]{} 0 \quad \text{and} \quad d_{\alpha_n} \xrightarrow[n \to +\infty]{} +\infty.$$

We have,

$$\frac{1}{d_{\alpha_n}} \sum_{\alpha \text{ root of } P_{\alpha_n}} \delta_\alpha \xrightarrow[n \to +\infty]{} \rho_R.$$

Baker–Rumely, Chabert-Loir, Favre–RL, 2006;
Application: Equidistribution of periodic points (Lyubich, 1983).

Dynamical heights

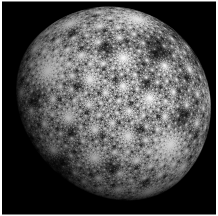


Figure : Density invariant by a rational map, by Chéritat.

Mandelbrot height

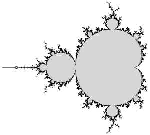
$c \in \mathbb{C}$,

$$P_c : \mathbb{C} \to \mathbb{C},\ P_c(z) := z^2 + c;$$
$$K_c := \{z \in \mathbb{C} : (P_c^n(z))_{n\geq 1} \text{ is bounded}\}.$$

Filled Julia set of P_c.

$$\mathscr{M} := \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$

The Mandelbrot set.



Mandelbrot height

Definition

The **Mandelbrot height** $h_{\mathscr{M}} : \overline{\mathbb{Q}} \to \mathbb{R}$ is,

$$h_{\mathscr{M}}(c) := h_{P_c}(c).$$

Comparison: Uniformization of $\widehat{\mathbb{C}} \setminus \mathscr{M}$.

- $h_{\mathscr{M}}(c) \geq 0$ with equality if and only if P_c is post-critically finite.

$\Leftrightarrow$ The orbit of the critical point of P_c is finite.

Theorem

The asymptotic distribution of small points for the Mandelbrot *set is given by the harmonic measure of the* Mandelbrot *set.*

Application: Equidistribution of post-critically finite parameters, Levin 1980s.

Proving equidistribution

α: Algebraic number;

$P_\alpha =$ Minimal polynomial of α (with integer coefficients);

$d_\alpha = \deg(P_\alpha)$;

$h_W(\alpha) = \frac{1}{d_\alpha} \log M(P_\alpha) = \frac{1}{d_\alpha} \int \log|P_\alpha(z)|\, \mathrm{d}\lambda(z)$.

When α is an algebraic integer ($\Leftrightarrow P_\alpha$ is monic):

$$h_W(\alpha) = \frac{1}{d_\alpha} \int \sum_{\alpha' \in \mathcal{O}(\alpha)} \log|z - \alpha'|\, \mathrm{d}\lambda(z)$$
$$= \iint \log|z - z'|\, \mathrm{d}\lambda(z)\, \mathrm{d}\delta_{\langle\alpha\rangle}(z').$$

$\mathcal{O}(\alpha) :=$ Set of roots of P_α;

$\delta_{\langle\alpha\rangle} := \frac{1}{d_\alpha} \sum_{\alpha' \in \mathcal{O}(\alpha)} \delta_{\alpha'}$.

Proving equidistribution

ρ, ρ': (Signed) measures on the Riemann sphere $\widehat{\mathbb{C}}$.

$$(\rho, \rho') := -\iint_{\mathbb{C}\times\mathbb{C}\setminus\mathrm{Diag}} \log|z - z'|\, \mathrm{d}\rho(z)\, \mathrm{d}\rho'(z').$$

Potential energy.

$$h_W(\alpha) = -\left(\lambda, \delta_{\langle\alpha\rangle}\right)$$
$$= \frac{1}{2}\left(\lambda - \delta_{\langle\alpha\rangle}, \lambda - \delta_{\langle\alpha\rangle}\right) + \frac{1}{d_\alpha^2} \log|\Delta(\alpha)|.$$

$\Delta(\alpha) :=$ discriminant of P_α (a nonzero integer).

Morally, Bilu's theorem follows from:

Cauchy–Schwarz *inequality*:
ρ regular, and $\rho\left(\widehat{\mathbb{C}}\right) = 0 \Rightarrow$

$$(\rho, \rho) \geq 0, \quad \textit{with equality if and only if } \rho = 0.$$

Details: Case α is not an integer (adelic formula);
$\delta_{\langle\alpha\rangle}$ is not regular (convolution, and error estimate).

Adelic heights

ρ: Regular probability measure on $\widehat{\mathbb{C}}$.

$h_\rho :=$ height such that for every algebraic integer α,

$$h_\rho(\alpha) = \frac{1}{2}\left(\rho - \delta_{\langle\alpha\rangle}, \rho - \delta_{\langle\alpha\rangle}\right) + \frac{1}{d_\alpha^2}\log|\Delta(\alpha)|.$$

Adelic height associated to ρ.

λ: uniform measure on S^1,

$$h_\lambda = h_W, \text{ the Weil height;}$$

$\mu_{\mathscr{M}}$: harmonic measure of the Mandelbrot set,

$$h_{\mu_{\mathscr{M}}} = h_{\mathscr{M}}, \text{ the Mandelbrot height.}$$

ρ_R: measure of maximal entropy of (some) $R \in \mathbb{Q}(z)$,

$$h_{\rho_R} = h_R, \text{ canonical height associated to } R;$$

For R with "good reduction at every prime".

Adelic heights

Zhang's inequality (1995)

The essential minimum of an adelic height is nonnegative.

Definition

An adelic height h_ρ is **quasi-canonical** if its essential minimum is equal to 0.

Theorem (Yuan, 2008)

If h_ρ is quasi-canonical, then the asymptotic distribution of Small points for h_ρ is given by ρ.

In dimension 1: Baker–Rumely, Chabert-Loir, Favre–RL, 2006;

The previous equidistribution results follow by observing:

The Weil height, the dynamical heights, and the Mandelbrot height are all quasi-canonical.

Beyond quasi-canonical heights

ω: Spherical measure on the Riemann sphere $\widehat{\mathbb{C}}$;

h_ω: Adelic height associated to ω.

Spherical height.

Theorem (Sombra, 2005)

The spherical height is not quasi-canonical. In fact

$$\textit{essential minimul of } h_\omega = \frac{1}{2}\log 2.$$

Theorem (Burgos–Phillipon–Sombra, 2015)

Among "toric" heights (= heights with radial symmetry), the only quasi-canonical height is the Weil height (!!).

Toric heights

Theorem (Burgos–Philippon–RL–Sombra, arXiv 2015)

ρ: *Regular probability on $\widehat{\mathbb{C}}$ with radial symmetry.*

Centered case: $\mathrm{supp}(\rho) \supset S^1$.

Equidsitribution to λ.

Bipolar case: $\mathrm{supp}(\rho)$ *disjoint from S^1, but intersecting both hemispheres.*

Non-radial, but centered limit measures.

Totally unbalanced case: $\mathrm{supp}(\rho)$ *disjoint from S^1, contained in a hemisphere.*

Non-radial and non-centered limit measures.