On accessibility of hyperbolic components of the tricorn

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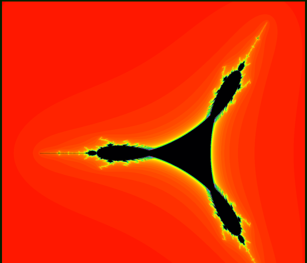
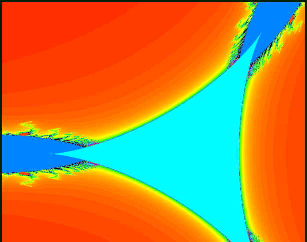
The tricorn family

- $f_c(z) = \bar{z}^2 + c$.
- $f_c^2(z) = (z^2 + \bar{c})^2 + c$: real-analytic 2-parameter family of biquadratic (quartic) polynomials.
- $K_c = \{z \in \mathbb{C};\ f_c^n(z) \not\to \infty\}$: filled Julia set.
- $J_c = \partial K_c$: Julia set.
- $\mathcal{M}^* = \{c \in \mathbb{C};\ K_c : \text{connected}\}$: The **tricorn**.
- **Periodic points:**
  - x: p-periodic point.
  - λ: multiplier of $x \overset{\text{def}}{\Leftrightarrow}$ multiplier for $\boldsymbol{f_c^2}$.
  - k: odd $\Rightarrow \lambda = (\frac{\partial f_c^k}{\partial \bar{z}}(x))\overline{(\frac{\partial f_c^k}{\partial \bar{z}}(x))} \geq 0$.



Hyperbolic components

- Hyperbolic component = (bounded) connected component of the hyperbolicity locus $\mathrm{Hyp}^* \subsetneq \mathrm{int}\, \mathcal{M}^*$.
  - **Remark:** $\neq$ component of $\mathrm{int}\, \mathcal{M}^*$.
  - e.g., period 1 and 2 hyperbolic components are contained in the same component of $\mathrm{int}\, \mathcal{M}^*$ (Crowe et al.).
- $\mathcal{H}$: hyperbolic component, p: period.
- $c \in \partial\mathcal{H} \Rightarrow f_c$ has an indifferent fixed point of f_c^p.
- p: odd $\Rightarrow \partial\mathcal{H}$ consists of 3 **parabolic arcs** and 3 **cusps**.
  - parabolic arc $\Leftrightarrow \exists$ simple 1-parabolic p-periodic point
    (1 attracting petal).
  - cusp $\Leftrightarrow \exists$ double 1-parabolic p-periodic point
    (2 invariant attracting petals).



It is a "1.5-dim family"!

- The tricorn is connected (Nakane).
  - $\exists \Phi : \mathbb{C} \setminus \mathcal{M}^* \to \mathbb{C} \setminus \overline{\mathbb{D}}$: real-analytic diffeomorphism.
- Therefore, we can define external rays (parameter rays) and do some combinatorics with them as in the case of the Mandelbrot set.
  - Parameter rays are stretching rays.
- Even iterate is holomorphic $\rightsquigarrow$ 1D phenomena.
  - discrete parabolic maps,
  - baby Mandelbrot sets.
- Odd iterate is anti-holomorphic $\rightsquigarrow$ 2D phenomena.
  - parabolic arcs,
  - baby tricorn-like sets,
  - wiggly features,
  - discontinuous straightening maps (baby tricorn-like sets are not homeomorphic to the tricorn).

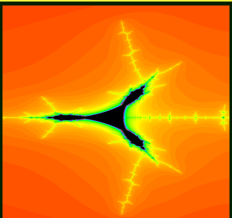
2D-phenomena: Wiggly features

The existence of parabolic arcs (arcs consisting parabolic parameters) induces many wiggly features:

- Non-landing umbilical cords (Hubbard-Schleicher, I, I-Mukherjee).
- Non-landing parameter rays (I-Mukherjee).
- baby tricorn-like sets are NOT (dynamically) homeomorphic to the tricorn (I-Mukherjee).

Conjecture

No pair of baby tricorn-like sets are dynamically homeomorphic unless they are symmetric (in which case they are trivially affinely homeomorphic).



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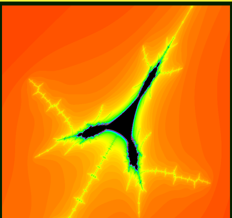

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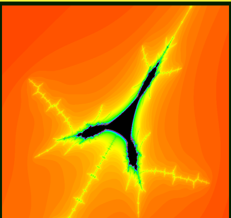
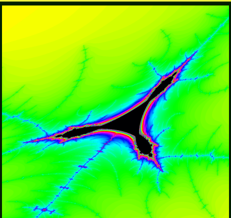

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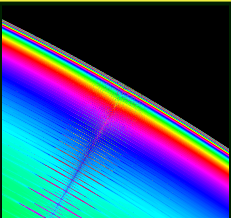
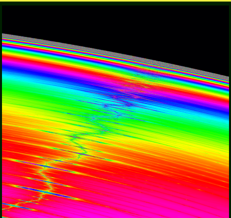




Accessible/inaccessible hyperbolic components

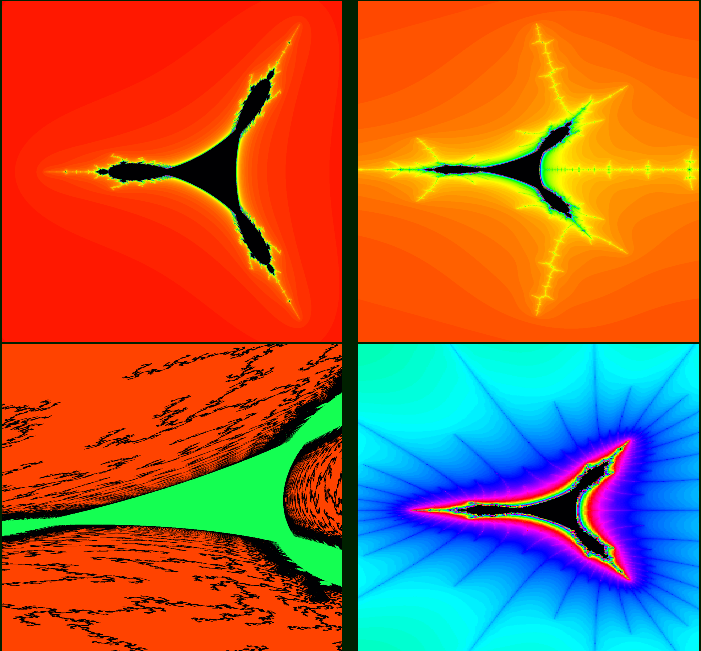
We say a hyperbolic component $\mathcal{H}$ is **accessible** if there is a path $\gamma : (0,1] \to \mathbb{C} \setminus \mathcal{M}^*$ such that $\gamma(t)$ converges to a point in $\partial\mathcal{H}$ as $t \searrow 0$.

Theorem 1 (I-Mukherjee)

Any hyperbolic component of period 1 and 3 in $\mathcal{M}^*$ are accessible.

- Seems reasonable to conjecture that “most” hyperbolic components are inaccessible.
- An attempt to find infinitely many accessible hyperbolic component converging to the Chebyshev map f_{-2} (I-Kawahira, in progress).

Accessible/inaccessible hyperbolic components

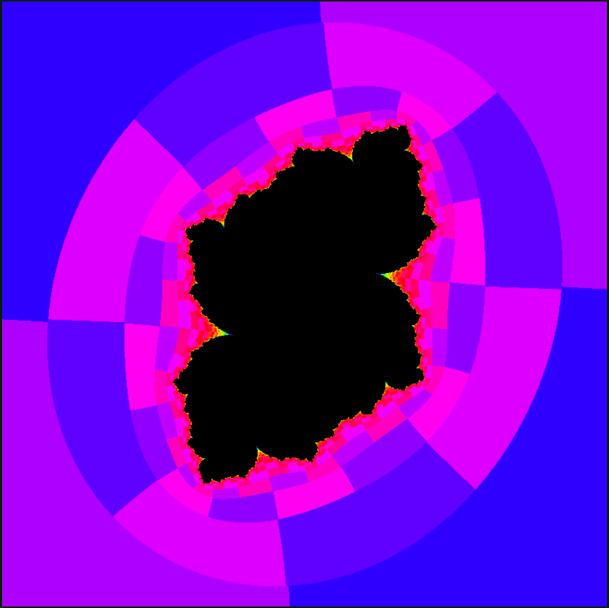


Fatou coordinates and Lavaurs maps

- $\mathcal{H}_0$: a hyperbolic component in $\mathcal{M}^*$.
- $\mathcal{C}_0 \subset \partial\mathcal{H}_0$: parabolic arc of a hyperbolic component of odd period p.
- $c \in \mathcal{C}_0$.
- $\phi_{c,*}$ ($* = \mathrm{attr}, \mathrm{rep}$): normalized attracting/repelling Fatou coordinate, i.e.,
$$\phi_{c,*}(f_c(z)) = \overline{\phi_{c,*}(z)} + \frac{1}{2}.$$
$$\mathrm{Re}\,\phi_{c,\mathrm{attr}}(0) = 0.$$
- **Remark:** $\mathbb{R}$ is invariant by $z \mapsto \bar{z} + \frac{1}{2}$, hence it follows that $\phi_{c,*}$ is unique up to **real** translation.
- Therefore, $\mathrm{Im}\,\phi_{c,*}$ is well-defined (**Ecalle height**).
- Fact: $\mathcal{C}_0$ is analytically parametrized by $\mathrm{Im}\,\phi_{c,\mathrm{attr}}(c)$ (the **critical Ecalle height**).

- $T_\tau(z) = z + \tau$.
- $g_{c,\tau} = \phi_{c,\mathrm{rep}}^{-1} \circ T_\tau \circ \phi_{c,\mathrm{attr}} : \operatorname{int} K_c \to \mathbb{C}$: Lavaurs map with phase τ.
- We only consider the case $\tau \in \mathbb{R}$ (reason explained later).
- Thus $g_{c,\tau} \circ f_c = f_c \circ g_{c,\tau}$.
- $K_{c,\tau} = K(f_c, g_{c,\tau}) = \{z \in \mathbb{C};\ (f_c, g_{c,\tau})\text{-orbit of } z \text{ is bounded}\}$: filled Julia-Lavaurs set.
- $J_{c,\tau} = J(f_c, g_{c,\tau}) = \partial K(f_c, g_{c,\tau})$: Julia-Lavaurs set.

Julia-Lavaurs set



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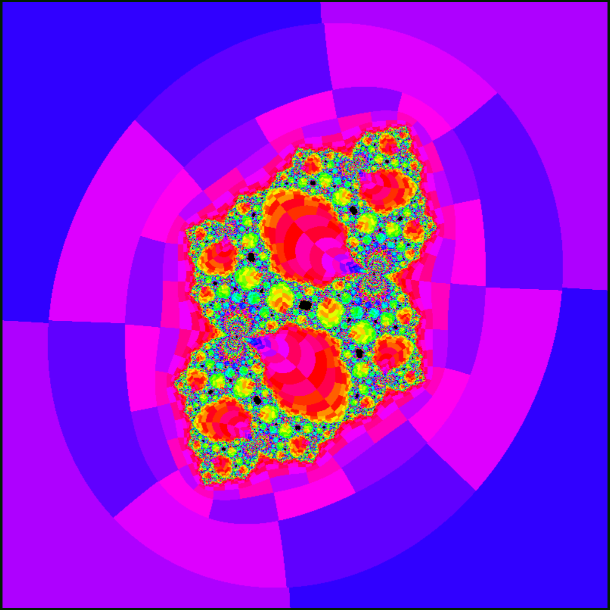
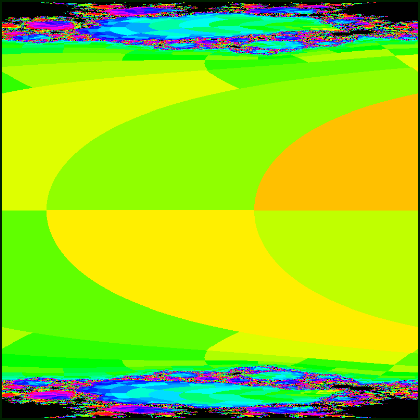


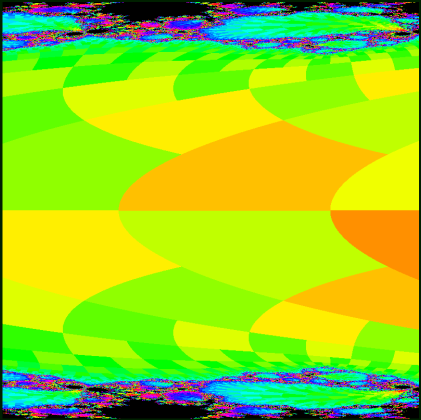
Geometric limits and Lavaurs maps

- Let $c_n \notin \overline{\mathcal{H}_0} \to c_0 \in \mathcal{C}_0$.
- ϕ_{c_n}: normalized Fatou coordinate s.t. $\phi_{c_n} \to \phi_{c,\mathrm{rep}}$.
- Assume $\exists k_n \to \infty$ s.t. $\phi_{c_n}(f_{c_n}^{2pk_n}(0)) \to \tau$.
- Then we have
$$f_{c_n}^{2pk_n} \to g_{c,\tau} \quad (n \to \infty).$$
- Notice: $\tau \in \mathbb{R}$!
  - $\phi_{c_n} - \alpha_n \to \phi_{c,\mathrm{attr}}$ for some $\boldsymbol{\alpha_n} \in \mathbb{R}$.
  - Therefore,
    $\tau \leftarrow \operatorname{Im} \phi_{c_n}(f_{c_n}^{2k_n}(0)) = \operatorname{Im}(\phi_{c_n}(f_{c_n}^{2k_n}(0)) - \alpha_n) \to \operatorname{Im} \phi_{c,\mathrm{attr}}(0)$.
- **Hubbard-Schleicher** (implicitly proved):
$$\{c_n\} \mapsto (c_0, \tau) \in \mathcal{C}_0 \times \mathbb{R}/\mathbb{Z}$$
is surjective.

Parameter space of Julia-Lavaurs sets



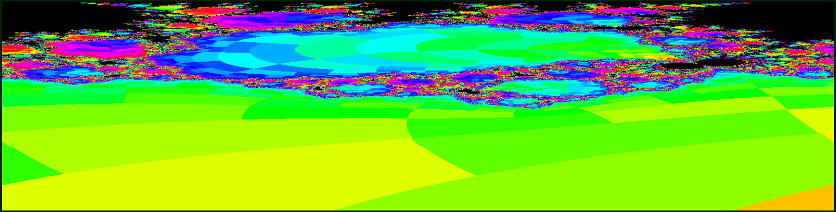
Julia-Lavaurs family



normalized tricorn family

The vertical direction is a parametrization of $\mathcal{C}_0$, and the horizontal direction is (an approximation of) the phase.

Parameter space of Julia-Lavaurs sets



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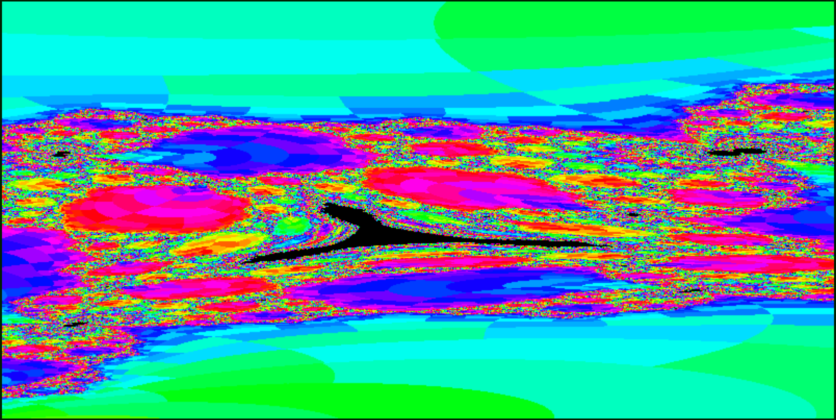


Inaccessibility for the family of geometric limits

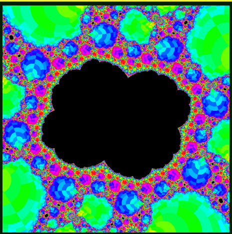
$\tilde{\mathcal{H}} \subset \mathcal{C}_0 \times S^1$: "primitive" hyperbolic component.

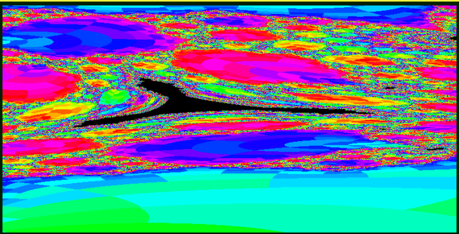
Lemma 2

1. The attractive basins are inaccessible from the escape region for $(c, \tau) \in \tilde{\mathcal{H}}$.
2. $\tilde{\mathcal{H}}$ is inaccessible from the escape locus.

Remark

- Indeed, there is no path **accumulating** to the boundary.
- The first statement still holds for the parabolic basin of $(c, \tau) \in \partial\tilde{\mathcal{H}}$.





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Proof of 1.

- Assume $f_c^l \circ g_{c,\tau}^m$ has an attracting fixed point.
- a polynomial-like restriction $h_{c,\tau} := f_c^l \circ g_{c,\tau}^m : U'_{c,\tau} \to U_{c,\tau}$ exists.
- Let $K_n = \{z \in \operatorname{int} K_c;\ g_{c,\tau}^k(z) \in \operatorname{int} K_c\ (k = 1, \ldots, n)\}$.
- $K_n \supset h_{c,\tau}^{-n}(U_{c,\tau})$ is a neighborhood of $K_{c,\tau}$.
- ∂K_n is contained in $J_{c,\tau}$.
- Therefore, the attractive basin is inaccessible from $\mathbb{C} \setminus K_n$.
- Escape region for $(f_c, g_{c,\tau}) = \bigcup(\mathbb{C} \setminus K_n)$.

$\square$

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Proof of 2.

- Assume $f_c^l \circ g_{c,\tau}^m$ has an attracting fixed point.
- $\exists \mathcal{U}$: nbd of $\tilde{\mathcal{H}}$ where polynomial-like restriction $h_{c,\tau} := f_c^l \circ g_{c,\tau}^m : U'_{c,\tau} \to U_{c,\tau}$ exists.
- Let $\mathcal{A}_n = \{(c, \tau);\ g_{c,\tau}^k(0) \in \operatorname{int} K_{c,\tau}\ (k = 1, \dots, n)\}$ and
- $\mathcal{C}_n = \{(c, \tau) \in \mathcal{U};\ g_{c,\tau}^k(0) \in U_{c,\tau}\ (k = 1, \dots, n)\}$.
- $\mathcal{C}_n$ is a neighborhood of $\mathcal{C}(\tilde{\mathcal{H}}) = \bigcap_k \mathcal{C}_k \supset \tilde{\mathcal{H}}$.
- $\partial \mathcal{A}_n$ is contained in the bifurcation locus.
- Therefore, $\tilde{\mathcal{H}}$ is inaccessible from $(\mathcal{C}_0 \times S^1) \setminus \mathcal{A}_n$.
- Escape locus $= \bigcup((\mathcal{C}_0 \times S^1) \setminus \mathcal{A}_n)$.

$\square$

Criterion for inaccessible hyperbolic components

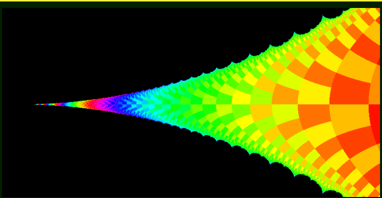
Let $\mathcal{H} \subset \mathrm{Hyp}^*$ be a hyperbolic component of odd period.

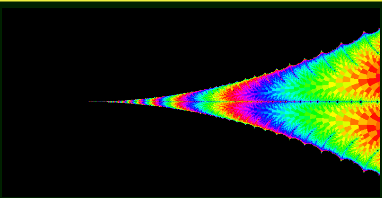
Lemma 3

Assume for any non-cusp $c \in \partial\mathcal{H}$ the following holds:

- Let $E_1, \ldots, E_K$ be the connected components of $K_c \cap \mathrm{Dom}(\phi_{c,\mathrm{rep}})$ such that the parabolic periodic point is in ∂E_k.
- $I_k := \mathrm{int}\,\mathrm{Im}\,\phi_{c,\mathrm{rep}}(E_k) \subset \mathbb{R}$ (Im $\phi_{c,\mathrm{rep}}$: Ecalle height).
- $\{I_k\}_{k=1}^{K}$ is an open cover of $\mathbb{R}$.

Then $\partial\mathcal{H}$ is inaccessible from the escape locus.





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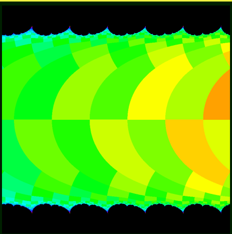

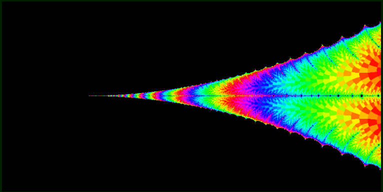

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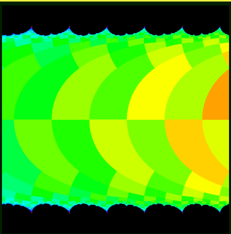
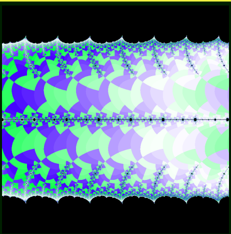
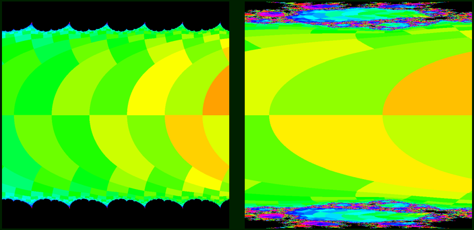
 

- The bifurcation locus near a parabolic arc looks like the blow-up of the Julia set at the parabolic periodic point w.r.t. the Ecalle height.



- Since any parabolic-attracting (virtually attracting) $c \in \mathcal{C}$ (or cusp) lies in the interior of $\mathcal{M}^*$ (in the common boundary arc with another hyperbolic components of double period), we need only check the assumption for c in the compact subarc of $\mathcal{C}$ consisting of non-parabolic-attracting parameters.

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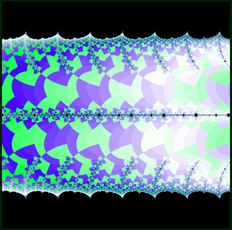
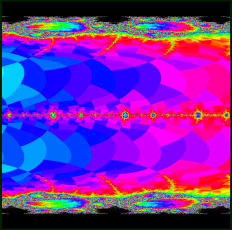

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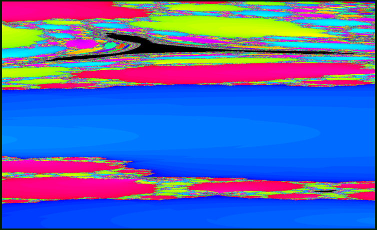
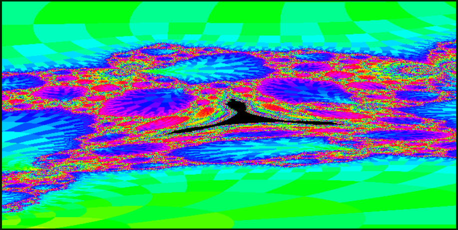
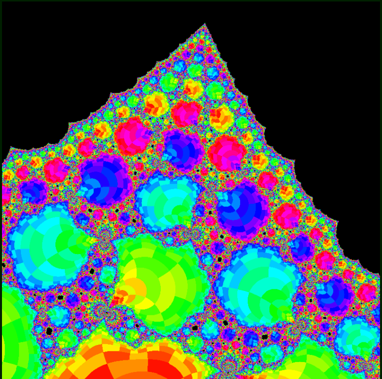
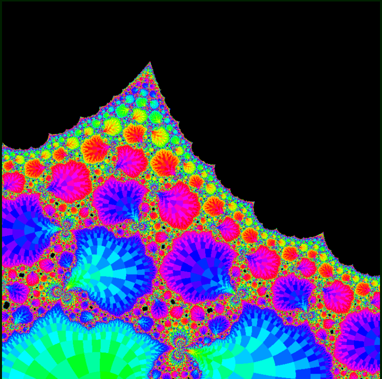
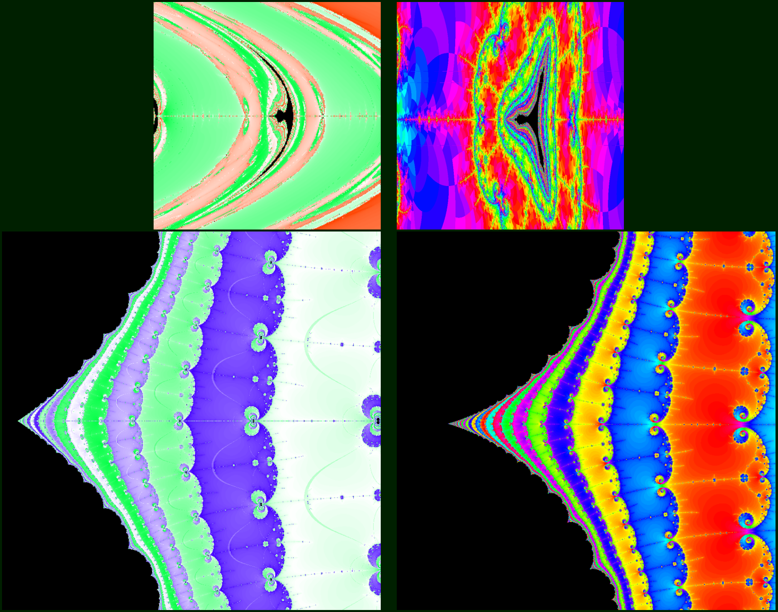
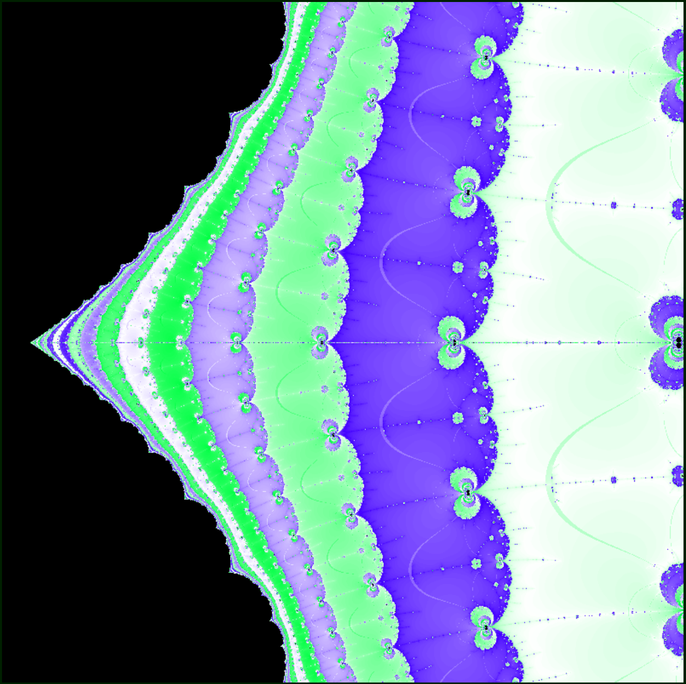
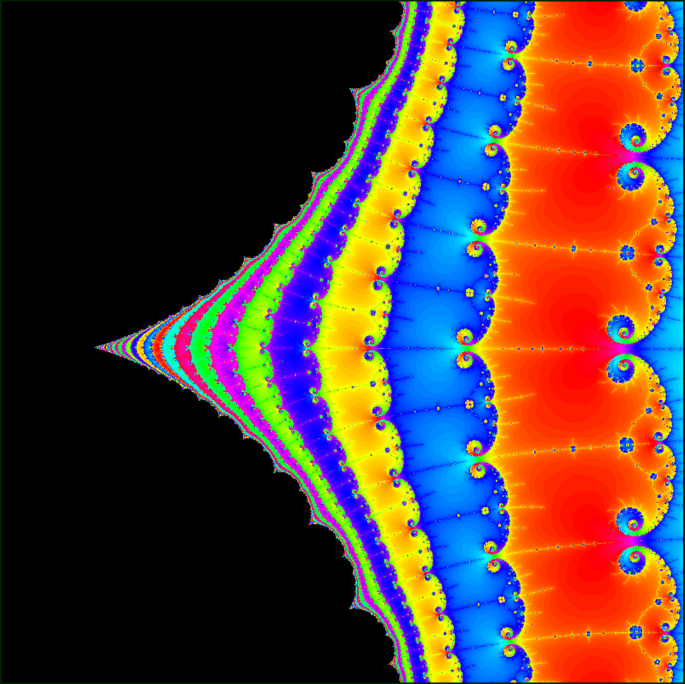
Perturbations

Assume $\mathcal{H}_n \subset \mathrm{Hyp}^*$ satisfy $\mathcal{H}_n \to \tilde{\mathcal{H}}$ "geometrically".







Inaccessible hyperbolic components for $\mathcal{M}^*$

Theorem 4 (I-Mukherjee, in progress)

- $\mathcal{H}_0 \subset \mathrm{Hyp}^*$: **real** hyperbolic component of odd period $p > 1$.
- $\mathcal{C} \subset \partial\mathcal{H}_0$: the root arc (i.e., the "umbilical cord" converges to it).
- $\tilde{\mathcal{H}} \subset \mathcal{C} \times S^1$: **real** hyperbolic component in the space of geometric limits.
- $\mathcal{H}_n \subset \mathrm{Hyp}^*$ converges to $\tilde{\mathcal{H}}$ geometrically.

Then for sufficiently large n, the root arc of $\mathcal{H}_n$ is inaccessible from the escape locus.

Outline of proof

- For any $c_n \in \partial\mathcal{H}_n$, there are three connected components $E_* = E_*(c_n)$ $(* = \pm, 0)$ of $K_c \cap \mathrm{Dom}(\phi_{c,\mathrm{rep}})$ in Lemma 3.
- Let $I_* = I_*(c_n) = \operatorname{int} \operatorname{Im} \phi_{c,\mathrm{rep}}(E_*)$.
  - I_+ is unbounded above,
  - I_0 is bounded,
  - I_- is unbounded below.
- Consider a geometric limit of a sequence $\mathcal{H}_n \ni c_n \to (c_0, \tau)$.
- $(f_{c_0}, g_{c_0,\tau})$ has an inaccessible parabolic basin.
- Therefore, $\operatorname{int} \operatorname{Im} \phi_{c_0,\tau,\mathrm{rep}}(K_{c_0,\tau}) = \mathbb{R}$.

Outline of proof (continued)

- Moreover, by assumption that $\mathcal{H}_0$ and $\tilde{\mathcal{H}}$ is real, E_+ and E_- must touch in the limit, so we have
$$\lim_{n\to\infty} \inf I_+ \le \lim_{n\to\infty} \sup I_-.$$
- $I_- = -I_+$ and $I_0 = -I_0$ by symmetry w.r.t. $z \mapsto \bar{z} + \frac{1}{2}$, so
$$\lim_{n\to\infty} \inf I_+ \le 0 < \lim_{n\to\infty} \sup I_0,$$
$$\lim_{n\to\infty} \inf I_0 < 0 \le \lim_{n\to\infty} \sup I_-.$$
- Hence for sufficiently large n, $\{I_*\}_{*=\pm,0}$ is an open covering of $\mathbb{R}$, and we can apply Lemma 3.

$\square$

Remarks

The assumptions that

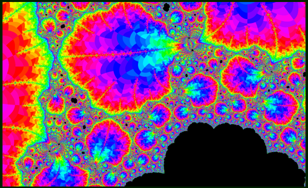
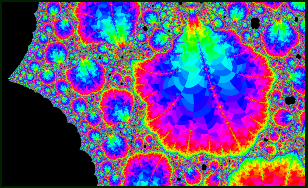
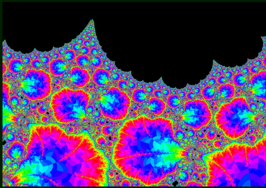
- the period $p > 1$,
- hyperbolic components are real, and
- the parabolic arc is the root arc

seems unnecessary, but we do not know how to exclude "degenarate" case:

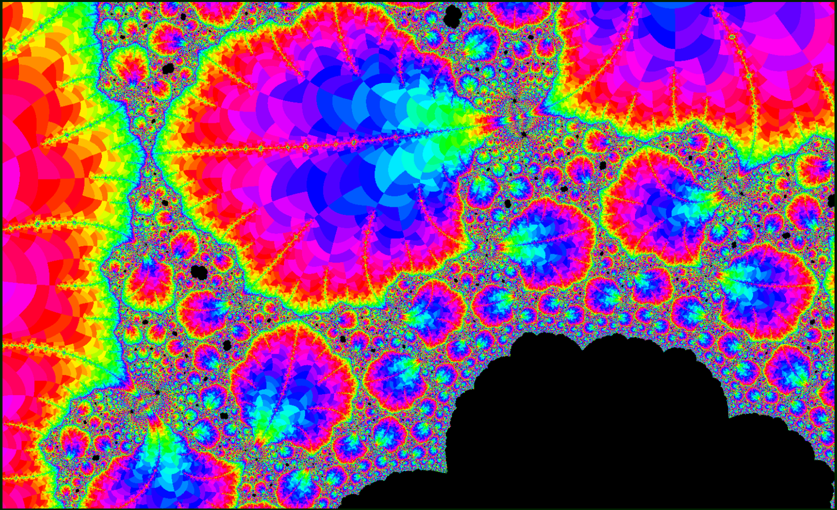
On root arcs: $\liminf I_+ = \limsup I_0, \quad \liminf I_0 = \limsup I_-$.

On co-root arcs: $\liminf I_+ = \limsup I_-$.

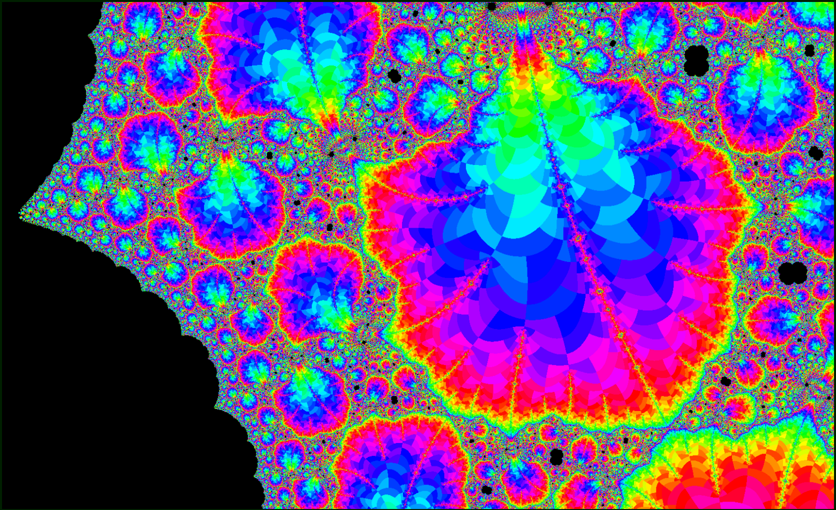
($E_0 = I_0 = \emptyset$ in this case.)

On root arc



On co-root arcs



On co-root arcs