Universality for the golden mean Siegel Disks, and existence of Siegel cylinders

Denis Gaidashev, Uppsala University,
(joint work with Michael Yampolsky)
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Universality for Siegel disks

Preliminaries

Consider a function, holomorphic on a nbhd of 0:

$$f(z) = \lambda z + az^2 + bz^3 ...$$

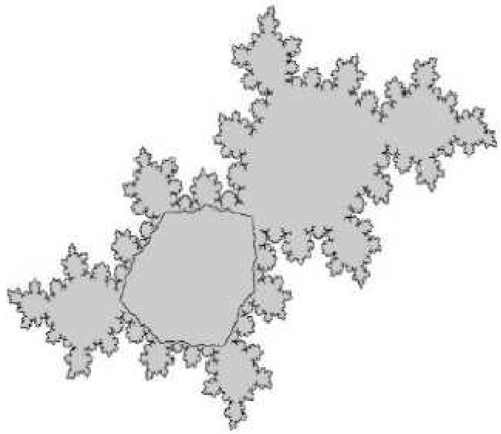
Question: Can one linearize this function on a nbhd of 0:

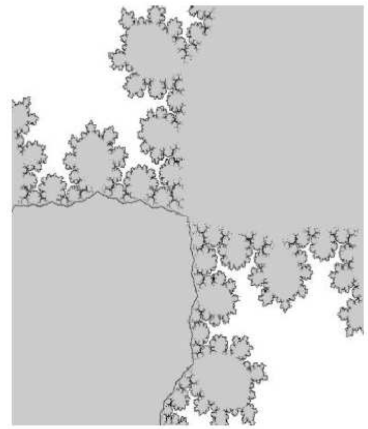
$$\phi^{-1} \circ f \circ \phi = \lambda \quad (?)$$

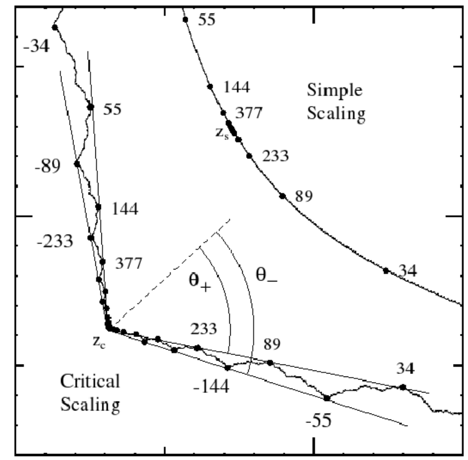
The answer is positive when $|\lambda| \neq 1$. The following addresses the case $\lambda = e^{2\pi i\theta}$:

Theorem. *(Siegel) f can be linearized by a local holomorphic change of coordinates for a. e. λ in $\mathbb{T}$.*

In particular, f is linearizable when θ is Diophantine: $\left|\theta - \frac{p}{q}\right| \geq \frac{\epsilon}{q^k}$, $k \geq 2$. The maximal domain of linearization is called the Siegel disk, $\triangle$.







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Consider a quadratic polynomial: $f_{\theta^*}(z) = e^{2\pi i \theta^*} z(1-0.5z)$, $\theta^* = \frac{\sqrt{5}-1}{2}$.

The boundary is self-similar at the critical point (Manton-Nauenberg; McMullen)

$$2\theta_- \approx 107.3, \quad 2\theta_+ \approx 120.0, \quad \lim_{n\to\infty} \frac{|f^{q_{n+1}}(z_c) - z_c|}{|f^{q_n}(z_c) - z_c|} = \lambda, \quad \lambda \approx 0.7419...$$

Conjecture. *Given an eventually periodic number* $\theta = [b_0, b_1, b_2, \ldots, b_n, a_1, a_2, \ldots, a_s, a_1, \ldots]$ *the self-similar geometry of the boundary of the Siegel disk is identical for all quadratic-like analytic maps defined on some neighborhood of zero with the multiplier* $e^{2\pi i\theta}$.

C. McMullen's renormalization for commuting pairs

- if θ is a quadratic of the bounded type, g and h - quadratic-like, with a multiplier $e^{2\pi i\theta}$ then,
  - $\exists$ a $hybrid\ conjugacy$ ϕ between g and h;
  - the complex derivative $\phi'(1)$ exists for all $z \in P(g)$, and is uniformly $C^{1+\alpha}$-conformal on $P(g)$:

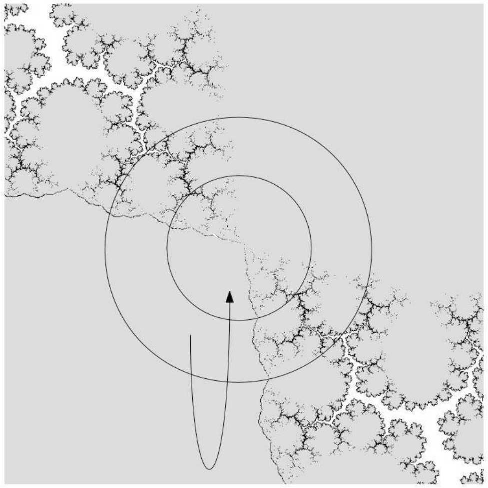
$$\phi(z+t) = \phi(z) + \phi'(z)t + O(|t|^{1+\alpha}$$

- Rescaled iterates $\lambda^{-n} \circ {P_\theta}^{q_{n+1}} \circ \lambda^n$ converge (C. McMullen).

• $\exists$ a nbhd U of 1, and $\epsilon > 0$, a function ψ on $U \cap \overline{\Delta_\theta}$, **conjugates $P_\theta{}^{q_n}$ to $P_\theta{}^{q_{n+1}}$**, and is conformal in $\Delta \cap U$ and $C^{1+\epsilon}$-anticonformal at 0:

$$\psi(z) = \begin{cases} 1 + \lambda(z-1) + O(|z-1|^{1+\epsilon}), & s \quad \text{is} \quad \text{even}, \\ 1 + \lambda\overline{(z-1)} + O(|z-1|^{1+\epsilon}), & s \quad \text{is} \quad \text{odd}, \end{cases}$$

here, $\lambda = \psi'(1)$ is the scaling ratio.

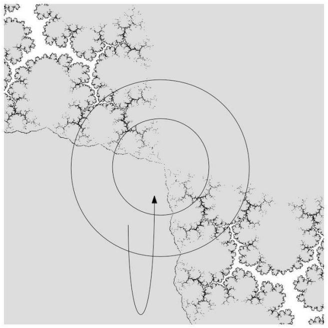


from X. Buff and Ch. Henriksen, 1999

The linearization of ψ at 1 will be called λ:

How do these results address the universal self-similarity of the Siegel disks?

– existence of the $C^{1+\alpha}$-conformal similarity map ψ implies that small-scale geometry of Δ_θ for P_θ is asymptotically linearly self-similar.



– existence of the $C^{1+\alpha}$-conformal hybrid conjugacy ϕ implies that the small-scale geometry of $P(g)$ for any quadratic-like g with the correct multiplier is asymptotically a linear copy of the small-scale geometry of $\partial\Delta_\theta$ for P_θ.

What does not follow from McMullen's theory is that $\overline{P(g)} = \partial\Delta$ for non-polynomial maps.

- Renormalization

Let $f : \mathcal{X} \mapsto \mathcal{X}$. Choose a subset $\mathcal{Y} \subset \mathcal{X}$, such that every point $y \in \mathcal{Y}$ returns to $\mathcal{Y}$ after $n(y)$ iterations. The map

$$R_f : y \mapsto f^{n(y)}(y)$$

is called a return map.

Next, suppose there is a "meaningful" rescaling A that "blows up" $\mathcal{Y}$ to the "size" of $\mathcal{X}$. We call

$$R[f] = A \circ R_f \circ A^{-1}$$

a renormalization of f.

Self-similarity of geometry for f is usually obtained from convergence of the iterations $f \mapsto R[f] \mapsto R^2[f] \mapsto$ .....

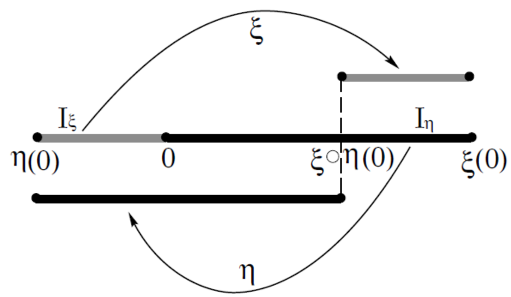
To demonstrate universality for golden mean Siegel disks, construct a renormalization operator R such that

$$R^k[f] \underset{k\to\infty}{\rightarrow} f^*$$

for "all" maps f with $f'(0) = e^{2\pi i\theta^*}$.

Renormalization for Commuting Pairs

• A *commuting pair* $\zeta = (\eta, \xi)$ consists of two C^2 orientation preserving homeos $\eta : I_\eta := [0, \xi(0)] \mapsto \eta(I_\eta)$, $\xi : I_\xi := [\eta(0), 0] \mapsto \xi(I_\xi)$, where



1) η and ξ have homeomorphic extensions to interval nbhds of their domains which commute: $\eta \circ \xi = \xi \circ \eta$;

2) $\xi \circ \eta(0) \in I_\eta$;

3) $\eta'(x) \neq 0 \neq \xi'(y)$ for all $x \in I_\eta \setminus \{0\}$ and all $y \in I_\xi \setminus \{0\}$.

• Regard $I = [\eta(0), \xi \circ \eta(0)]$ as a circle, identifying $\eta(0)$ and $\xi \circ \eta(0)$, and set $f_\zeta : I \mapsto I$:

$$f_\zeta(x) = \begin{cases} \eta \circ \xi(x), & x \in [\eta(0), 0] \\ \eta(x), & x \in [0, \xi \circ \eta(0)]. \end{cases}$$

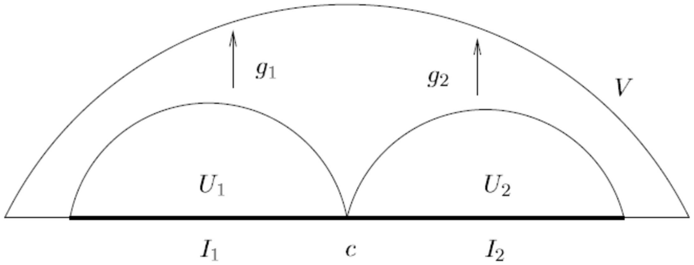
• A C^2 critical circle homeo f generates commuting pairs
$\zeta_n := \left(f^{q_n}|_{[f^{q_{n+1}}(c), c]}, f^{q_{n+1}}|_{[c, f^{q_n}(c)]}\right)$.

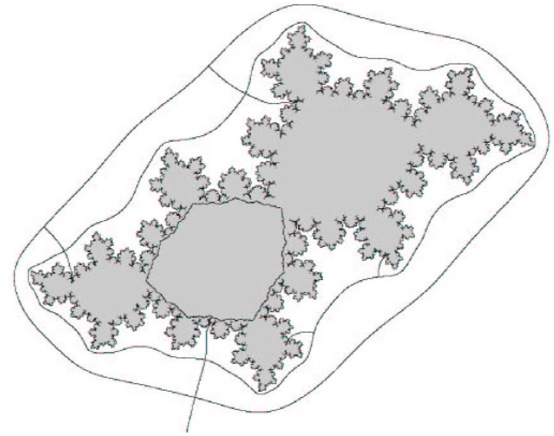
• For a pair $\zeta = (\eta, \xi)$ we denote by $\tilde{\zeta}$ the pair $(\tilde{\eta}|_{\tilde{I}_\eta}, \tilde{\xi}|_{\tilde{I}_\xi})$, where tilde means rescaling by $\lambda = -\frac{1}{|I_\eta|}$.

• The *renormalization* of a (golden mean) commuting pair ζ is

$$\mathcal{R}\zeta = \left(\widetilde{\eta \circ \xi}|_{\tilde{I}_\xi}, \tilde{\eta}|_{[0, \widetilde{\eta(\xi(0))}]}\right)$$

A problem: The space of commuting pairs is not nice, not a Banach manifold, in particular, impossible to work with on a computer.





McMullen's holomorphic commuting pairs

Let $\eta : \Omega_1 \mapsto \Sigma$ and $\xi : \Omega_2 \mapsto \Sigma$ be two univalent maps between quasidisks in $\mathbb{C}$, with $\Omega_i \subset \Sigma$. Suppose η and ξ have homeomorphic extensions to the boundary of domains Ω_i and Σ. We say that such a pair $\zeta = (\eta, \xi)$ a McMullen holomorphic pair if

1) $\Sigma \setminus \overline{\Omega_1 \cup \Omega_2}$ is a quasidisk;

2) $\overline{\Omega_i} \cap \partial\Sigma = I_i$ is an arc;

3) $\eta(I_1) \subset I_1 \cup I_2$ and $\xi(I_2) \subset I_1 \cup I_2$;

4) $\overline{\Omega_1} \cap \overline{\Omega_2} = \{c\}$, a single point.

Renormalization for Almost Commuting Pairs

- Let (η, ξ) be a pair of maps defined and holomorphic on open sets $Z \ni 0$ and $W \ni 0$, $Z \cap W \neq \emptyset$, in $\mathbb{C}$.

- Assume

$$\eta = \phi \circ q_2, \quad \xi = \psi \circ q_2,$$

where $q_2(z) := z^2$ and ϕ and ψ are univalent on $q_2(Z)$ and $q_2(W)$ respectively. The Banach space of such pairs will be denoted $\mathcal{E}(Z, W)$.

- The subset of pairs in $\mathcal{E}(Z, W)$ that satisfy

$$(\eta \circ \xi)^{(n)}(0)) = (\xi \circ \eta)^{(n)}(0)), \ n = 0, 1, 2, \tag{0}$$

will be referred to as *almost commuting symmetric pairs* and will be denoted $\mathcal{M}(Z, W)$.

Proposition. *$\mathcal{M}(Z, W)$ is a Banach submanifold of $\mathcal{E}(Z, W)$.*

• Let $c(z) := \bar{z}$. A pair $\zeta = (\eta, \xi) \in \mathcal{M}(Z, W)$ will be called *renormalizable*, if

$$\begin{aligned} \lambda(c(W)) &\subset Z, \\ \lambda(c(Z)) &\subset W, \\ \xi(\lambda(c(Z))) &\subset Z, \end{aligned}$$

where $\lambda(z) = \xi(0) \cdot z$, while the renormalization of a pair $\zeta = (\eta, \xi)$ will be defined as

$$\mathcal{P}\zeta = (\eta \circ \xi, \eta), \quad \mathcal{R}\zeta = c \circ \lambda^{-1} \circ \circ \mathcal{P}\zeta \circ \lambda \circ c. \tag{-3}$$

• Equivalently, if an almost commuting symmetric pair is renormalizable, we can defined the renormalization of its univalent factors as follows:

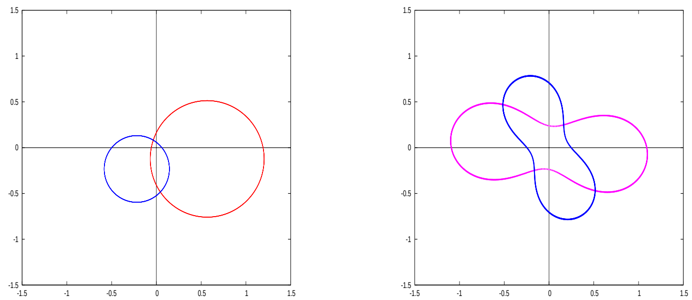
$$\mathcal{R}(\phi, \psi) = \left(c \circ \lambda^{-1} \circ \phi \circ q_2 \circ \psi \circ \lambda^2 \circ c, c \circ \lambda^{-1} \circ \phi \circ \lambda^2 \circ c\right). \tag{-3}$$

Lemma. $\mathcal{R}$ *preserves the Banach manifold of almost commuting pairs*

- Specific domains: U and V, domains of analyticity of ϕ and ψ are

$$U := \mathbb{D}_{r_\eta}(c_\eta), \quad V = \mathbb{D}_{r_\xi}(c_\xi), \quad \text{where}$$

$$c_\eta = \quad 0.5672961438978619 - 0.1229664702397770 \cdot i, \; r_\eta = 0.636,$$
$$c_\xi = -0.2188497414079558 - 0.2328147240271490 \cdot i, \; r_\xi = 0.3640985354093064.$$



Domains U and V (left) and Z and W (right).

- $\mathcal{A}_1(U, V)$ is the Banach manifold of all factors

$$\phi(z) = \sum_{i=0}^{\infty} \phi_i \left(\frac{z - c_\eta}{r_\eta} \right)^i, \quad \psi(z) = \sum_{i=0}^{\infty} \psi_i \left(\frac{z - c_\xi}{r_\xi} \right)^i,$$

univalent on U and V, respectively, and bounded in the following norm,

$$\|(\eta, \xi)\| := \sum_{i=1}^{\infty} \left(|\eta_i| + |\xi_i| \right).$$

• We iterate a Newton map $\mathcal{N}$ for the operator $\mathcal{R}$ and obtain a good approximation of the fixed point a.c.s. pair ζ_0.

• $D\mathcal{R}$ is first calculated as analytic formulas, implemented in a code, and diagonalized numerically as a a linear operator in a tangent space to $\mathcal{A}_1(U,V)$.

• The Newton map is $\mathcal{N}$ is shown to be a metric contraction, with a small bound on $D\mathcal{N}$ in a nbhd of ζ_0, so that the hypothesis of the Contraction Mapping Principle is fulfilled. Existence of the fixed point ζ_* of $\mathcal{R}$ follows.

• Bounds on the expanding eigenvalues (one relevant, one irrelevant, associated with translation changes of coordinates) are obtained; the rest of the spectrum is shown to be contractive, specifically, $\|D\mathcal{R}\zeta_*|_{T_{\zeta_*}\mathcal{W}}\| < 0.85$.

Immediate consequences of the computer assisted proof

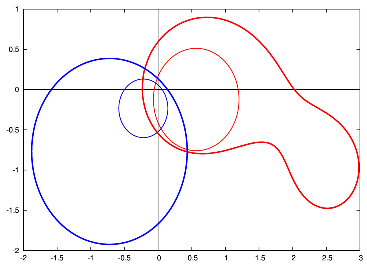
• There exists a nbhd $\mathcal{B}$ of (ϕ_*, ψ_*) in $\mathcal{A}_1(U,V)$ such that all almost commuting symmetric pairs $\zeta = (\phi \circ q_2, \psi \circ q_2)$ with $(\phi, \psi) \in \mathcal{B}$ and $\rho(\zeta) = \theta_*$ form a local stable manifold W^s in $\mathcal{A}_1(U,V)$.

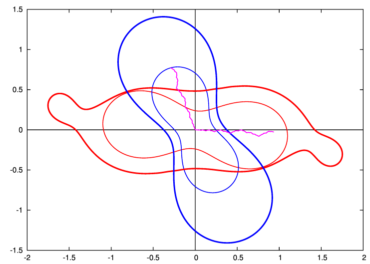
• Consider the nonsymmetric pairs:

$$\zeta = (\phi \circ q_2 \circ \alpha, \psi \circ q_2 \circ \beta),$$

where $\alpha(x) = x + O(x^2)$ and $\beta(x) = x + O(x^2)$ are close to identity. If $\eta(\xi(x)) - \xi(\eta(x)) = o(x^3)$ then ζ is an almost commuting pairs. $C(Z,W)$ - the space of such maps (a finite codimension manifold in the Banach space of pairs holomorphic on $Z \cup W$, equipped with the sup-norm).

• There exist domains $Z \subset \tilde{Z}$ and $W \subset \tilde{W}$, and a neighborhood a nbhd B' of ζ_* in $C(Z,W)$ such that





1) $\mathcal{R}$ is analytic from B' to $C(\tilde{Z}, \tilde{W})$;

2) The linearization of $\mathcal{R}$ is still hyperbolic.

Consequences: An invariant quasi-arc.

We abbreviate

$$\zeta^{\bar{s}} \equiv \xi^{b_n} \circ \eta^{a_n} \circ \cdots \circ \xi^{b_2} \circ \eta^{a_2} \circ \xi^{b_1} \circ \eta^{a_1}, \tag{-5}$$

A partial order on multi-indices: $\bar{s} \succ \bar{t}$ if

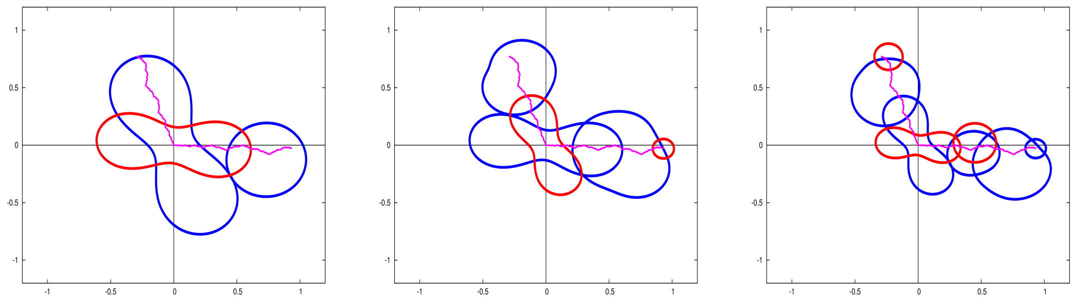
$$\bar{s} = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n), \quad \bar{t} = (a_1, b_1, \ldots, a_k, b_k, c, d),$$

where $k < n$ and either $c < a_{k+1}$ and $d = 0$ or $c = a_{k+1}$ and $d < b_{k+1}$.
Consider the n-th pre-renormalization of $\zeta \in \mathcal{W}^s(\zeta_*)$:

$$p\mathcal{R}^n \zeta = \zeta_n = (\eta_n|_{Z_n}, \xi_n|_{W_n}) \equiv (\zeta_n^{\bar{s}_n}|_{Z_n}, \zeta_n^{\bar{t}_n}|_{W_n},).$$

Define a *dynamical partition*

$$\mathcal{V}_n \equiv \{\zeta^{\bar{w}}(Z_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } \zeta^{\bar{w}}(W_n) \text{ for all } \bar{w} \prec \bar{t}_n\}.$$



Lemma. *There exist $K > 1$, $k \in \mathbb{N}$, and $C > 0$ such that the following properties hold.*

1. *For every n and every $W \in \mathcal{V}_n$, the domain W is a K-bounded distortion image of one of the sets $D_1^* = \eta_*(Z_1)$ or $B_n^* = \xi_*(W_1)$.*
2. *Let T_{n+k} be an interval in the dynamical partition $\mathcal{P}_{n+k}$ which is contained inside the interval $S_n \in \mathcal{P}_n$. Let $P_{n+k} \in \mathcal{V}_{n+k}$ and $Q_n \in \mathcal{V}_n$ be the elements with the same multi-indices as T_{n+k} and S_n. Then $P_{n+k} \Subset Q_n$ and*

$$\operatorname{mod}(Q_n \setminus P_{n+k}) > C.$$

3. *Let Q_1 and Q_2 belong to the n-th partition $\mathcal{V}_n$ and $Q_1 \cap Q_2 \neq \emptyset$. Then Q_1 is K-commensurable with Q_2.*

Theorem. *Consider a pair* $\zeta \in W^s(\zeta_*)$. *Assume further that* $\zeta = (\eta, \xi)$ *is a* **commuting**. *Then there exists a* ζ*-invariant quasi-symmetric Jordan arc* $\gamma \ni 0 = \overline{P(\zeta)}$ *such that* $\zeta|_\gamma$ *is q.-s. conjugate to the pair* $H = (f|_I, g|_J)$, *where*

$$f(x) = T^2(x) - 1 \text{ and } g(x) = T(x) - 1, \quad T(x) = x + \theta_*$$

by a conjugacy that maps 0 *to* 0.

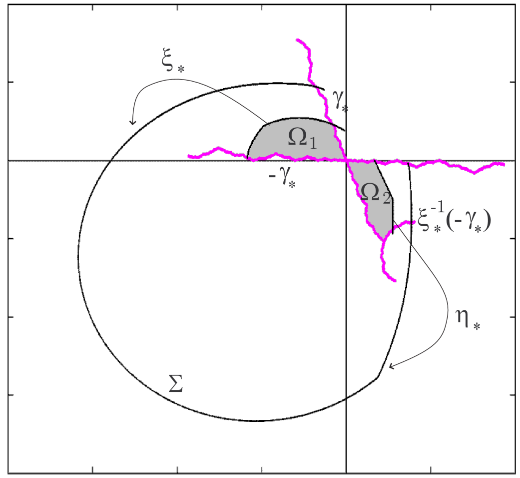


Figure 1: The domains of the McMullen holomorphic pair extension of ζ_*.

Consequence: ren. fixed point is McMullen

We use rigorous computer-assisted estimates to prove the following:

Proposition. *There exists a neighborhood W in $C(Z, W)$, such that for all $\zeta \in W^s(\zeta_*) \cap W$, the pair $\mathcal{P}^3\zeta$ has an extension to a McMullen holomorphic pair*

$$\tilde{\zeta} = (\tilde{\xi} : \Omega_1 \to \Sigma, \tilde{\eta} : \Omega_2 \to \Sigma).$$

Corollary 1. *For every* $\zeta \in W^s(\zeta_*) \cap W$ *the McMullen holomorphic pair extension* $\tilde{\zeta}$ *of* $\mathcal{P}^3\zeta$ *is quasiconformally conjugate to McMullen's fixed point* $\hat{\zeta}$*, this conjugacy is conformal on* $\operatorname{int}(K(\tilde{\zeta}))$*.*

Corollary 2. *The critical point* 0 *is a measurable deep point if* $K(\tilde{\zeta})$*, i.e.*

$$\operatorname{area}(\mathbb{D}_r(0) - K(\tilde{\zeta})) = O(r^{2+\epsilon}).$$

We say that a quasiconformal map ϕ of subset of $\mathbb{C}$ is $C^{1+\alpha}$-conformal at point z if $\phi'(z)$ exists and

$$\phi(z+t) = \phi(z) + \phi'(z)t + O(|t|^{1+\alpha}).$$

Corollary 3. *The conjugacy is* $C^{1+\alpha}$*-conformal at* 0*.*

Consequence 3: 2 D maps (joint with R. Radu and M. Yampolsky)

• M. Lyubich, A. de Carvalho, M. Martens, P. Hazard extended renormalization for unimodal maps to the Hénon-like maps. They showed that the dissipative Hénon-like maps are in the same universality class as the 1D maps.

Q: Can the same be done for maps with Siegel disks?

A: Yes, the extension of the operator to pairs of 2D maps $\Sigma = (A, B)$ has the form

$$\mathcal{R}^n \Sigma \equiv (\bar{A}, \bar{B}) = \Pi_2 \Pi_1 H \circ \left(\Sigma^{\bar{l}_n}, \Sigma^{\bar{m}_n} \right) \circ H^{-1}(x, y),$$

where

- Π_1 and Π_2 are certain projections that normalize the critical point of $\mathcal{R}^n \Sigma(x, 0)$ and enforce almost commutativity. $\Pi_i = id$ if the maps commute.
- H is a change of coordinates which makes the renormalization even more dissipative.

Lemma. *There exists an $n \in \mathbb{N}$, and a choice of an appropriate space $\mathcal{B}_\delta$ of 2D δ-perturbations of 1D maps, such that*

$$\mathrm{dist}(\mathcal{R}^n \Sigma, \iota(\mathcal{H}(\lambda_n(\hat{Z}_1), \lambda_n(\hat{W}_1)))) < C\delta^2$$

whenever $\Sigma \in \mathcal{B}_\delta$.

Theorem.

- *The point* $\iota(\zeta_*) = \left(\begin{pmatrix} \eta_* \\ \eta_* \end{pmatrix}, \begin{pmatrix} \xi_* \\ \xi_* \end{pmatrix} \right)$ *is a renormalization fixed point in a nbhd of* $\iota(C(Z,W)$ *in an appropriate Banach space of 2D maps.*
- *The linear operator* $N = D_{\iota(\zeta_*)} \mathcal{R}^2$ *is compact. The spectrum of* N *coincides with the spectrum of one-dimensional renormalization.*

Consider the complex quadratic Hénon map

$$H_{c,a}(x,y) = (x^2 + c + ay, ax), \quad \text{for } a \neq 0.$$

We say that a dissipative Hénon map $H_{c,a}$ has a *semi-Siegel fixed point* if the eigenvalues of the linear part of $H_{c,a}$ at that fixed point are $\lambda = e^{2\pi i\theta}$, with $\theta \in (0,1) \setminus \mathbb{Q}$ and μ, with $|\mu| < 1$, and $H_{c,a}$ is locally biholomorphically conjugate to the linear map $L(x,y) = (\lambda x, \mu y)$:

$$H_{\lambda,\mu} \circ \phi = \phi \circ L, \quad \phi : \mathbb{D} \times \mathbb{C} \to \mathbb{C}^2$$

sending $(0,0)$ to the semi-Siegel fixed point, such that the image $\phi(\mathbb{D} \times \mathbb{C})$ is *maximal*

We call $\phi(\mathbb{D} \times \mathbb{C})$ the *Siegel cylinder*; it is a connected component of the interior of K^+. Let

$$\Delta = \phi(\mathbb{D} \times \{0\}),$$

and by analogy with the one-dimensional case call it the *Siegel disk* of the Hénon map.

$$\mathcal{RG}\Sigma = \Lambda_\Sigma^{-1} \circ H_\Sigma \circ \left(\Sigma^{\tilde{s}_n}, \Sigma^{\tilde{t}_n}\right) \circ H_\Sigma^{-1} \circ \Lambda_\Sigma = L_\Sigma^{-1} \circ \hat{p}\mathcal{R}^n \Sigma \circ L_\Sigma,$$

on $\Omega \cup \Gamma \subset \mathbb{C}^2$.

$$L_\Sigma = H_\Sigma^{-1} \circ \Lambda_\Sigma.$$

For each multi-index

$$\bar{w} \prec \tilde{s}_{ln} \text{ or } \bar{w} \prec \tilde{t}_{ln}$$

we define a domain

$$Q^i_{\bar{w}} = \Sigma^{\bar{w}} \circ L_\Sigma \circ L_{\mathcal{RG}\Sigma} \circ \ldots \circ L_{\mathcal{RG}^{l-1}\Sigma}(\Upsilon^i), \ i = 1 \text{ or } 2, \Upsilon^1 = \Omega, \Upsilon^2 = \Gamma \qquad (-5)$$

Given $\Sigma \in W^s_{\text{loc}}(\zeta_\lambda)$, consider the following collection of functions defined on $\Omega \cup \Gamma$:

$$\Psi^\Sigma_{\bar{w}} = \Sigma^{\bar{w}} \circ L_\Sigma.$$

Given a collection of index sets $\{\bar{w}^i\}$, $\bar{w}^i \prec \bar{s}_n$ or $\bar{w}^i \prec \bar{t}_n$, consider the following *renormalization microscope*

$$\Phi^k_{\bar{w}^0,\bar{w}^1,\bar{w}^2,\ldots,\bar{w}^{k-1},\Sigma} = \Psi^\Sigma_{\bar{w}^0} \circ \Psi^{\mathcal{RG}\Sigma}_{\bar{w}^1} \circ \ldots \circ \Psi^{\mathcal{RG}^{(k-1)}\Sigma}_{\bar{w}^{k-1}}.$$

Lemma. *The renormalization microscope is a uniform contraction and maps a set Υ^i onto an element of partition $\mathcal{Q}_{kn}$ for Σ.*

Proof.

Proposition. *There exists* $\epsilon > 0$ *such that the following holds. Let* $|\mu| < \epsilon$*, and*

$$H_{\lambda_1,\mu} \in W^s(\zeta_\lambda) \text{ where } \lambda_1 = e^{2\pi i\theta}.$$

Denote Ω_n, Γ_n *the domains of definition of the* n*-th pre-renormalization* $p\mathcal{R}^n H_{\lambda_1,\mu}$*. Then there exists a curve* $\gamma_* \subset \mathbb{C}^2$ *such that the following properties hold:*

γ_* *is a homeomorphic image of the circle;*

$\gamma_* \cap \Omega_n \neq \emptyset$ *and* $\gamma_* \cap \Gamma_n \neq \emptyset$ *for all* $n \geq 0$*;*

there exists a topological conjugacy

$$\varphi_* : \mathbb{T} \to \gamma_*$$

between the rigid rotation $x \mapsto x + \theta_1 \mod \mathbb{Z}$ *and* $H_{\lambda_1,\mu}|_{\gamma_*}$*;*

there exists m *such that* $G^m(\theta_1) = \theta$*;*

the conjugacy φ_* *is not* C^1*-smooth.*

Theorem (G, R. Radu, M. Yampolsky) *There exists $\epsilon > 0$ such that the following holds. Let $H_{\lambda_1,\mu} \in W^s(\zeta_\lambda)$ with $|\mu| < \epsilon$ and let γ_* be the invariant curve. Then γ_* bounds a Siegel disk for $H_{\lambda_1,\mu}$. The eigenvalue*

$$\lambda_1 = e^{2\pi i \theta_1} \text{ with } \theta = G^m(\theta_1) \text{ for some } m \geq 0. \tag{-5}$$

Finally, there exists $\epsilon_1 > 0$ such that for all $|\mu| < \epsilon_1$ and for all λ_1 satisfying (), we have $H_{\lambda_1,\mu} \in W^s(\zeta_\lambda)$.

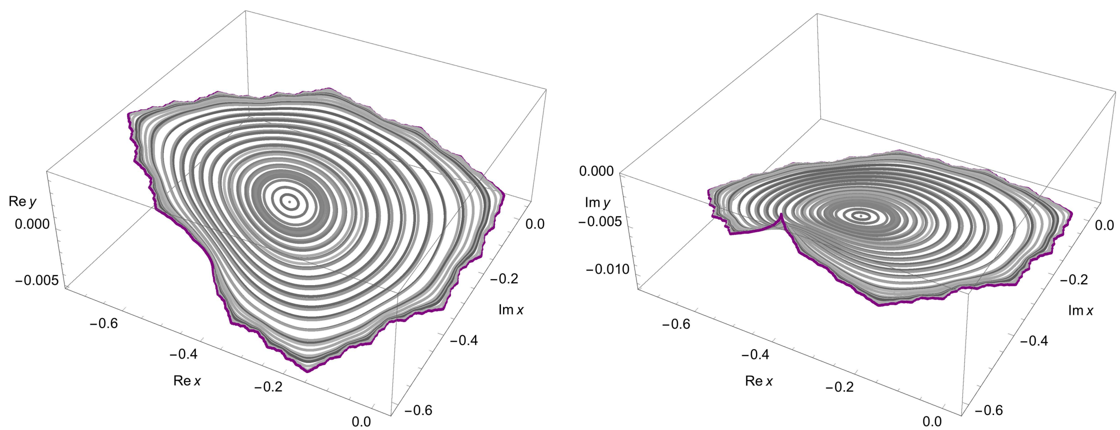


Figure 2: A three dimensional plot of the Siegel disk and its boundary for a Hénon map with a semi-Siegel fixed point with the golden mean rotation number. The parameter $a = 0.01 + 0.01i$. The three axes are as follows: TOP: Re(x), Im(x) and Re(y); BOTTOM: Re(x), Im(x) and Im(y).