Distribution of PCF cubic polynomials

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The parameter space of polynomials

Poly_d: degree d polynomials modulo conjugacy by affine transformations.

- $\{P = a_d z^d + \cdots + a_0,\ a_d \neq 0\}/\{P \sim \phi \circ P \circ \phi^{-1}, \phi(z) = az + b\}$
- $\mathbb{C}^* \times \mathbb{C}^d / \mathrm{Aff}(2, \mathbb{C})$.

Complex affine variety of dimension $d - 1$: finite quotient singularities.

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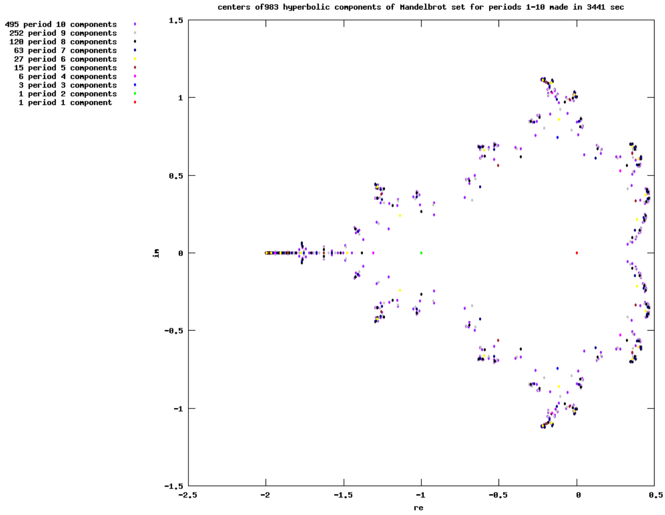
PCF maps are defined over $\overline{\mathbb{Q}}$:

$$c = 0,\ c^2 + c = 0,\ (c^2 + c)^2 + c = 0, \ldots$$

Distribution of hyperbolic quadratic PCF polynomials



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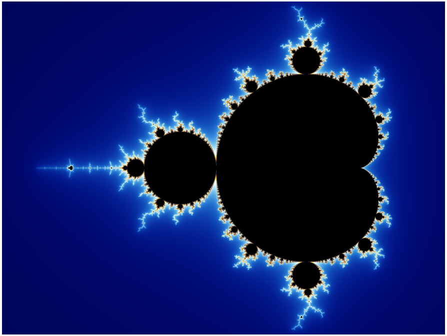

Distribution of quadratic PCF polynomials

$\mathrm{PCF}(n) = \{P_c, \text{ the orbit of 0 has cardinality } \leq n\}$.

Theorem (Levin, Baker-H'sia, F.-Rivera-Letelier, ...)

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- Levin: potential theoretic arguments
- Baker-H'sia: exploit adelic arguments, and work over all completions of $\mathbb{Q}$ both Archimedean and non-Archimedean: $\mathbb{C}, \mathbb{C}_p$ for p prime.

The cubic case Poly_3

- Parameterization: $P_{c,a}(z) = \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3$, $a, c \in \mathbb{C}$.
- $\mathrm{Crit}(P_{c,a}) = \{0, c\}$
- $P_{c,a}(0) = a^3$, $P_{c,a}(c) = a^3 - \frac{c^3}{6}$.

$$\mathrm{PCF}(n, m) = \{\text{orbit of 0 has cardinality } \le n\} \;\&\; \{\text{orbit of } c \text{ has cardinality } \le m\}$$

Finiteness of PCF maps

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  - if $|a| > \max\{1, |c|\}$, then $|P_{c,a}(0)| = |a|^3$, $|P^2_{c,a}(0)| = |a|^9$ and, $|P^n_{c,a}(0)| = |a|^{3^n} \to \infty$;

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  - $\mathrm{PCF}(n,m) \subset \{|c|, |a| \leq 1\}$.

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Proposition (Ingram)

For any finite extension $K/\mathbb{Q}$ *the set* $\mathrm{PCF} \cap K^2$ *is finite.*

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Theorem (F.-Gauthier)

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$\mu = \text{Monge-Ampère}(G_{\mathcal{C}}) = (dd^c)^2 G_{\mathcal{C}}$.

The adelic approach

Interpretation of the Green function in the parameter space:

$$G_{\mathcal{C}} = \max\{G_{c,a}(c), G_{c,a}(0)\}$$

- Dynamical Green function:

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Key observation: $P_{c,a} \in \mathrm{PCF}$ iff

$$\mathrm{Height}(c,a) := \frac{1}{\deg(c,a)} \sum_p \sum_{c',a'} G_{\mathcal{C},p}(c',a') = 0$$

$\longrightarrow$ Apply Yuan's theorem!

Special curves

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Theorem (Baker-DeMarco, Ghioca-Ye, F.-Gauthier)

Suppose C is an irreducible algebraic curve in Poly_3 *such that* $\mathrm{PCF} \cap C$ *is infinite. Then there exists a persistent critical relation.*

The curves $\mathrm{Per}_n(\lambda)$: DeMarco's conjecture

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Theorem (F.-Gauthier)

The set $\mathrm{PCF} \cap \mathrm{Per}_n(\lambda)$ *is infinite iff* $\lambda = 0$.

$n = 1$ by Baker-DeMarco

Scheme of proof

C irreducible component of $\mathrm{Per}_n(\lambda)$ containing infinitely many PCF: $\lambda \in \bar{\mathbb{Q}}$.

1. One of the critical point is persistently preperiodic on C.
2. C contains a unicritical PCF polynomial $\Longrightarrow |\lambda|_3 < 1$.
3. There exists a *quadratic* PCF polynomial having λ as a multiplier $\Longrightarrow |\lambda|_3 = 1$.

Step 2

$P_{c,a}(z) = \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3$

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a) Existence of the PCF unicritical polynomial by Bezout and Step 1.

b) Suppose $P(z) = z^3 + b$ is PCF

- $|b|_3 \leq 1$
- periodic orbits are included in $|z|_3 \leq 1$
- $|P'(z)|_3 = |3z^2|_3 < 1$ on the unit ball hence $|\lambda|_3 < 1$

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1. Apply the previous theorem: there exists a persistent critical relation.
2. Suppose $P^{m'}_{c,a}(0) = P^{m}_{c,a}(c)$ for all $c, a \in C$.
   - a branch at infinity $\mathfrak{c}$ of C induces a cubic polynomial $\mathfrak{P} = P_{c(t),a(t)} \in \mathbb{C}((t))[z]$
   - both points 0 and c tend to ∞ on $\mathfrak{c}$
   - Kiwi & Bezivin $\Longrightarrow$ all multipliers of $\mathfrak{P}$ are repelling
   - contradiction with the existence of a periodic point with constant multiplier λ

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$$\mathfrak{P}^m(z) = a_0(t) + a_1(t)z + \ldots + a_k(t)z^k$$

$\mathfrak{P}^m\{|z| \leq 1\} = \{|z| \leq 1\}$ implies $a_i(t) \in \mathbb{C}[[t]]$

The renormalization is $a_0(0) + a_1(0)z + \ldots + a_k(0)z^k$.

Dynamics of the quadratic renormalization

- The renormalization is a quadratic PCF polynomial Q.
- One multiplier of Q is λ: non-repelling orbits pass through $\overline{B}$ (Kiwi)
- All multipliers of Q have $|\cdot|_3 = 1$.

Happy Birthday Sebastian!