Connectivity of Julia sets of Newton maps: A unified approach

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Newton's method in the complex plane

Given $f(z)$ a complex polynomial, or an entire transcendental map, its **Newton's method** is defined as

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

N_f is either a rational map or a transcendental meromorphic map, generally with infinitely many poles and singular values.

- It is one of the oldest and best known root-finding algorithms.
- It was one of the main motivations for the classical theory of holomorphic dynamics.
- It belongs to the special class of meromorphic maps: Those with

NO FINITE, NON-ATTRACTING FIXED POINTS

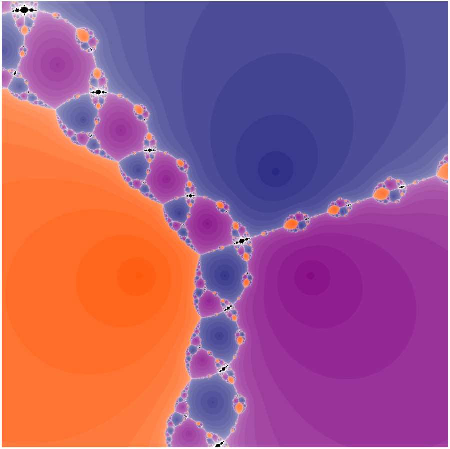
Newton's method in the complex plane

As all complex dynamical systems, its phase space decomposes into two totally invariant sets:

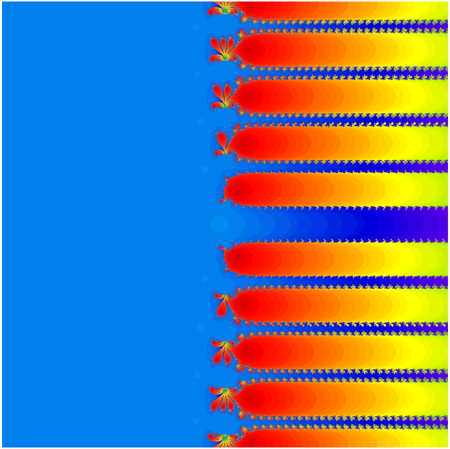
- **The Fatou set (or stable set)**:
  - Basins of attraction of attracting or parabolic cycles,
  - Siegel discs (irrational rotation domains),
  - Herman rings (irrational rotation annuli),
  - Wandering domains ($N^n(U) \cap N^m(U) = \emptyset$) or
  - Baker domains ($\{N^{pn}\}$ converges locally uniformly to ∞, for some $p > 0$ and $n \to \infty$, and ∞ is an essential singularity).

- **The Julia set (or chaotic set)** = closure of the set of repelling periodic points = closure of prepoles of all orders = boundary between the different stable regions ....

Newton's method N_f for $f(z)$.



$f(z) = z(z-1)(z-a)$



$f(z) = z + e^z$

Main Theorem

The study of the distribution and topology of the basins of attraction has recently produced efficient algorithms to locate all roots of P. [Hubbard, Schleicher and Sutherland '04 '11].

- Goal: To present a new unified proof of the following theorem.

Theorem

Let f be a polynomial or an ETF. Then, all Fatou components of its Newton's method N_f are simply connected. (Equivalently, $\mathcal{J}(N_f)$ is connected.)

- In particular, there are no Herman rings: only basins and Siegel disks (if f polynomial) or additionally Baker or wandering domains (if f transcendental), all of them simply connected.

History of the problem

- f polynomial
  - Partial results from Przytycki '86, Meier '89, Tan Lei ...
  - A **more general theorem on meromorphic maps** by Shishikura'90, closing the problem. ▸ Shishikura's Theorem
- f entire transcendental; N_f Newton's method.
  - Mayer + Schleicher '06: Basins of attraction and "virtual immediate basins" are simply connected.
- f entire transcendental, generalization of Shishikura's general theorem:
  - Bergweiler + Terglane '96: case where U is a wandering domain.
  - F + Jarque + Taixés '08: case where U is an attracting basins or a preperiodic comp.
  - F + Jarque + Taixés '11: case where U is a parabolic basin.
  - Baranski, F., Jarque, Karpinska '14 case where U is a Baker domain and no Herman rings, closing the problem.

History and goal

- Shishikura's proof (of the general theorem) and its extensions were heavily based on surgery. The transcendental case was quite delicate.

- To conclude the problem, new tools were developed in [BFJK'14]:

  - Existence of absorbing regions inside Baker domains (as it is the case for attracting or parabolic basins).

  - New strategy for the proof, different from all the previous ones, based on the existence of fixed points under certain situations.

We now use these new tools to give a UNIFIED proof of the connectivity of $\mathcal{J}(N_f)$ in all settings at once – rational and transcendental; DIRECT – not as a corollary of the general result; and therefore SIMPLER.

Tools: Existence of absorbing regions (in Baker domains)

Absorbing Theorem ([BFJK'14])

Let F be a transcendental meromorphic map and U be an invariant Baker domain. Then there exists a domain $W \subset U$, which satisfies:

(a) $\overline{W} \subset U$,

(b) $F^n(\overline{W}) = \overline{F^n(W)} \subset W$ *for every* $n \geq 1$,

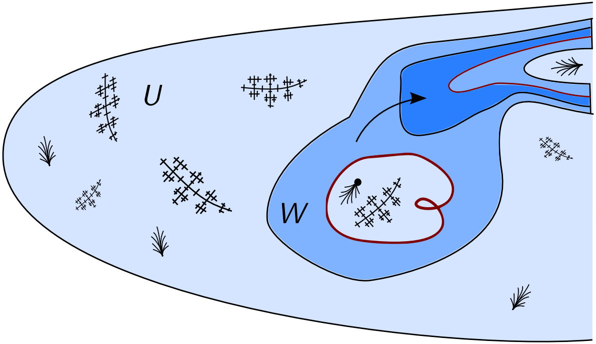
(c) $\bigcap_{n=1}^{\infty} F^n(\overline{W}) = \emptyset$,

(d) *W is absorbing in U for F, i.e., for every compact set $K \subset U$, there exists $n_0 \in \mathbb{N}$ such that $F^n(K) \subset W$ for all $n > n_0$.*

Moreover, F is locally univalent on W.

- The theorem holds for any $p-$**cycle of Baker domains**, just taking F^p.
- It is well known that basins of attraction contain simply connected absorbing regions. ▸ Idea of the proof

Absorbing regions inside Baker domains, in general, are NOT **simply connected** (König '99, BFJK '13).



Happy birthday! Per molts anys!! Gefeliciteerd!!!

Theorem (Shishikura'90)

Let g be a rational map. If $\mathcal{J}(g)$ is disconnected, then g has two weakly repelling fixed points (multiplier $\lambda = 1$ or $|\lambda| > 1$).

- Notice that every rational map has at least one weakly repelling fixed point.
- In the case of Newton maps, infinity is the only non-attracting fixed point and there are no others. Hence $\mathcal{J}(N)$ is connected.
- The proof is based on several different surgery constructions.

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Existence of absorbing domains

Cowen's Theorem

We have the following commutative diagram [Baker+Pomerenke'79; Cowen'81].

- G holomorphic w/o fixed pts
- T Möbius transf.
- $\Omega \in \{\mathbb{H}, \mathbb{C}\}$
- $V, \varphi(V)$ simply connected
- $\varphi : \mathbb{H} \to \Omega$ semiconjugacy
- φ univalent in V.

$$\begin{array}{ccccc} \varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\ \big\downarrow \varphi^{-1} & & \big\uparrow \varphi & & \big\uparrow \varphi \\ V & \subset & \mathbb{H} & \xrightarrow{G} & \mathbb{H} \\ & & \big\downarrow \pi & & \big\downarrow \pi \\ & & U & \xrightarrow{F} & U \end{array}$$

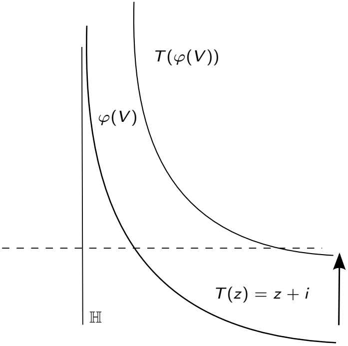
Moreover, $\{\varphi, T, \Omega\}$ depends only on (the speed to infinity of the orbits of) G.
This solves the case of U simply connected, taking π the Riemann map.

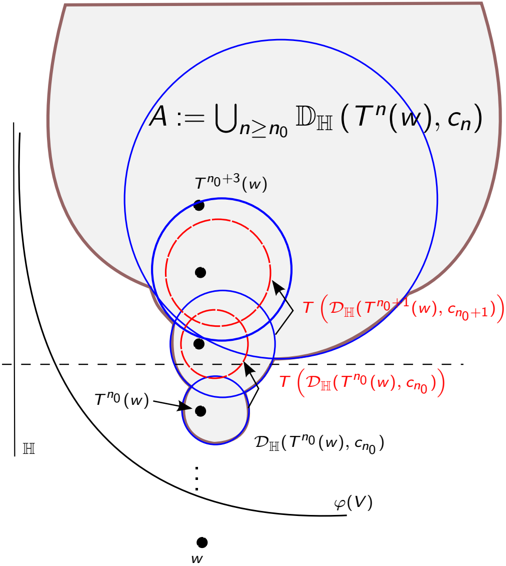
Idea of the proof

- In general we cannot guarantee that $\overline{\pi(V)} \subset U$.
- So we define a set $A \subset \varphi(V)$ small enough and absorbing to ensure that $W := \pi\left(\varphi^{-1}(A)\right)$ has the desired properties.

$$
\begin{array}{ccccccc}
A & \subset & \varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\
\big\downarrow{\scriptstyle \varphi^{-1}} & & \big\downarrow{\scriptstyle \varphi^{-1}} & & \big\uparrow{\scriptstyle \varphi} & & \big\uparrow{\scriptstyle \varphi} \\
\varphi^{-1}(A) & \subset & V & \subset & \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\
\big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\
W := \pi\left(\varphi^{-1}(A)\right) & \subset & \pi(V) & \subset & U & \xrightarrow{F} & U
\end{array}
$$

Defining the set A (case $\Omega = \mathbb{H}$, $T(z) = z + i$)





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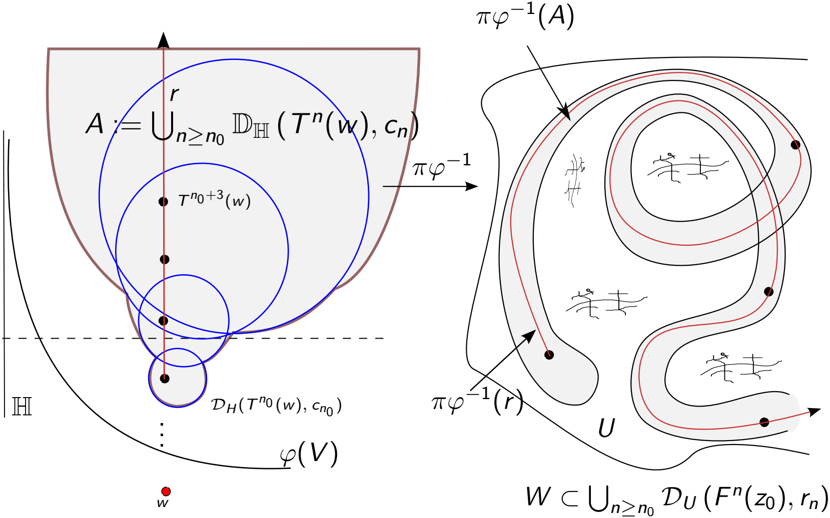

$W \subset \bigcup_{n \geq n_0} \mathcal{D}_U(F^n(z_0), r_n)$

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