Non-density of stability for holomorphic endomorphisms of $\mathbb{CP}^k$

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Foreword

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- This is work in progress and some details still need to be checked.
- I restrict to $k = 2$. Similar results in higher dimensions can be obtained easily (e.g. by taking products).

Plan

1. Basic facts on endomorphisms of $\mathbb{CP}^2$
2. Stability and bifurcations in dimension 1
3. Review on bifurcations in higher dimension and main results
4. Two mechanisms for robust bifurcations :
   - Mechanism 1 : robustness from topology
   - Mechanism 2 : robustness from fractal geometry
5. Further settings and perpectives

Holomorphic maps on $\mathbb{P}^2$

Let $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ holomorphic (no indeterminacy points), and $d = \deg(f)$, which equals $\deg(f^{-1}(L))$ for a generic line L. From now on $d \geq 2$.

Given homogeneous coordinates $[z_0 : z_1 : z_2]$, f expresses as

$$[P_0(z_0, z_1, z_2) : P_1(z_0, z_1, z_2) : P_2(z_0, z_1, z_2)]$$

where the P_i are homogeneous polynomials of degree d without common factor. Note $f^{-1}(L) = \{aP_1 + bP_2 + cP_3 = 0\}$

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In particular the space $\mathcal{H}_d$ of holomorphic maps on $\mathbb{P}^2$ is a Zariski open set in $\mathbb{P}^N$ with $N = 3\frac{(d+2)!}{2d!} - 1$

Dynamics of holomorphic maps on $\mathbb{P}^2$

For generic x, $\#f^{-1}(x) = d^2$ (Bézout) so the topological degree is $d_t = d^2$.

Theorem (Yomdin, Gromov)

Topological entropy $h_{top}(f) = \log d_t = 2\log d$.

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Preimages equidistribute towards a canonical invariant measure μ_f.

Theorem (Fornæss-Sibony)

There is a unique probability measure μ_f s.t. for generic $x \in \mathbb{P}^2$,

$$\frac{1}{d^{2n}} \sum_{f^n(y)=x} \delta_y \to \mu_f.$$

and μ_f is invariant and mixing.

- Denote $J^* = \mathrm{Supp}(\mu_f)$ and J the Julia set (in the usual sense).
- Typically $J^* \subsetneq J$.

Trivial example : $f(z, w) = (p(z), q(w))$ where p and q are polynomials of degree d. Then

$$J^* = \pi_1^{-1}(J_p) \cap \pi_2^{-1}(J_q) \text{ and } J = \pi_1^{-1}(J_p) \cup \pi_2^{-1}(J_q)$$

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Polynomial maps in $\mathbb{C}^2$ of the form $f(z, w) = (p(z, w), q(z, w))$ and such that $p_d^{-1}(0) \cap q_d^{-1}(0) = \{0\}$, extend as holomorphic maps on $\mathbb{P}^2$. Then J^* is a compact subset in $\mathbb{C}^2$ while J is unbounded.

Dynamics of holomorphic maps on $\mathbb{P}^2$

The canonical measure μ_f concentrates a lot of the dynamics of f.

Theorem (Briend-Duval)

- μ_f is the unique measure of maximal entropy $h_\mu(f) = h_{top}(f)$;
- periodic points (resp. repelling periodic points) equidistribute towards μ_f ;
- μ_f is (non-uniformly) repelling : its (complex) Lyapunov exponents satisfy $\chi_1, \chi_2 \geq \frac{1}{2} \log d$

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Theorem (De Thélin)

If $X \Subset \mathbb{P}^2 \setminus \mathrm{Supp}(\mu_f)$ then $h_{top}(f|_X) \leq \log d$.

Stability and bifurcations in dimension 1

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Theorem (Mañé-Sad-Sullivan, Lyubich)

Let $(f_\lambda)_{\lambda\in\Lambda}$ as above, and $\Omega \subset \Lambda$ be a connected open subset. The following properties are equivalent :

1. periodic points do not change type (attracting, repelling, indifferent) in Ω ;
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Then we say that the family is stable over Ω, and from this we get a decomposition

$$\Lambda = \mathrm{Stab} \cup \mathrm{Bif}\,.$$

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Since $N_{att} \leq 2d - 2$ the procedure stops after finitely many steps, so we ultimately obtain f_k belonging to $\mathrm{Stab} \cap U$. $\square$

Remark : this argument cannot be generalized to higher dimensions...

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Two main ideas :

- Construction of hyperbolic repellers of large Hausdorff dimension from bifurcations of parabolic points.
- At a Misiurewicz bifurcation there is similarity between dynamical and parameter space.

Also, the Douady-Hubbard theory of polynomial-like mappings shows that copies of the Mandelbrot set are abundant in Bif

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Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic mappings of degree $d \geq 2$ on $\mathbb{P}^2$, and $\Omega \subset \Lambda$ be a connected open set. TFAE :

1. J^*-repelling cycles do not bifurcate along Ω ;
2. J^* moves holomorphically (in a weak sense) in Ω ;
3. $\lambda \mapsto \chi_1(f_\lambda) + \chi_2(f_\lambda)$ is harmonic on Ω ;
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This yields a parameter dichotomy $\Lambda = \mathrm{Stab} \cup \mathrm{Bif}$.

Note that the definition depends on Λ : if $\Lambda \subset \Lambda'$ it may happen that $f \in \mathrm{Stab}\,|_\Lambda$ but $f \in \mathrm{Bif}\,|_{\Lambda'}$.

Misiurewicz bifurcations

Definition

A Misiurewicz bifurcation occurs at λ_0 if there exists $N \ni \lambda_0$, a holomorphically moving repelling periodic point $N \ni \lambda \mapsto \gamma(\lambda)$ and an integer k such that :

1. $\gamma(\lambda_0) \in f_{\lambda_0}^k(\mathrm{Crit}(f_{\lambda_0}))$ and $\gamma(\lambda_0) \in J_{\lambda_0}^*$
2. for some $\lambda \in N$, $\gamma(\lambda) \notin f_\lambda^k(\mathrm{Crit}(f_\lambda))$, i.e. $\gamma(\lambda)$ does not persistently belong to the post-critical set.

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Remark : the condition that $\gamma(\lambda) \in J^*_\lambda$ is open in parameter space.

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Lemma

(generalized) Misiurewicz bifurcations are contained (and dense) in the bifurcation locus.

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Note : families with open subsets of bifurcations were recently constructed by Bianchi and Taflin.

Remark : case of holomorphic automorphisms

Theorem (Buzzard)

$\mathring{\mathrm{Bif}}$ is non empty in $\mathrm{Aut}_d(\mathbb{C}^2)$ for sufficiently large d.

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Remark : one can embed the dynamics of a complex Hénon map into a holomorphic map of $\mathbb{P}^2$

$$(z, w) \mapsto (aw + p(z), az + \varepsilon w^d)$$

This cannot be used to produce robust bifurcations in $\mathcal{H}_d$ because the maximal measure is disjoint from the Hénon-like dynamics.

Robust bifurcations from topology

Start with a "one-dimensional" mapping of the form

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Note that from the 1D theory $f \notin \mathring{\mathrm{Bif}}$.

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Suppose that p has the property that there exists a critical point c and $k \geq 1$ s.t. $p^k(c) \in E$, where E is a basic repeller for p which disconnects the plane (and $p^j(c) \notin \mathrm{Crit}(p)$ for $0 < j < k$)

Then f is accumulated by robustly bifurcating parameters in $\mathcal{H}_d$.

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Taking a small perturbation of p the assumption of the theorem is satisfied with $E = \partial A$.

Then there exists $\varepsilon_j \to 0$ such that

$$(p(z) + \varepsilon_j(w-1), w^d) \in \mathring{\mathrm{Bif}}.$$

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Theorem

Let $f(z,w) = (p(z), w^d + \kappa)$. Assume there exists $c \in \mathrm{Crit}(p)$ such that $p(c)$ is a repelling fixed point with

$$1 < \left|p'(p(c))\right| < 1.01$$

Then if κ is large enough, f is accumulated by robust bifurcations : $f \in \overline{\mathring{\mathrm{Bif}}}$.

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The mechanism underlying the proof is based on the idea of blenders (Bonatti-Diaz, etc.).

Note : again the assumption requires $d \geq 3$.

Perspectives

The previous construction is reminiscent from :

Theorem (Shishikura)

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ and assume $f \in \mathrm{Bif}$. Then there exists $f_j \to f$ such that

$$\text{hyp-dim}(f_j) \to 2.$$

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Question : Assume $f(z,w) = (p(z), q(w)) \in \mathrm{Bif}$. Does there exist $\mathcal{H}_d \ni f_j \to f$ possessing (repelling) blenders ? Does $f \in \overline{\mathring{\mathrm{Bif}}}$?

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More generally : assume $f \in \mathrm{Bif}$ and $\text{hyp-dim}(f) > 2$. Does $f \in \overline{\mathring{\mathrm{Bif}}}$?

Perspectives

Special case : recall that a Lattès map is an endomorphism semiconjugate to a multiplication on a complex torus.

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "\times k"] \arrow[d, "\pi"'] & \mathbb{T}^2 \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, "f"'] & \mathbb{P}^2 \end{tikzcd}$$

In particular it admits basic repellers of dimension $\geq (4 - \varepsilon)$ for every $\varepsilon > 0$.

Question : Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a Lattès map. Does $f \in \overline{\mathring{\text{Bif}}}$?

Thanks !