Diabolical Entropy

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Parameter Problems in Analytic Dynamics

Topological Entropy and the Quadratic Family

$$f_a : x \mapsto x^2 + a, \qquad a \in \mathcal{A} = [-2, 1/4]$$

- $h(a) := \lim_{n\to\infty} \frac{1}{n} \log \# f_a^{-n}(0)$.
- $h(a)$ exists, $a \mapsto h(a)$ is continuous and monotone. [MS, DHS, MT, BvS]
- $\lambda(a) := \lim_{n\to\infty} \frac{1}{n} \log |Df_a^n(a)|$
- If $\lambda(a_0) < 0$, a_0 is hyperbolic, there is a periodic attractor, $a \mapsto h(a)$ is locally constant at a_0. [LPS]
- the hyperbolic set Hyp is open and dense. [GŚ, Ly]
- for almost every $a \in \mathcal{A}$, $\lambda(a)$ exists, $\lambda(a) \neq 0$. [AM,Ly]
- for pos. measure set of parameters, $\lambda(a) > 0$. [J, BC, AM]

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Regularity of Toplogical Entropy

$\mathrm{WR}(a) : \lim_{\delta \searrow 0} \liminf \frac{1}{n} \log \sum_{|f_a^j(a)|<\delta,\ j \leq n} \log |f^j(a)| = 0$

- Tsujii's weak regularity condition : "does not return too close, too soon, too often"
- $\mathcal{W} := \{a : \lambda(a)$ exists and $\lambda(a) > 0$ and $\mathrm{WR}(a)\}$
- full measure in Hyp^c [AM,Ly,L,T]
- $\mathcal{A}_{\mathrm{NLC}} = \{a : \{a\} = h^{-1}(h(a))\}$ – positive measure set

Theorem (D, Mihalache)

Suppose $a \in \mathcal{W}$ *and* $\{a\} = h^{-1}(h(a))$. *Then*

$$\lim_{t \to 0} \frac{\log |h(a+t) - h(a)|}{\log t} = \frac{h(a)}{\lambda(a)}.$$

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Visible measures of maximal entropy

μ_{acip} : absolutely continuous invariant probability measure
$\mu_{\max}$: measure of maximal entropy
Lyapunov exponent $\chi(\mu) := \int \log |Df| d\mu$.

Theorem (D, Mihalache)

Let g be a real-analytic unimodal map with non-degenerate critical point. Then $\mu_{\max} = \mu_{\text{acip}}$ if and only if g is analytically conjugate to $x \mapsto x^2 - 2$.

[Shub-Sullivan, Martens de Melo] Expanding maps : abs cns conjugacy upgrades to smooth/analytic conjugacy.
[D] expanding induced map : upgrades to smooth conjugacy on an interval... implies pre-Chebyshev.
Analytic conjugacy between renormalised map and $x \mapsto x^2 - 2$, contradiction if renormalised (too many critical points).

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Escalier du diable

Corollary

$a \neq -2$ implies

$$\frac{h(a)}{\chi(\mu^a_{\mathrm{acip}})} > 1, \qquad \frac{h(a)}{\chi(\mu^a_{\max})} < 1.$$

For $a \neq -2$

$$\chi(\mu_{\mathrm{acip}}) = h(\mu^a_{\mathrm{acip}}) < h(\mu^a_{\max}) = h(a) < \chi(\mu^a_{\max}).$$

Corollary

Moreover, $h'(a) = 0$ almost everywhere.

For almost every $a \in \mathcal{W}$, $\quad \lambda(a) = \chi(\mu^a_{\mathrm{acip}}) = h(\mu^a_{\mathrm{acip}})$ [AM].
Thus, almost everywhere, Hölder exponent > 1.
Uniformity ?

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Uniformity

Theorem (Misiurewicz Szlenk, Raith, D Todd, D Mihalache)

$s \mapsto g_s$ continuous family of S-unimodal maps, each with positive topological entropy. Then

$$s \mapsto h_{\text{top}}(g_s), \quad s \mapsto \mu_{\max}^{g_s}, \quad s \mapsto \chi(\mu_{\max}^{g_s})$$

are continuous.

Pressure $P_s(t) = \sup_\mu h(\mu) - t \int \log |Dg_s| d\mu$

- pressure is analytic on a nbd of zero [based on DT]
- pressure functions converge on a nbd of zero [DT]
- slope of pressure at zero is $-\chi(\mu_{\max}^{g_s})$
- Therefore $s \mapsto \chi(\mu_{\max}^{g_s})$ is continuous.

Uniformity II

Lemma

If $g_k \to g_0$, S-unimodal, and $h_{\rm top}(g_k)/\chi(\mu_{\rm acip}^{g_k}) \to 1$.
Then $\mu_{\max}^{g_0} = \mu_{\rm acip}^{g_0}$.

Lemma

In a neighbourhood of a_F (Feigenbaum), there exists $\varepsilon > 0$ with

$$\frac{h(a)}{\chi(\mu_{\max}^a)} < 1 - \varepsilon, \quad \frac{h(a)}{\chi(\mu_{\rm acip}^a)} > 1 + \varepsilon.$$

.

- Take a sequence a_n converging to a_F, f_{a_n} is $(m_n + 1)$ times Feigenbaum renormalisable.
- subsequence of (rescaled) m_n-renormalised maps converge to some S-unimodal map g [Sullivan]
- by 2nd Theorem, $\mu_{\max} \neq \mu_{\rm acip}$.

Uniformity III

Summing up :

Theorem

Given $\varepsilon > 0$*, there exists* $\delta > 0$ *for which*

- *for all* $a \in (-2+\varepsilon, a_F)$*, if* μ^a_{acip} *exists then*

$$h(a)/\chi(\mu^a_{\text{acip}}) > 1+\delta$$

- *for all* $a \in (-2+\varepsilon, a_F)$*,*

$$h(a)/\chi(\mu^a_{\max}) < 1-\delta.$$

Recall first theorem : Suppose $a \in \mathcal{W}$ and $\{a\} = h^{-1}(h(a))$. Then

$$\lim_{t\to 0} \frac{\log|h(a+t)-h(a)|}{\log t} = \frac{h(a)}{\lambda(a)}.$$

Uniformity and Dimension

$$X_\varepsilon := \{a \in \mathcal{A}_{\mathrm{NLC}} \cap \mathcal{W} : a > -2+\varepsilon, \quad \lambda(a) = \chi(\mu^a_{\mathrm{acip}})\}$$
$$Y_\varepsilon := \{a \in \mathcal{A}_{\mathrm{NLC}} \cap \mathcal{W} : a > -2+\varepsilon, \quad \lambda(a) = \chi(\mu^a_{\max})\}.$$

$$\lim_{t\to 0} \frac{\log|h(a+t)-h(a)|}{\log t} = \frac{h(a)}{\lambda(a)}.$$

Theorem

$\dim_{\mathrm{H}}(h(X_\varepsilon)) < 1, \quad \dim_{\mathrm{H}}(Y_\varepsilon) < 1.$

- $\cup_\varepsilon X_\varepsilon$ has full measure in $\mathcal{A}_{\mathrm{NLC}}$ [Avila Moreira Lyubich Levin Tsujii...]
- $\cup_\varepsilon h(Y_\varepsilon)$ has full measure in $[0, \log 2]$. [Bruin Sands]

Proof of main theorem

$$\lim_{t\to 0} \frac{\log |h(a+t)-h(a)|}{\log t} = \frac{h(a)}{\lambda(a)}.$$

- h monotone, cns, suffices to prove for t_n with $\log t_n / \log t_{n+1}$ aribtrarily close to 1.
- Tent map $T_b : x \mapsto 1 - b|x|$, turning point at 0, entropy $\log b$
- $\xi_n(a) = f_a^n(a), \quad \phi_n(b) = T_b^n(1)$
- $\frac{1}{n}\log |D\xi_n(a+t)| \approx \lambda(a)$ for a subsequence of n, for a neighbourhood which gets mapped to the large scale
- $\frac{1}{n}\log |D\phi_n(b)| \approx \log b_0 = h(a)$ on corresponding nbds
- Use conjugacy with tent map to measure change of entropy
- $\log |t| \approx -n\lambda(a), \quad \log |h(a+t) - h(a)| \approx -nh(a)$

Tsujii's Lemma

Lemma

Suppose $a_0 \in \mathcal{W}$. *Let* $\delta > 0$. *There exist* $r_0 > 0, m \geq 1$, *a sequence* $(k_n)_{n\geq 0}$ *and decreasing neighbourhoods* $\omega_n \ni a_0$ *for which*

- $\frac{k_{n+1}}{k_n} \leq 1 + \delta$
- $\xi_{k_n}(\omega_n) \supset B(\xi_{k_n}(a_0), r_0)$
- ξ_j *has bounded distortion on* ω_n *for* $j = m, m+1, \ldots, k_n$
- $\frac{1}{k_n}|D\xi_{k_n}(a)| \approx \lambda(a)$.

Requires Weak Regularity, Transversality [L], Collet-Eckmann.

HAPPY BIRTHDAY Sebastian