Double Parabolic Renormalization in the Quadratic Family

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joint work with A. Epstein and C. Petersen

Quadratic rational maps

- rat_2 is the space of conjugacy classes of quadratic rational maps.
- the elementary symmetric functions of the multipliers of the fixed points are
  - $\sigma_1 := \mu_1 + \mu_2 + \mu_3$,
  - $\sigma_2 := \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$ and
  - $\sigma_3 := \mu_1\mu_2\mu_3$.

Proposition (Milnor)

- rat_2 *is isomorphic to* $\mathbb{C}^2$*.*
- σ_1 *and* σ_2 *provide global coordinates.*
- $\sigma_3 = \sigma_1 - 2$*.*

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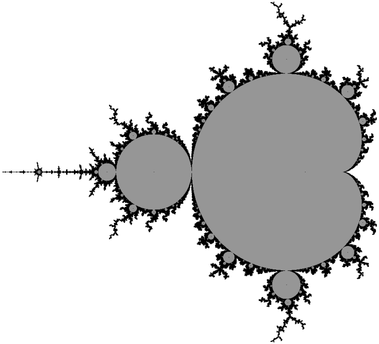
- $\mathrm{Per}_1(\mu) \subset \mathrm{rat}_2 = \big\{[f]$ having a fixed point with multiplier $\mu\big\}$.
- $\mathrm{Per}_1(\mu)$ is the line $\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_3 = 0$.

- $\mathrm{Bif}(\mu) := \overline{\{[f] \text{ having a cycle with multiplier } 1\}}$.



$\mathrm{Bif}(0)$

Bifurcation loci

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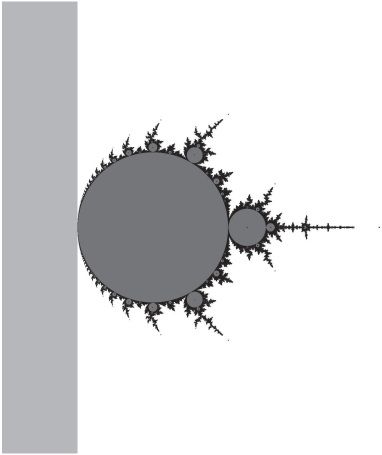


$\mathrm{Bif}(1)$

Germs with a parabolic fixed point

Assume

$$g(z) = \mathrm{e}^{2\pi\mathrm{i}\frac{p}{q}} z + \mathcal{O}(z^2).$$

Then

- $$g^{\circ q}(z) = z + Cz^{\nu q+1} + \mathcal{O}(z^{\nu q+2}) \text{ with } C \neq 0.$$
- g is formally conjugate to

$$\mathrm{e}^{2\pi\mathrm{i}\frac{p}{q}} z \cdot \left(1 + z^{\nu q} + \alpha z^{2\nu q}\right) \text{ with } \alpha \in \mathbb{C}.$$

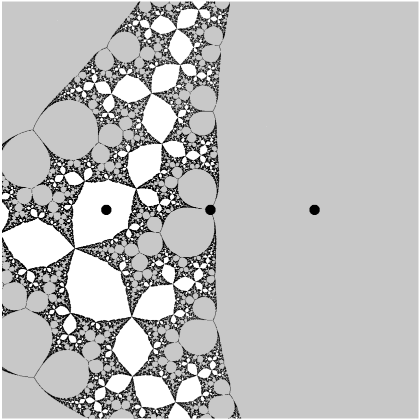
Definition

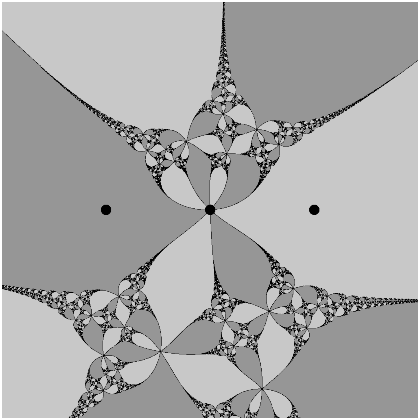
The résidu itératif of g is

$$\text{résit}(g) = \frac{\nu q + 1}{2} - \alpha.$$

Quadratic rational maps with parabolic fixed points

- $g_{p/q,a}(z) = \mathrm{e}^{2\pi\mathrm{i}\frac{p}{q}} \cdot \dfrac{z}{1 + az + z^2}$.
- $g_{p/q,a}$ and $g_{p/q,-a}$ are conjugate via $z \mapsto -z$.





Quadratic rational maps with parabolic fixed points

- $g_{p/q,a}^{\circ q}(z) = z + C_{p/q}(a)z^{q+1} + O(z^{q+2})$.
- Set $R_{p/q}(a) := \text{résit}(g_{p/q,a})$ when the parabolic point is not degenerate (i.e., $C_{p/q}(a) \neq 0$).

Theorem (B., Ecalle, Epstein)

For $q \geq 2$,

- *$C_{p/q}$ is a polynomial of degree $q-2$ having simple roots.*
- *$R_{p/q}$ is a rational map of degree $2q-2$ which only has double poles : infinity and the zeroes of $C_{p/q}$.*

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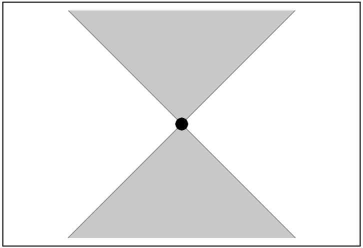
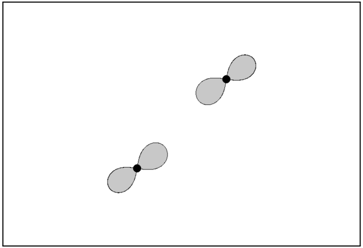
Question

How does $R_{p/q}$ depend on p/q?

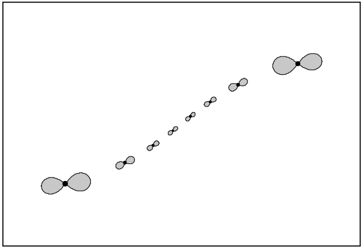
Parabolic degeneracy



$p/q = 1/1$

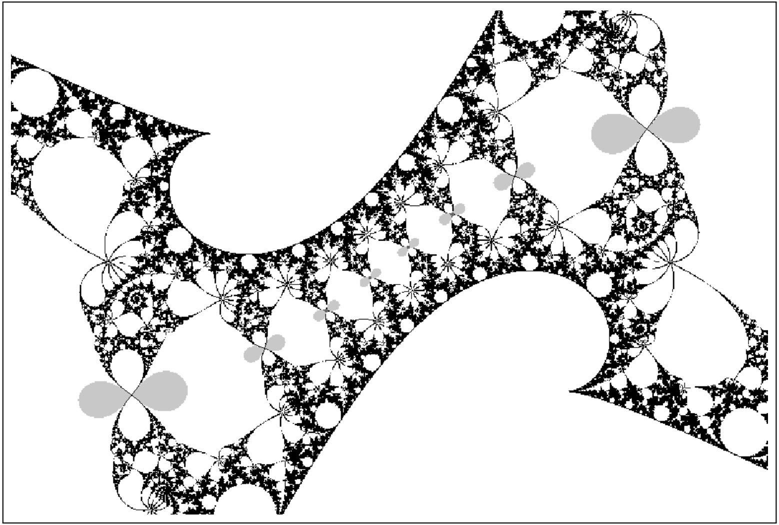


$p/q = 1/4$



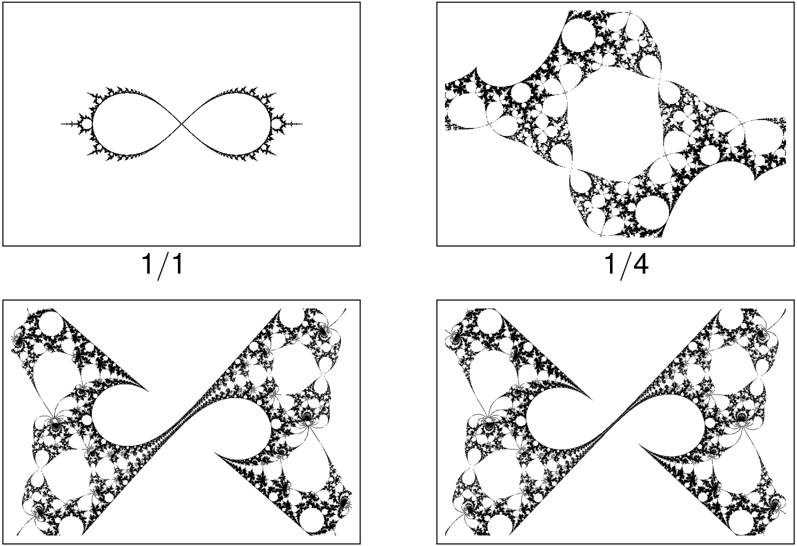
$p/q = 1/10$

Bifurcation locus



$p/q = 1/10$

Limit of bifurcation loci



1/1 1/4 1/50 1/100

Limit of résidus itératifs

- Consider $R_{p/q}$ as a function $R_{p/q} : \mathrm{Per}_1(e^{2\pi i p/q}) \to \widehat{\mathbb{C}}$.
- Given $r/s \in \mathbb{Q}$ and $k \in \mathbb{N}$, set $p_k/q_k := 1/(k + r/s)$.

Theorem (B., Ecalle, Epstein)

As $k \to +\infty$, the sequence of meromorphic functions

$$\left(\frac{p_k}{q_k}\right)^2 R_{p_k/q_k} : \mathrm{Per}_1(e^{2\pi i p/q}) \to \widehat{\mathbb{C}}$$

converges to a meromorphic transcendental function

$$\mathfrak{R}_{r/s} : \mathrm{Per}_1(1) \setminus \{[g_0]\} \to \widehat{\mathbb{C}}.$$

The function $\mathfrak{R}_{r/s}$ has a double pole at infinity, an essential singularity at $[g_0]$ and infinitely many poles (which are double poles) accumulating this singularity.

Fatou coordinates

- Set $b = 1/a^2$.
- The map

$$g_a(z) = \frac{z}{1 + az + z^2}$$

is conjugate via $Z = 1/(az)$ to

$$F_b(Z) = Z + 1 + \frac{b}{Z}.$$

- The sequence $\big(F_b^{\circ n}(Z) - n - b \log n\big)$ converges to an attracting Fatou coordinate Φ_b.
- The sequence $\big(F_b^{\circ n}(Z - n + b \log n)\big)$ converges to a repelling Fatou parametrization Ψ_b.

Horn maps

- The lifted horn map $\mathfrak{H}_b = \Phi_b \circ \Psi_b$ commutes with the translation by 1.
- It projects via $Z \mapsto z = \mathrm{e}^{2\pi \mathrm{i} Z}$ to a germ $\mathfrak{h}_b$ fixing 0 with multiplier $\mathrm{e}^{2\pi^2 b}$.
- The holomorphic map

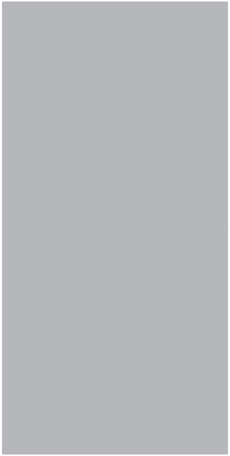
$$\mathfrak{f}_{r/s,b} = \mathrm{e}^{2\pi \mathrm{i} \frac{r}{s}} \cdot \mathrm{e}^{-2\pi^2 b} \cdot \mathfrak{h}_b$$

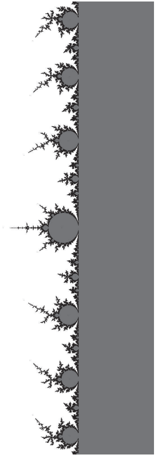
  fixes 0 with multiplier $\mathrm{e}^{2\pi \mathrm{i} \frac{r}{s}}$.
- $\mathfrak{R}_{r/s}(b)$ is the résidu itératif of $\mathfrak{f}_{r/s,b}$ when the parabolic point is not degenerate.

Where are the poles?

Proposition

The poles of $\mathfrak{R}_{r/s}$ *belong to the strip* $\{0 < \mathrm{Re}(b) < 1/2\}$.





Where are the poles?

Theorem (B., Epstein, Petersen)

For each r/s*, the poles of* $\mathfrak{R}_{r/s}$ *form* s *sequences* $(b_{j,n})$, $j \in [\![1,s]\!]$, *satisfying*

$$b_{j,n} = \frac{n}{2\pi \mathrm{i}} + \frac{1}{4} - \frac{\sigma_j}{2\pi \mathrm{i}} + \mathrm{o}(1), \quad \text{as } n \to +\infty,$$

where

- *the numbers* $\mu_j := \mathrm{e}^{2\pi \mathrm{i}\sigma_j}$ *are distinct,*
- $\prod \mu_j = (-1)^s$ *and*
- *the set* $\{\mu_j,\ j \in [\![1,s]\!]\}$ *is invariant by the map* $\mu \mapsto \mathrm{e}^{2\pi \mathrm{i} r/s}/\mu$.

Where are the poles?

Corollary

The poles of $\mathfrak{R}_{0/1}$ form a sequence (b_n) with the asymptotic behavior

$$b_n = \frac{n}{2\pi i} + \frac{1}{4} - \frac{1}{4\pi i} + o(1).$$

Corollary

The poles of $\mathfrak{R}_{1/2}$ form a sequence (b_n) with the asymptotic behavior

$$b_n = \frac{n}{4\pi i} + \frac{1}{4} - \frac{1}{8\pi i} + o(1).$$

Where are the poles?

Corollary

For r/s with s odd, among the poles of $\mathfrak{R}_{r/s}$, there is a sequence (b_n) which has the asymptotic behavior

$$b_n = \frac{n}{2\pi i} + \frac{1}{4} - \frac{1 - r + r/s}{4\pi i} + o(1).$$

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For r/s with s even, among the poles of $\mathfrak{R}_{r/s}$, there is a sequence (b_n) which has the asymptotic behavior

$$b_n = \frac{n}{4\pi i} + \frac{1}{4} - \frac{r/s}{4\pi i} + o(1).$$

Elements of the proof

- The proof consist in controlling the asymptotic behavior of the horn maps $\mathfrak{H}_b$ as $\mathrm{Im}(b) \to \pm\infty$.
- For $a = 0$, the map $g_0(z) = z/(1 + z^2)$ is semi-conjugate to g_2 via $z \mapsto z^2$. Note that $a = 2$ corresponds to $b = 1/4$.

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- For $a = 0$, the map $g_0(z) = z/(1 + z^2)$ is semi-conjugate to g_2 via $z \mapsto z^2$. Note that $a = 2$ corresponds to $b = 1/4$.
- Set

$$\mathfrak{F}_b := \mathfrak{H}_b + \pi \mathrm{i} b.$$

- Let $\tau : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the holomorphic map defined by

$$\tau(b) := \Phi_b(\sqrt{b}) - \Phi_{1/4}(1/2) - \mathrm{i}\pi(b - 1/4).$$

- Set

$$\Lambda(b) := 2\pi\mathrm{i}(b - 1/4).$$

Proposition

As $\mathrm{Im}(b) \to +\infty$*,*

$$T_{-\tau(b)} \circ \mathfrak{F}_b \circ T_{\tau(b)} \to \mathfrak{F}_{1/4}$$

locally uniformly in the upper half-plane.

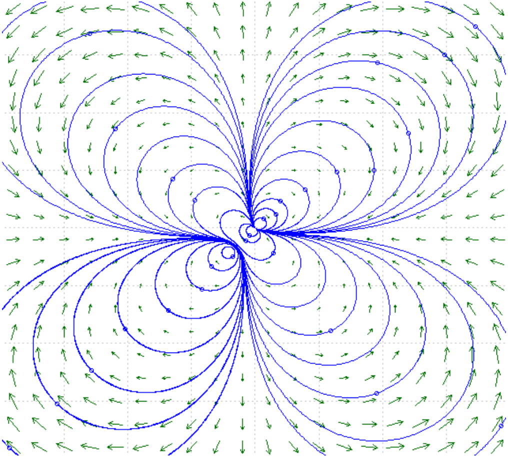
Proposition

As $\mathrm{Im}(b) \to -\infty$*,*

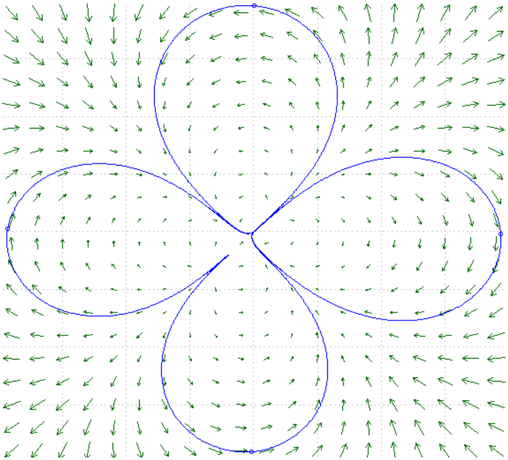
$$T_{-\tau(b)} \circ \mathfrak{F}_b \circ T_{\tau(b)} - T_{\Lambda(b)} \circ \mathfrak{F}_{1/4} \circ T_{-\Lambda(b)} \circ \mathfrak{F}_{1/4} \to 0$$

locally uniformly in the upper half-plane.

The dynamics when $\mathrm{Im}(b) \to -\infty$



Perturbed Fatou Coordinates



Elements of the proof

Corollary

As $\mathrm{Im}(b) \to +\infty$*, the following convergence holds locally uniformly in the unit disk:*

$$e^{-2\pi i\tau(b)}\mathfrak{g}_b\left(e^{2\pi i\tau(b)}z\right) \to \mathfrak{g}_{1/4}.$$

Corollary

As $\mathrm{Im}(b) \to +\infty$, $\mathfrak{R}_{r/s}(b) \to \mathfrak{R}_{r/s}(1/4)$.

Corollary

The entire map $\mathfrak{R}_{r/s}$ *has no poles with large positive imaginary part.*

- For $\mu \in \mathbb{C} \setminus \{0\}$, let $\mathcal{F}_\mu : \mathbb{D} \to \widehat{\mathbb{C}}$ be the *finite type analytic map* on $\widehat{\mathbb{C}}$ defined by

$$\mathcal{F}_\mu(z) := \frac{e^{2\pi r/s}}{\mu} \cdot \mathfrak{f}_{1/4}\big(\mu \cdot \mathfrak{f}_{1/4}(z)\big).$$

- Let $\mathcal{R} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the meromorphic transcendental function defined by

$$\mathcal{R}(\mu) := \text{résit}(\mathcal{F}_\mu)$$

when the parabolic point is not degenerate.

Elements of the proof

Let $\lambda : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be defined by

$$\lambda(b) := e^{2\pi i \Lambda(b)} = e^{4\pi^2(b-1/4)}.$$

Corollary

As $\mathrm{Im}(b) \to -\infty$, *the following convergence holds locally uniformly in the unit disk:*

$$e^{-2\pi i \tau(b)} \mathfrak{g}_b \left(e^{2\pi i \tau(b)} z\right) - \mathcal{F}_{\lambda(b)}(z) \to 0.$$

Corollary

As $\mathrm{Im}(b) \to -\infty$, $\mathfrak{R}_{r/s} - \mathcal{R} \circ \lambda \to 0$ *uniformly.*