Expanding Thurston maps

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Let S^2 be a topological 2-sphere. A map $f\colon S^2 \to S^2$ is a *branched covering map* iff

- it is continuous and orientation-preserving,
- near each point $p \in S^2$, it can be written in the form $z \mapsto z^d$, $d \in \mathbb{N}$, in suitable complex coordinates.

$d = \deg_f(p)$ *local degree* of f at p.

$C_f = \{p \in S^2 : \deg_f(p) \geq 2\}$ set of *critical points* of f.

Remark: Every rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere $\widehat{\mathbb{C}}$ is a branched covering map.

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The postcritical set

If $f\colon S^2 \to S^2$ is a branched covering map, then

$$P_f = \bigcup_{n\in\mathbb{N}} f^n(C_f)$$

is called the *postcritical set* of f. Here f^n is the nth-iterate of f.

Remarks: Points in P_f are obstructions to taking inverse branches of f^n. Each iterate f^n is a covering map over $S^2 \setminus P_f$.

Thurston maps

A map $f\colon S^2 \to S^2$ is called a *Thurston map* iff

- it is a branched covering map,
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Different viewpoints on Thurston maps:

- f well-defined only up to isotopy relative to P_f (one studies dynamics on isotopy classes of curves etc.), or
- f pointwise defined (one studies pointwise dynamics under iteration etc.).

Often one wants to find a "good representative" f in a given isotopy class.

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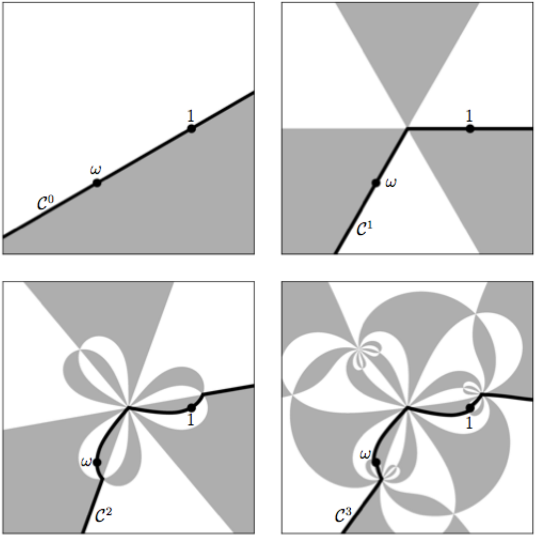
Thurston map g

$$g(z) = 1 + \frac{\omega - 1}{z^3}, \qquad \omega = e^{4\pi i/3}.$$

- $C_g = \{0, \infty\}$.
- Critical portrait: $0 \mapsto \infty \mapsto 1 \mapsto \omega \mapsto \omega$.
- $P_g = \{1, \omega, \infty\}$,
- $J = \mathcal{C}^0 =$ line through 1, ω, ∞.

Tiles for the map g



Tiles up to level 3 for g.

Tiles

Let $n \in \mathbb{N}_0$, $f: S^2 \to S^2$ be a Thurston map, and $J \subseteq S^2$ be a Jordan curve with $P_f \subseteq J$. Then a *tile of level n* or *n-tile* is the closure of a complementary component of $f^{-n}(J)$.

- tiles are topological 2-cells (=closed Jordan regions),
- tiles of a given level n form a cell decomposition $\mathcal{D}^n$ of S^2.
- the cell decompositions $\mathcal{D}^n$ for different levels n are usually not compatible.
- they are compatible (i.e., $\mathcal{D}^{n+1}$ is refinement of $\mathcal{D}^n$ for all $n \in \mathbb{N}_0$) iff $J \subseteq f^{-1}(J)$ equiv. $f(J) \subseteq J$ (i.e., J is f-invariant).

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Expanding Thurston maps

A Thurston map $f\colon S^2 \to S^2$ is *expanding* if the size of n-tiles goes to 0 uniformly as $n \to \infty$; so we require

$$\lim_{n\to\infty} \max_{n\text{-tile}\, X^n} \operatorname{diam}(X^n) = 0.$$

This is:

- independent of Jordan curve J,
- independent of the underlying base metric on S^2.

Remark: A rational Thurston map R is expanding iff R has no periodic critical points iff $\mathcal{J}(R) = \widehat{\mathbb{C}}$ for its Julia set.

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Invariant curves I

Problem. Let f be an expanding Thurston map. Does there exist an f-invariant Jordan curve J with $P_f \subseteq J$?

Answer: No, in general!

Example:

$$f(z) = i\frac{z^4 - i}{z^4 + i}, \quad P_f = \{-i, 1, i\}.$$

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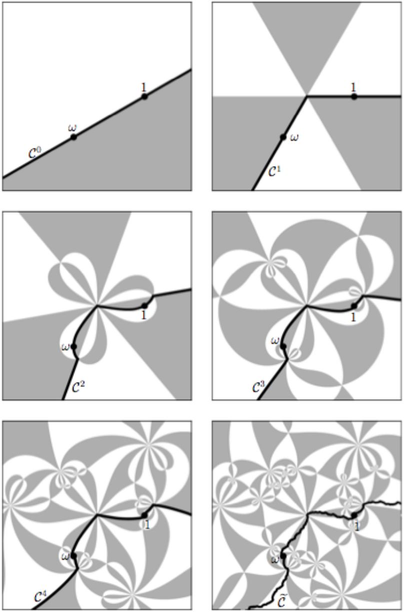
Iterative construction of invariant curve for g



Invariant curves II

Theorem. (B.-Meyer, Cannon-Floyd-Parry) Let f be an expanding Thurston map. Then for each sufficiently high iterate f^n there exists a (forward-)invariant quasicircle $\mathcal{C} \subseteq S^2$ with $P_f = P_{f^n} \subseteq \mathcal{C}$.

Corollary. (B.-Meyer, Cannon-Floyd-Parry) Let f be an expanding Thurston map. Then every sufficiently high iterate f^n is described by a subdivision rule.

Remark: If $J \subseteq S^2$ is an arbitrary Jordan curve with $P_f \subseteq J$, then there exists n, and a quasicircle $\mathcal{C}$ isotopic to J rel. P_f s.t. $f^n(\mathcal{C}) \subseteq \mathcal{C}$.

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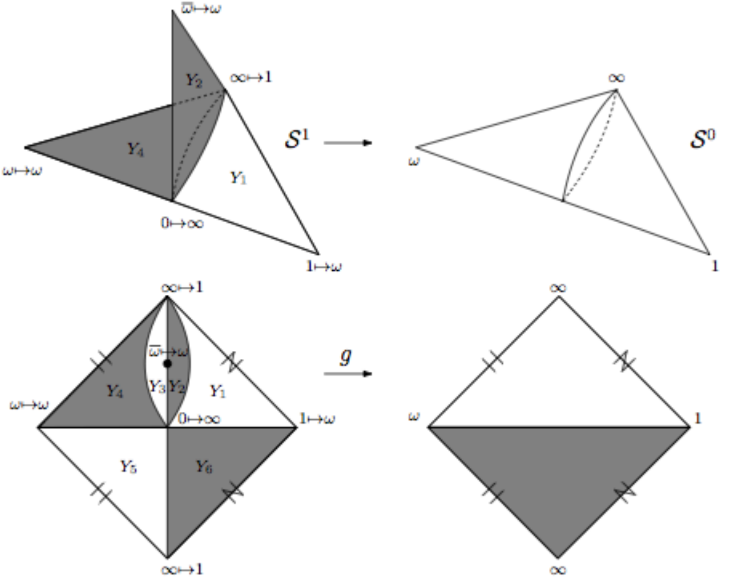
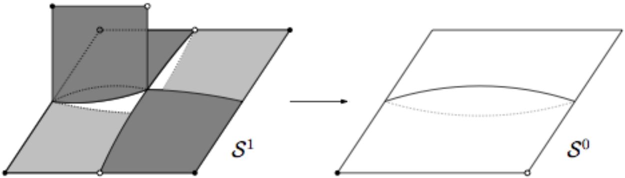
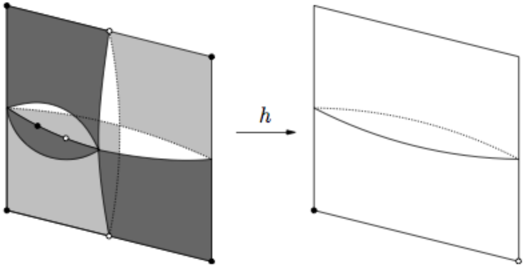
Subdivision rule for g



Thurston map h





Thurston map h

- $\#C_h = 6$,
- $\#P_h = 4$,
- Map h is described by a *two-tile subdivision rule*: Combinatorial data specifying how the two level-0 tiles are subdivided by 6 and 4 level-1 tiles, respectively.

A basic problem

When is an expanding Thurston map f conjugate to a rational map? So when is there a homeomorphism $\phi\colon S^2 \to \widehat{\mathbb{C}}$ and a rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ s.t.

$$\begin{array}{ccc} S^2 & \xleftrightarrow{\phi} & \widehat{\mathbb{C}} \\ \downarrow f & & \downarrow R \\ S^2 & \xleftrightarrow{\phi} & \widehat{\mathbb{C}} \end{array}$$

Remark: The map h in the previous example is not conjugate to a rational map.

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Proposition. Let f be an expanding Thurston map. Then there exists a *visual metric* ϱ on S^2 (unique up to snowflake equivalence) s.t. for all n-tiles X^n,

$$\varrho\text{-diam}(X^n) \simeq \Lambda^{-n},$$

where $\Lambda > 1$.

Two metrics ϱ_1 and ϱ_2 are *snowflake equivalent* iff there ex. $\alpha > 0$ s.t.

$$\varrho_1 \simeq {\varrho_2}^{\alpha}.$$

Definition: The *visual sphere* of f is (S^2, ϱ), where ϱ is a visual metric for f.

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The visual sphere of a Thurston map

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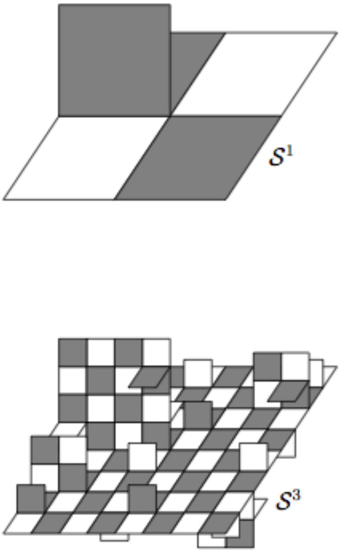
The visual sphere of h



Theorem. (B.-Meyer, Pilgrim-Haïssinsky)
Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and (S^2, ϱ) the visual sphere of f.
Then f is conjugate to a rational map if and only if f has no periodic crititical points and (S^2, ϱ) is quasisymmetrically equivalent to the standard sphere 2-sphere, i.e., $\widehat{\mathbb{C}}$ equipped with the chordal metric.

Quasisymmetric maps

A homeomorphism $f\colon X \to Y$ between metric spaces is *(weakly-) quasisymmetric* (=qs) if there exists $H \geq 1$ s.t.

$$|x-y| \leq |x-z| \Rightarrow |f(x)-f(y)| \leq H|f(x)-f(z)|$$

for all $x, y, z \in X$.

- f is quasisymmetric if it maps balls to "roundish" sets of uniformly controlled eccentricity.
- Quasisymmetry global version of quasiconformality.
- bi-Lipschitz $\Rightarrow$ qs $\Rightarrow$ qc.
- In $\mathbb{R}^n$, $n \geq 2$: qs $\Leftrightarrow$ qc.
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Cannon's conjecture

Version I: Suppose G is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. Then G admits an action on hyperbolic 3-space $\mathbb{H}^3$ that is discrete, cocompact, and isometric.

If true, the conjecture would give a characterization of fundamental groups $\pi_1(M)$ of closed hyperbolic 3-manifolds M from the point of view of geometric group theory.

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This is equivalent to:

Version II: Suppose G is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. Then $\partial_\infty G$ is qs-equivalent to $\mathbb{S}^2$.

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The quasisymmetric uniformization problem

Suppose X is a metric space homeomorphic to a "standard" metric space Y. When is X qs-equivalent to Y?

- Precise meaning of "standard" metric space depends on context.
- Examples: $Y = \mathbb{R}^n$, $\mathbb{S}^n$, standard $1/3$-Cantor set C, etc.
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Further directions

- What are the special properties of subdivison rules associated with rational Thurston maps?
- Can one reprove Thurston's characterization of rational maps using the combinatorial approach?
- An expanding Thurston map need not have an invariant Jordan curve containing the postcritical set P_f. Does there always exist an invariant graph $G \supseteq P_f$?
- Can one extend the theory of expanding Thurston maps to Thurston maps that are only expanding on their "Julia sets"? (Analog of subhyperbolic rational maps).

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