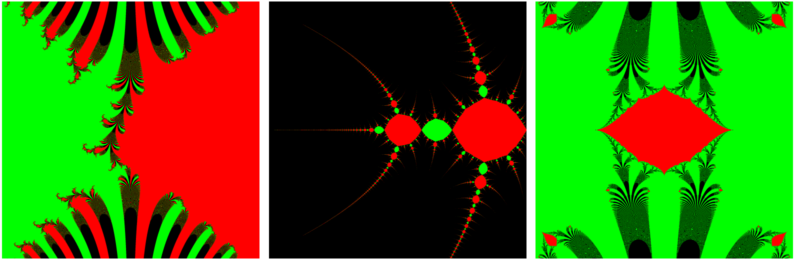
Lyapunov exponents and related concepts for entire functions

Walter Bergweiler

Christian-Albrechts-Universität zu Kiel

24098 Kiel, Germany

(joint work with Xiao Yao and Jianhua Zheng, Tsinghua University)



Parameter Problems in Analytic Dynamics

London, June 27 – July 1, 2016

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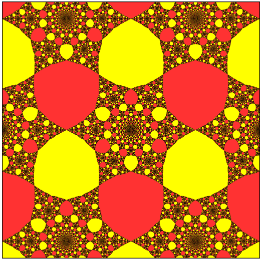

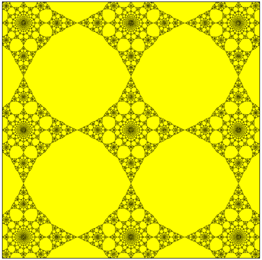

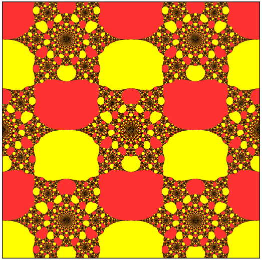

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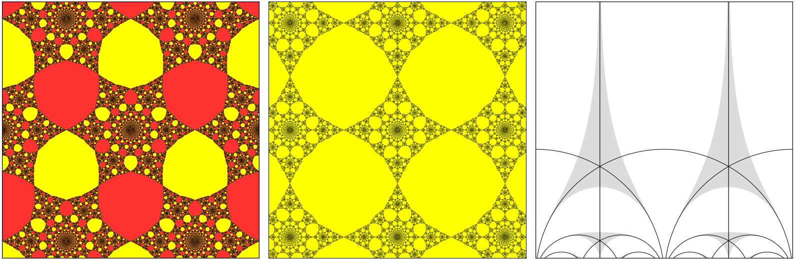

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It is more systematical to consider

$$f^*(z) = |f'(z)| \frac{1 + |z|^2}{1 + |f(z)|^2} = f^{\#}(z)(1 + |z|^2)$$

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$$\sup_{z\in\mathbb{C}} (f^n)^*(z)$$

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Barrett, Eremenko 2013: The constant $\dfrac{1}{2}$ is best possible here.

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Idea of proof: There exists a sequence (z_k) of periodic points with that $\chi(f, z_k) \to \infty$. Suppose for simplicity the z_k are fixed points.

Choose z near z_1 such that $f^n(z)$ stays near z_1 for a long time, then "jumps" close to z_2, stays there, jumps close to z_3, etc. $\square$

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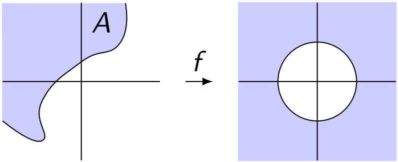

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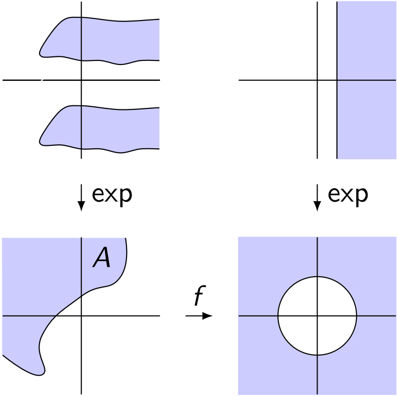

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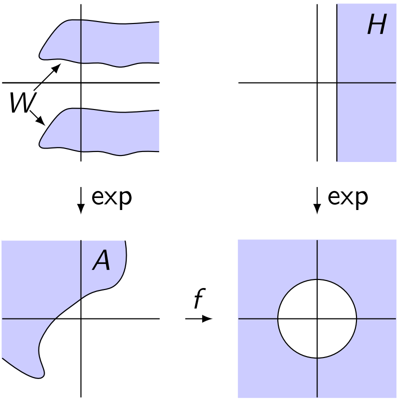

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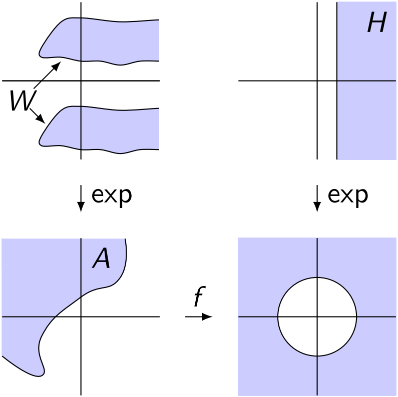

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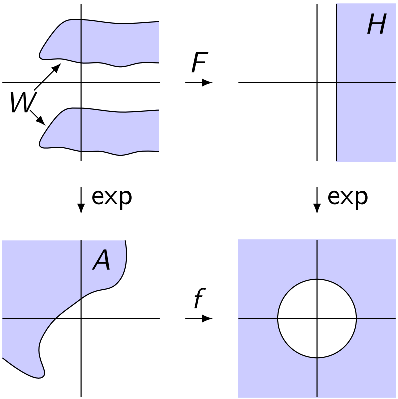

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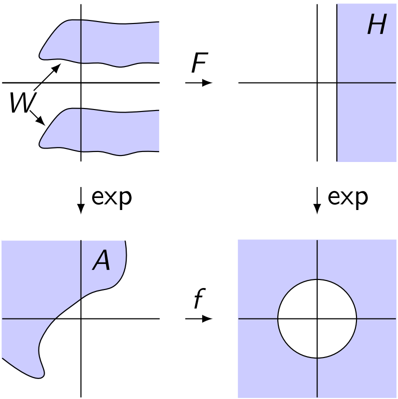

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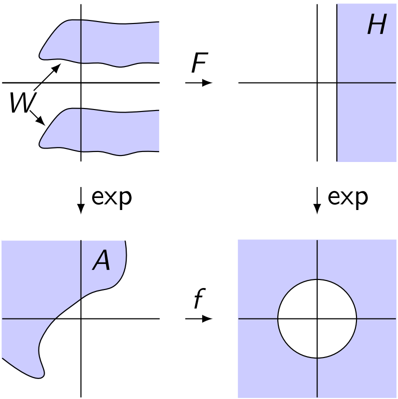

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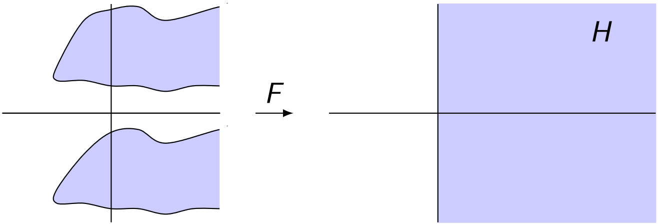

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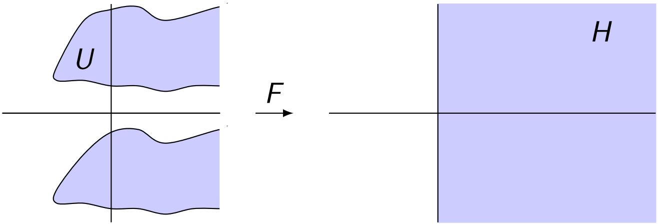

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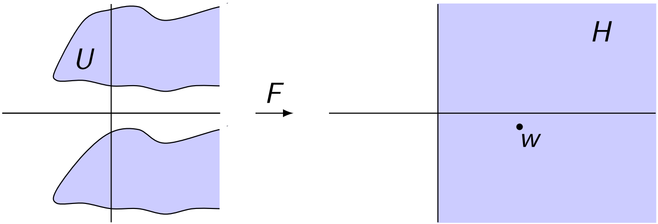

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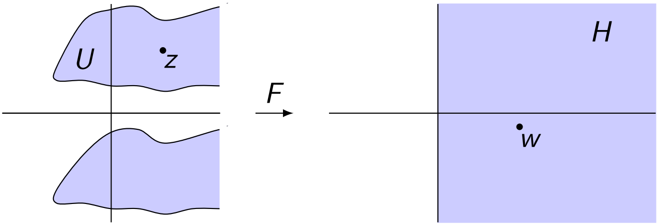

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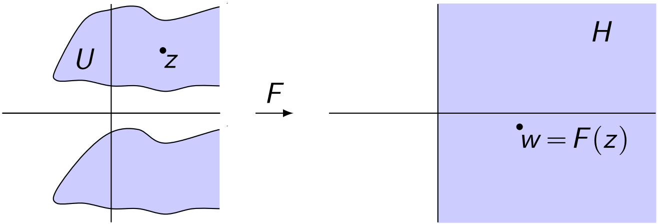

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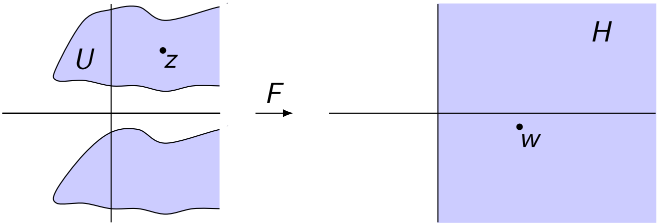

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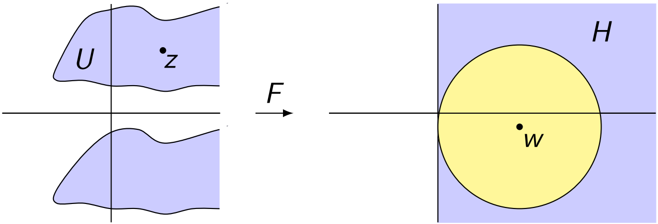

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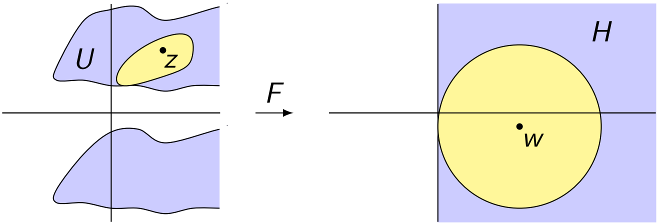

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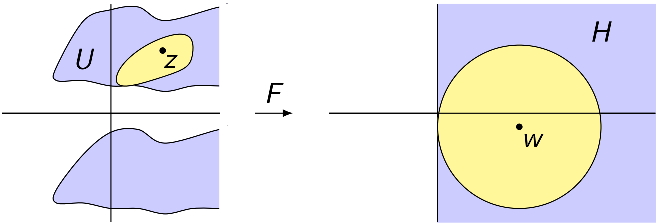

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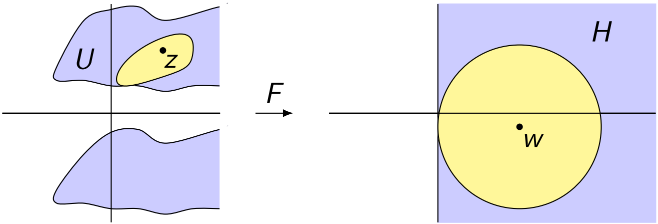

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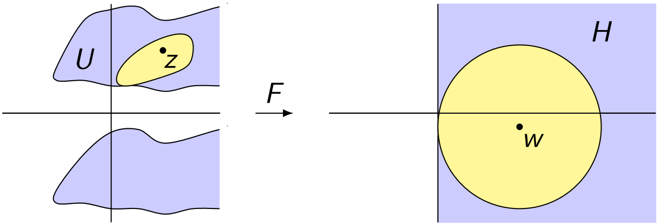

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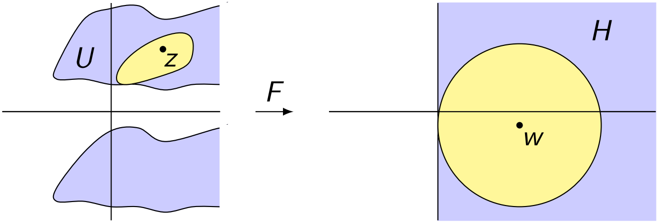

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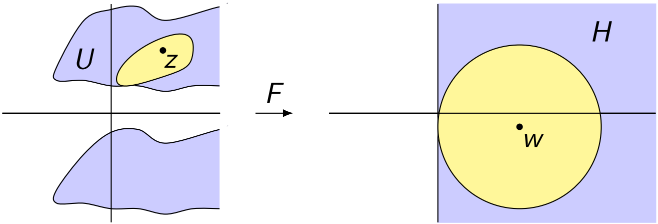

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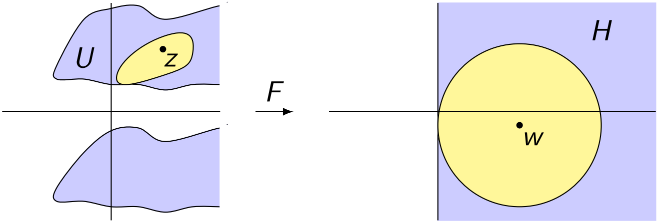

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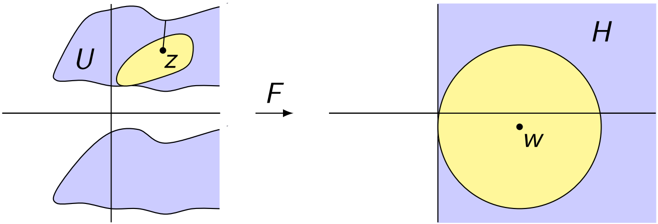

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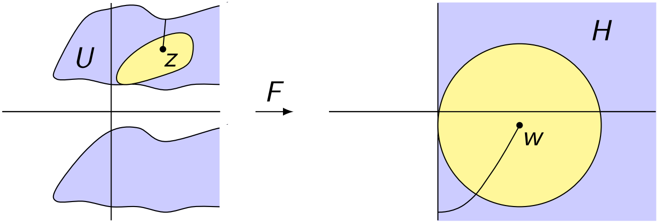

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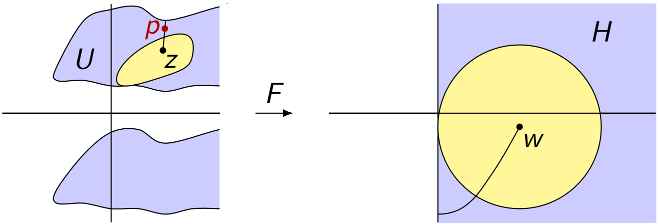

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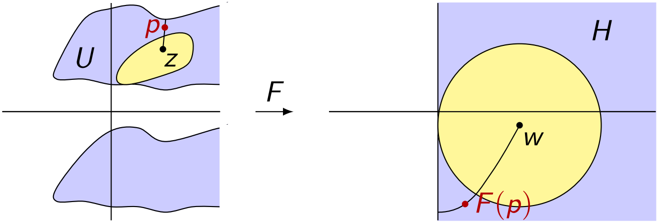

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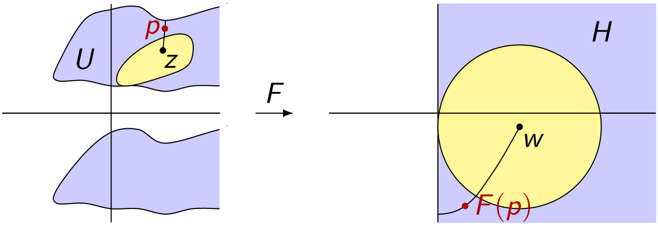

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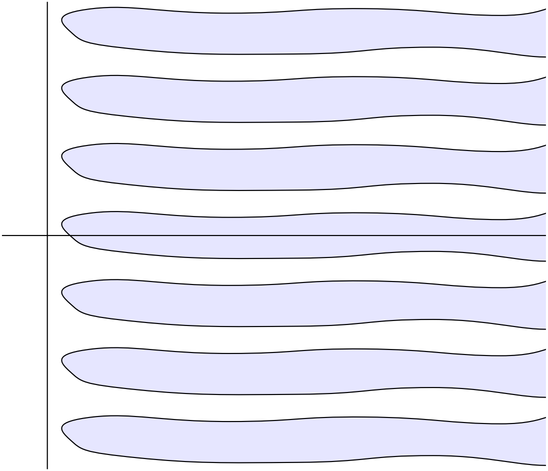

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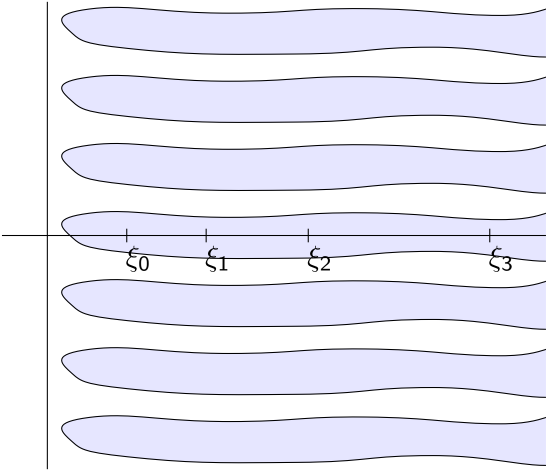

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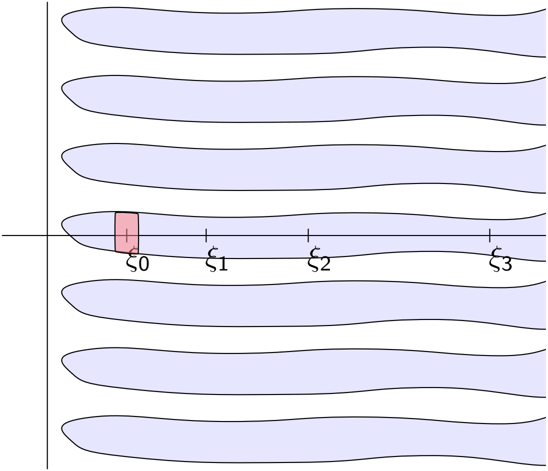

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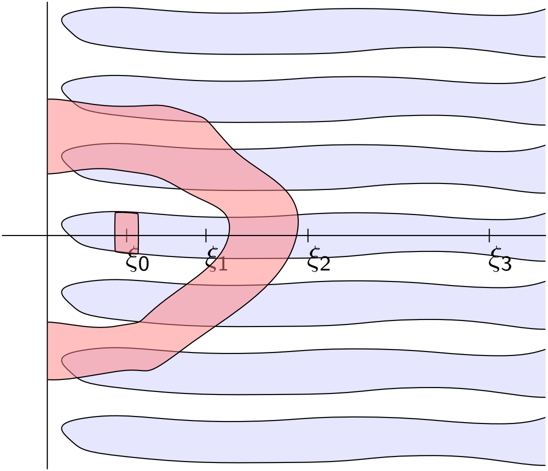

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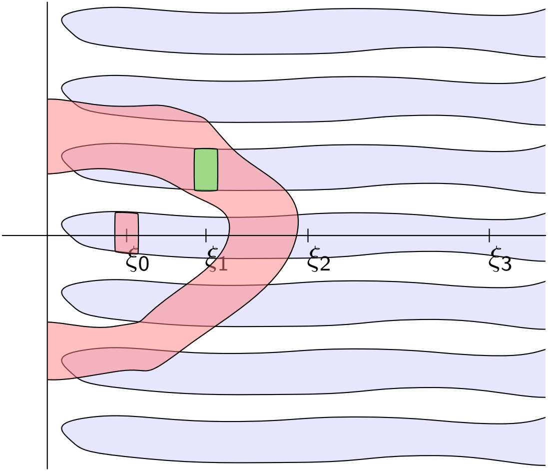

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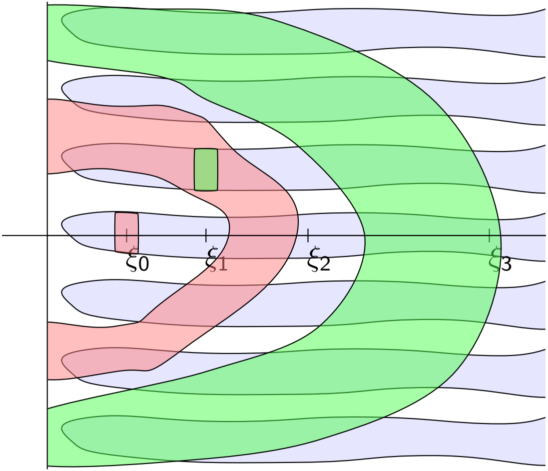

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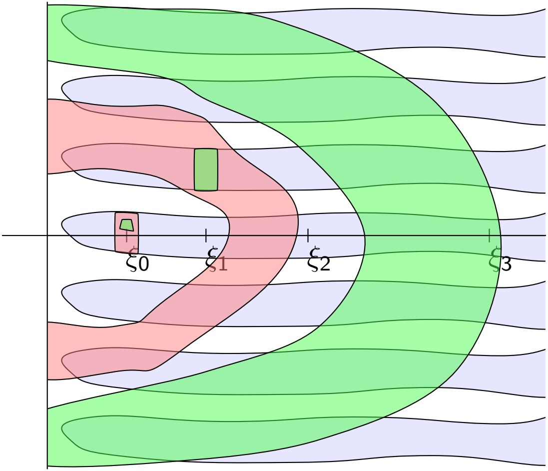

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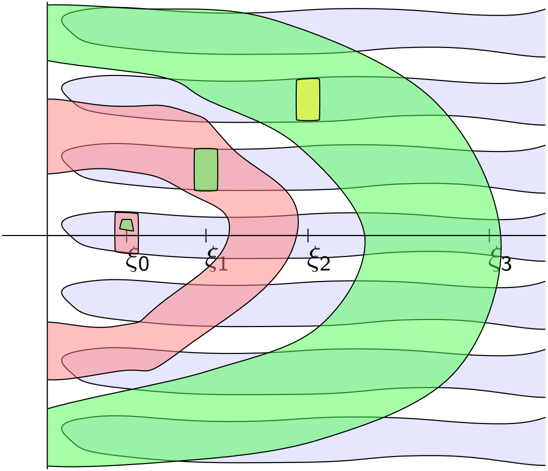

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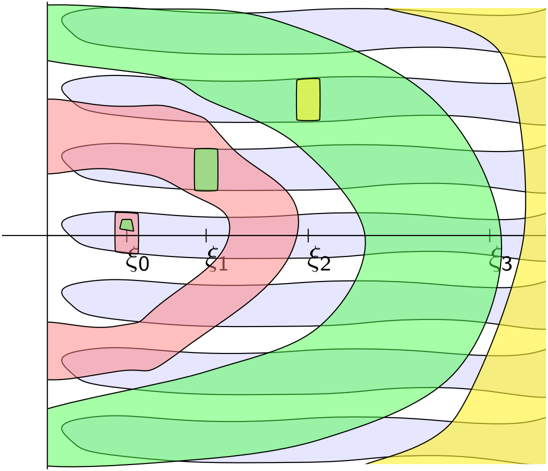

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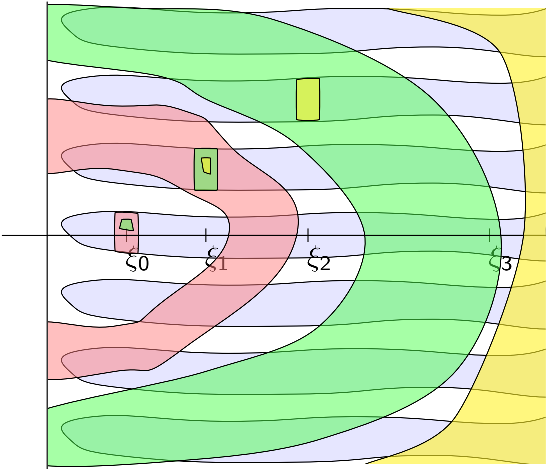

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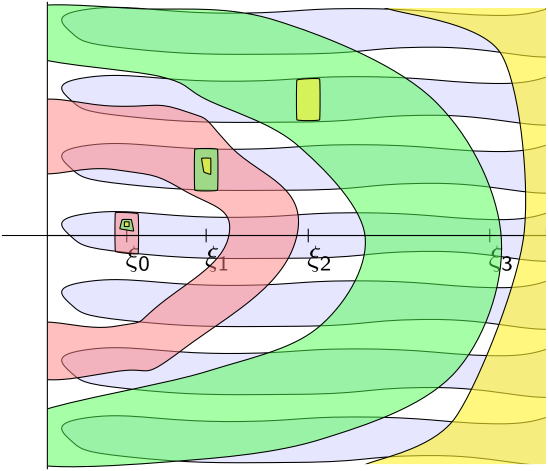

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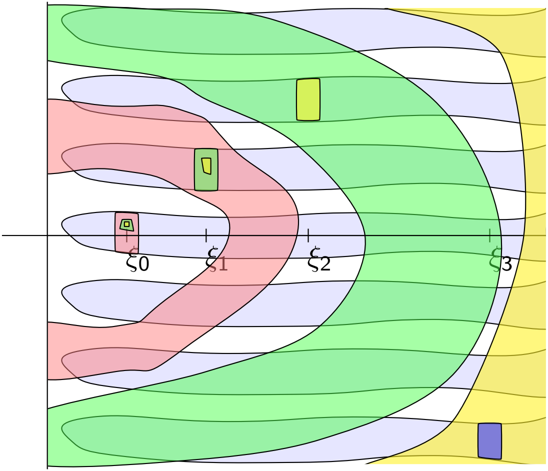

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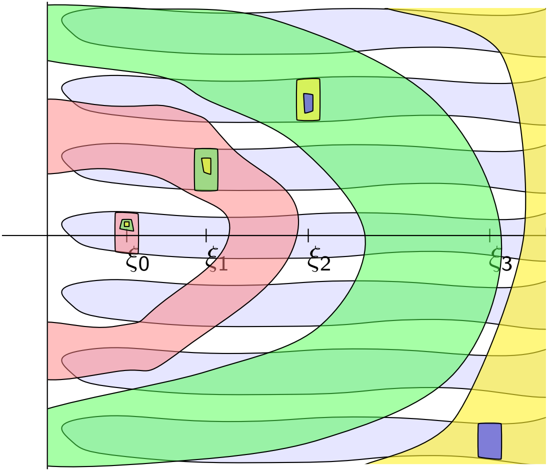

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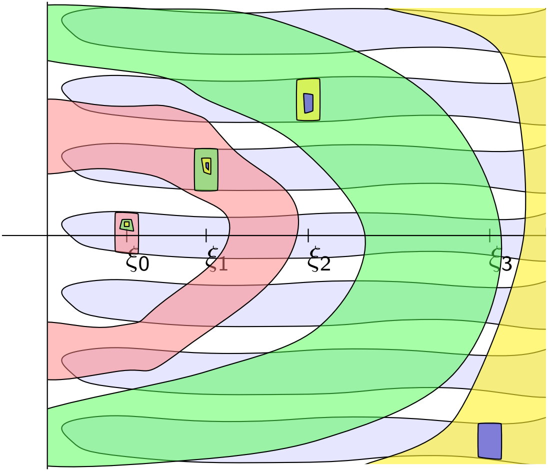

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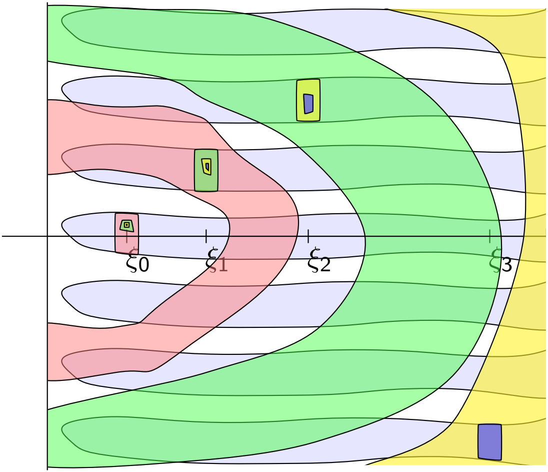

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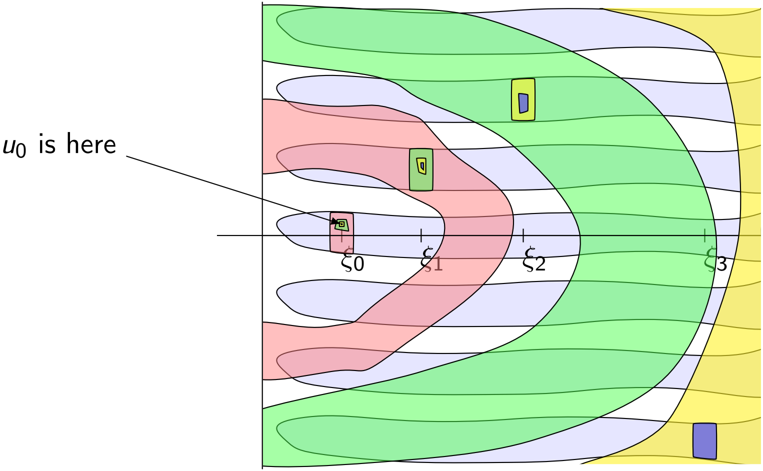

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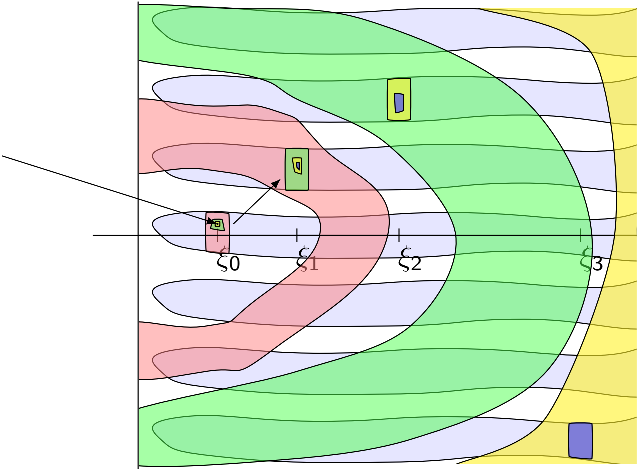

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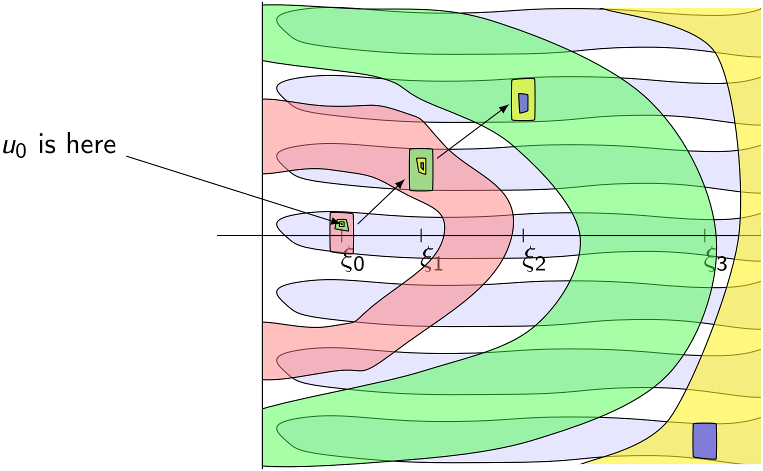

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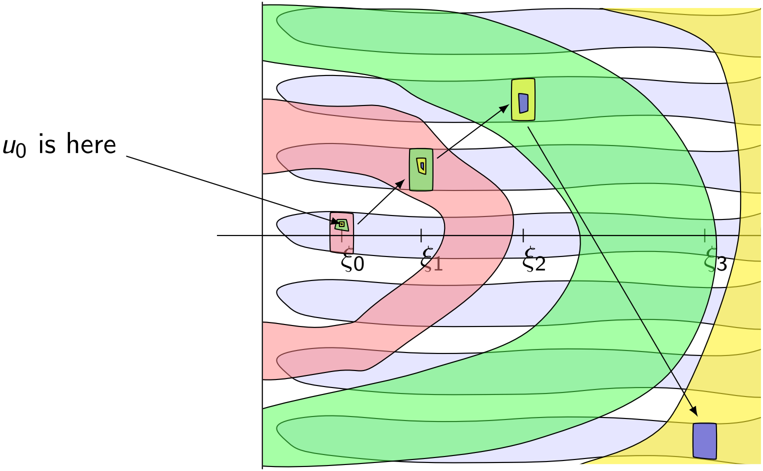

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$$|F'(u_k)| \gtrapprox \max_{\operatorname{Re} z = \operatorname{Re} u_k} \operatorname{Re} F(z) \gtrapprox e^{\beta \operatorname{Re} u_k} \approx \exp\left(\beta(1+\alpha)^k\right).$$

This implies that

$$|(F^n)'(u_0)| = \prod_{k=0}^{n-1} |F'(u_k)|$$

We want to find z_0 with

$$\liminf_{n\to\infty} \frac{1}{n} \log\log (f^n)^{\#}(z_0) \geqslant \log(1+\lambda(f)).$$

The example $f(z) = e^z$ suggests to choose $z_0 = e^{u_0}$ with $\operatorname{Re} F^n(u_0) \approx (1+\lambda(f))^n$. We will actually choose

$$\operatorname{Re} F^n(u_0) \approx (1+\alpha)^n \quad \text{with } \alpha < \lambda(f).$$

Let $\alpha < \beta < \lambda(f)$. Then $\log M(r, f) \geqslant r^{\beta}$, and this translates to

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This yields

$$(f^n)^{\#}(z_0) \approx \frac{|(f^n)'(z_0)|}{|f^n(z_0)|^2} \gtrapprox \exp\left(\left(\frac{\beta}{\alpha} - 1\right)(1+\alpha)^n\right).$$

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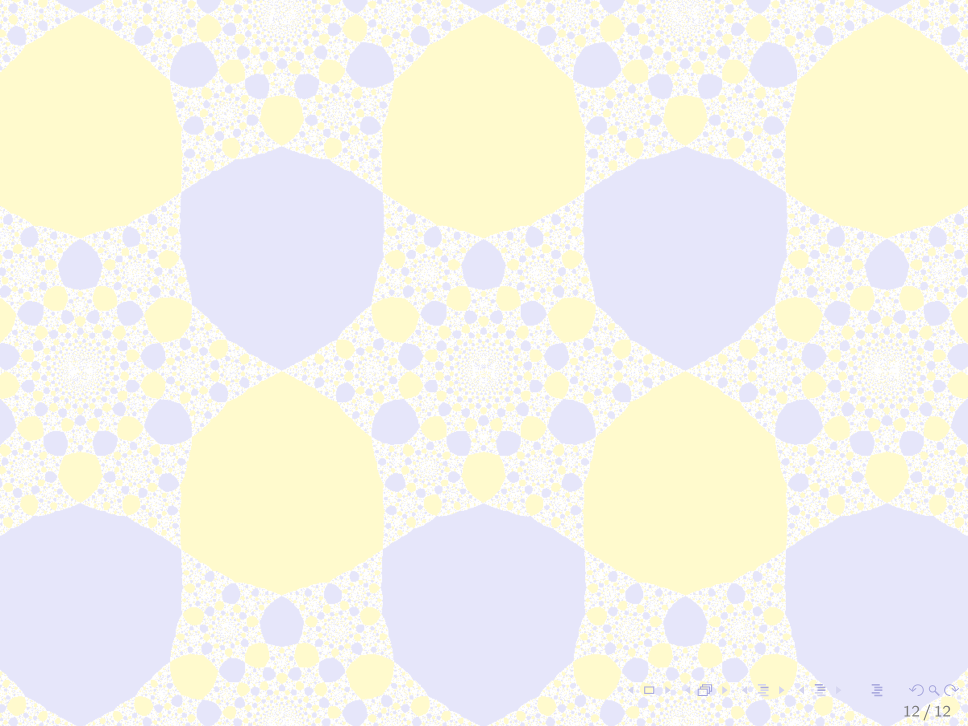
Actually will choose $\alpha = \alpha_n \to \lambda(f)$ and $\beta = \beta_n \to \lambda(f)$.



Thank you very much for your attention

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Gelukkige verjaardag,
Sebastian!