No Smooth Julia Sets for Complex Hénon Maps

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Dynamics of invertible polynomial maps of $\mathbb{C}^2$

If we want invertible polynomial maps, we must move to dimension 2.

One approach: Develop parallels between dynamics in dimensions 1 and 2.

Consider: $z \mapsto p(z)$ which can become invertible if we add another variable w:

$$(z, w) \mapsto (p(z) - w, z)$$

Starting with $(z, w) \mapsto (z, w)$ we may also construct

$$(z, w) \mapsto (z, w + p(z))$$

These maps behave very differently under iteration. How do we know what maps to study?

Another approach: Use Algebra
Jung's Theorem on the structure of $PolyAut(\mathbb{C}^2)$.

$\deg(x^j y^k) = j + k$, and $\deg((f_1, f_2)) = \max\{\deg(f_1), \deg(f_2)\}$.

$\deg(f)$ is only sub-multiplicative: $\deg(f \circ g) \leq \deg(f)\deg(g)$

$1 = \deg(f \circ f^{-1}) < \deg(f)\deg(f^{-1})$ unless f is linear

The *dynamical degree*

$$\mathrm{ddeg}(f) := \lim_{n\to\infty} \deg(f^n)^{1/n}$$

is invariant under conjugation. A *complex Hénon map* has the form

$$f(x, y) = (y, p(y) - \delta x)$$

with nonzero $\delta \in \mathbb{C}$ and $\deg(p) > 1$.

Theorem (Friedland-Milnor)

Complex Hénon maps minimize degree within their conjugacy classes. If $g \in PolyAut(\mathbb{C}^2)$ has $\mathrm{ddeg}(g) > 1$, then there are complex Hénon maps $f_1, \ldots, f_k$ such that g is conjugate to $f_1 \circ \cdots \circ f_k$.

Theorem (Friedland-Milnor)

Suppose that $f : \mathbb{C}^2 \to \mathbb{C}^2$ *is an invertible polynomial mapping. Then, modulo conjugacy by automorphisms,* f *is either:*

1. *affine or elementary:* $(x, y) \mapsto (\alpha x + \beta, \gamma y + p(x))$
2. *composition* $f = f_n \circ \cdots \circ f_1$*, where* f_j *is a generalized Hénon map* $f_j(x, y) = (y, p_j(y) - \delta_j x)$*, with* $d_j := deg(p_j) \geq 2$ *and nonzero* $\delta_j \in \mathbb{C}$

In case 1, the elementary maps preserve the set of vertical lines, and the dynamics is simple.
With f as above, we have $\mathrm{ddeg}(f) = \deg(f) = d_k \cdots d_1 = d$, and the complex Jacobian is $\delta = \delta_k \cdots \delta_1$.

Theorem (Friedland-Milnor, Smillie)

In case 2, the topological entropy is $\log(d)$.

We define the sets $K^+ = \{(x, y) \in \mathbb{C}^2 : \{f^n(x, y), n \geq 0\}$ is bounded$\}$ and $J^+ = \partial K^+$. (Similarly for K^- and J^-, replacing f by f^{-1}.)

J^+ is the set of points where the forward iterates are not locally normal. Equivalently, this the set where f is not Lyapunov stable in forward time.

In case 1 (affine or elementary map), J^+ is an algebraic set (possibly empty).

Theorem ([BS1], S=Smillie)

In Hénon case, if q is a saddle point, then $\overline{W^s(q)} = J^+$, i.e., J^+ is the closure of the stable manifold.

Remark

This is independent of the saddle point q, so all stable manifolds have the same closures.

Invitation to read the series [BS1–8]. If you have trouble finding them, send me email.

Let $p(z)$ be an expanding (hyperbolic) polynomial, and let $f(x, y) = (y, p(y) - \delta)$. For small δ, J^+ and J^- have laminar structure:

Theorem (Hubbard-ObersteVorth, Fornæss-Sibony)

If $|\delta| > 0$ is sufficiently small, then J^+ is laminated by Riemann surfaces, and the transversal slice looks locally like J_p. Further, J^- is laminated and transversal to J^+. J^- is locally the product of a disk and a Cantor set.

In general, a map f is *hyperbolic* if J is a hyperbolic set.

Theorem (BS1)

If f is a hyperbolic Hénon map, then there are at most finitely many sink orbits, and J^+ and J^- have laminar structure away from this finite set.

Problem

How can you recognize hyperbolicity in Hénon maps?

Especially in special cases?

J^+ can be a topological manifold of real dimension 3

Corollary

If f is in Case 2, then J^+ cannot be a manifold of real dimension 2.

Proof. If J^+ is a 2-manifold, it must be equal to $W^s(q)$. But there are more than one saddle point, so this is not possible.

For a polynomial $p(y)$ and small δ, define

$$f(x, y) = (y, p(y) - \delta x)$$

Theorem (Fornæss-Sibony, Hubbard-ObersteVorth)

Suppose that the Julia set $J_p \subset \mathbb{C}$ is a Jordan curve, and p is uniformly expanding on J_p. Then for sufficiently small $|\delta| > 0$, $J^+(f)$ is a 3-manifold.

Theorem (Radu-Tanase)

Similar result for quadratic, semi-parabolic maps.

Theorem (Fornæss-Sibony)

For generic h, the 3-manifold J^+ is not C^1 smooth.

What is the dynamical behavior on the Fatou set $\mathcal{F}^+$?

Jacobian$(f) = \det(Df) = \delta$ is a constant.

f is dissipative $\Leftrightarrow |\delta| < 1 \Leftrightarrow$ volume contracting

Dichotomy: dissipative vs. conservative

Problem

Can a dissipative map have a wandering Fatou component?

What about special maps? (hyperbolic case is known)

Theorem (Astorg-Buff-Dujardin-Peters-Raissy)

There is a (noninvertible) polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^2$ with a wandering Fatou component.

Remark

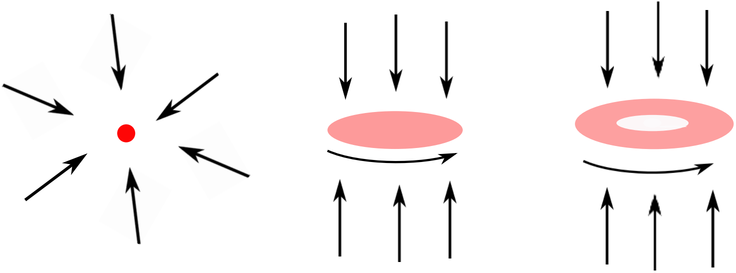
If a Hénon map has a parabolic fixed point, then it is conservative (not dissipative).

Invariant Fatou components: Dissipative case 1.

Suppose that Ω is a connected component of $\mathrm{int}(\mathcal{F}^+)$ and that $f(\Omega) = \Omega$.

Theorem (BS2)

Suppose that Ω is a recurrent Fatou component for a dissipative Hénon map. Then Ω must be one of three types of basin pictured. The basins are uniformized by $\mathbb{C}^2$, $\mathbb{C} \times \Delta$ and $\mathbb{C} \times A$, respectively.



Problem

Can the basin of the annulus actually occur?

Invariant Fatou components: Dissipative case 2.

Theorem (Lyubich-Peters)

Suppose that Ω is a non-recurrent Fatou component for a dissipative Hénon map. If $|\delta| < (deg(f))^{-2}$, then $\Omega = \mathcal{B}$ is the basin of a semi-parabolic fixed point, i.e., a fixed point with multipliers 1 and δ.

The structure of a map at a semi-parabolic fixed point has been described in detail by T. Ueda.

Theorem (Ueda, Hakim)

A semi-attracting basin is uniformized by $\mathbb{C}^2$. In fact $(f, \mathcal{B})$ is biholomorphically conjugate to $(T, \mathbb{C}^2)$, with $T(z, w) = (z + 1, w)$.

Problem

Can the dissipation condition be weakened to $|\delta| < 1$?

Invariant Fatou components: Conservative case 1

Theorem (Friedland-Milnor)

If $|\delta| = 1$, *then* $K = K^+ \cap K^- \subset \{|x|, |y| < R\}$.

Corollary

If Ω *is a component of int*(K), *then* Ω *is periodic, i.e.,* $f^p(\Omega) = \Omega$.

Corollary

In the conservative case, there are no wandering components.

Let $\Omega \subset \mathrm{int}(K) = \mathrm{int}(K^+) = \mathrm{int}(K^-)$ be fixed, i.e., $f(\Omega) = \Omega$.

Theorem (BS2)

$\mathcal{G}(\Omega) :=$ *limits of sequences* $f^{n_j}|_\Omega$ *is a (real) torus* $\mathbb{T}^\rho$ *with* $\rho = 1$ *or 2.*

Because of the torus action induced by f, we say that Ω is a *rotation domain*, and ρ is the *rank* of the domain.

Existence of Ω: Choose $L = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$, $|\mu_j| = 1$, suitable for linearization. If $f(p) = p$, $Df(p) = L$, then f can be linearized at p, and so there is a fixed component $\Omega \subset \text{int}(K)$.
Conversely, if Ω is a component of $\text{int}(K)$ with $f(\Omega) = \Omega$, and if there is a fixed point $p \in \Omega$, then f can be linearized in a neighborhood of p. We ask whether every component Ω must arise in this way (from a fixed point), or whether Ω can be like an annulus or something without fixed point? Simply:

Problem

Must there be a fixed point in Ω?

Problem

Is it possible that $\Omega = \text{int}(K)$? I.e., can the interior of K be connected?

Problem

What is Ω in terms of holomorphic uniformization? Can you show it is not (biholomorphically equivalent to) something familiar like the bidisk Δ^2 or the ball $\mathbb{B}^2$?

Let $U^+ := \mathbb{C}^2 - K^+$ be the points that escape to infinity in forward time. Then we also have $J^+ = \partial U^+$.

$$G^+ := \lim_{n\to\infty} \frac{1}{\deg^n} \log(||f^n|| + 1)$$

has the properties

$G^+ \circ f = \deg \cdot G^+$, G^+ is continuous and subharmonic on $\mathbb{C}^2$

$U^+ = \{G^+ > 0\}$, and G^+ is harmonic on U^+.

Fundamental currents $\mu^\pm := \frac{1}{2\pi} dd^c G^\pm$ $\quad J^\pm = \mathrm{supp}(\mu^\pm)$.

Let $\xi_q : \mathbb{C} \to W^u(q)$ be the uniformization of the unstable manifold with $\xi_q(0) = q$. It follows that

$$f \circ \xi_q(\zeta) = \xi_q(\beta_q \zeta)$$

and

$$G^+(\xi_q(\beta_q\zeta)) = \deg(f) \cdot G^+(\xi_q(\zeta))$$

How to see Hénon maps: the Hubbard picture

We may take a look at the sets J^+ which we will prove are not smooth.
Hubbard looked empirically at Hénon maps in terms of unstable slice pictures. The set $W^u(q) \cap K^+$ is invariant. This set may be displayed graphically by plotting level sets of $G^+ \circ \xi_p$ and its harmonic conjugate in the uniformizing coordinate $\zeta \in \mathbb{C}$. The gray/white shading gives the binary digits of G^+ and its harmonic conjugate.

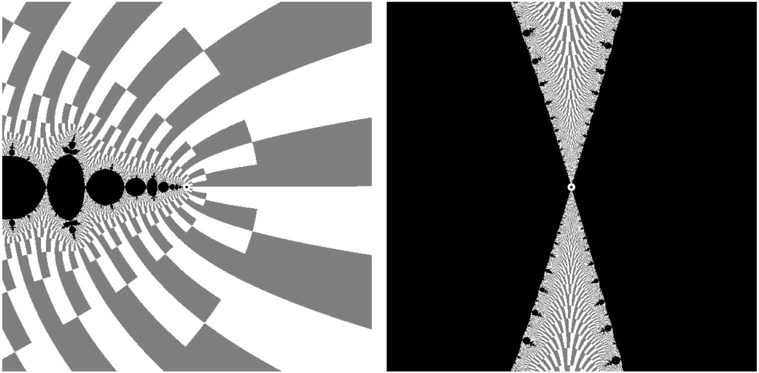
This produces self-similar picture (invariant under $\zeta \mapsto \beta_q \zeta$).

Several properties were suggested by looking at such pictures, and some of the corresponding Theorems were proved in [BS7].

There are infinitely many possible pictures – one for each saddle cycle, but all the pictures are closely related to each other. Zooming in closely at one of the pictures will reveal all of the other pictures.

Unstable slice pictures for the map $f(x,y) = (y, y^2 - 1.1 - .15x)$

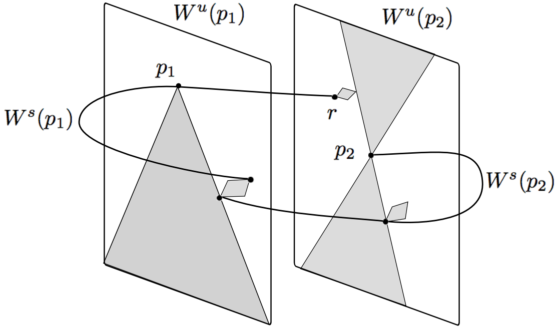
Self-similar picture with respect to the uniformizing parameter.
Gray/white regions give binary coding for G^+/ harmonic conjugate;
Black = K^+ (basin of attracting 2-cycle); boundary of black = J^+.



Unstable slices with centers (small dot) at the 2 fixed points:
Multipliers are ≈ 3.5 and ≈ -1.1

How unstable slices are connected by stable manifolds

Stylized picture shows stable manifolds $W^s(p_1)$ and $W^s(p_2)$. The transverse intersections $W^s(p_1) \cap W^u(p_2)$ are dense in $W^u(p_2) \cap J^+$. By Lambda Lemma at the saddle point p_2, the slice at the intersection point will look like the slice at p_2.



In the connected, dissipative, hyperbolic case, [BS7] gives converse.

Food for thought: two more unstable slices of the same map

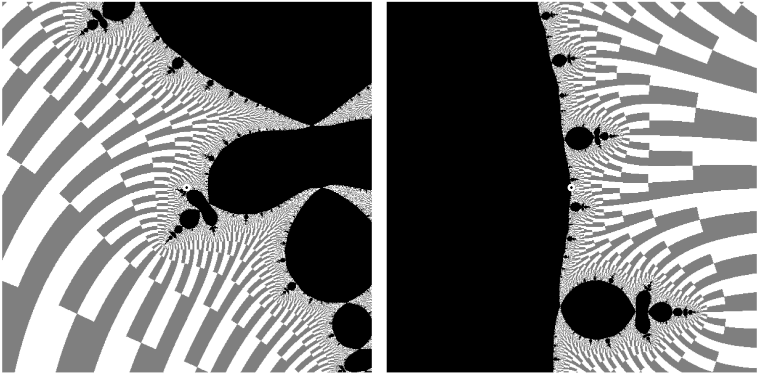


Image on the left: the saddle point has period 3 and multiplier $\sim 2.44918 + 4.43005i$. Since this multiplier is non-real, we see that the slice $W^u(p) \cap K^+$ spirals towards p. Complex conjugate also 3-cycle.

Image on the right: the saddle point has period 4 and multiplier ~ 6.26274. There is also a conjugate pair of (non-real) 4-cycles.

What can unstable slice pictures show? Basins

Suppose that p is an attracting fixed point for a dissipative map. We can find its basin $\mathcal{B}(p)$ without having to search the whole space $\mathbb{C}^2$. By [BS1–2], we know that $\mathcal{B}(p)\mathcal{W}^{\sqcap}(\text{II})$ will be an open subset of the unstable slice picture. In in fact, this will happen arbitrarily close to the origin.

The previous pictures showed the period 2 attracting basin (solid black) very prominently.

What can unstable slice pictures show? Connectivity

Theorem (BS6)

Suppose that f is dissipative, $|\delta| < 1$. Then the TFAE:

- *J is connected.*
- *K is connected.*
- *$\exists$ saddle point p: $W^u(p) \cap J^+$ is connected.*
- *$\forall$ saddle point p: $W^u(p) \cap J^+$ is connected.*

In drawing parallels between dimensions 1 and 2, we find that

$$\mathbb{C} - \overline{\Delta} \leftrightarrow \text{ the complex solenoid}$$

$$S^1 \leftrightarrow \text{the real solenoid } \Sigma_0$$

Theorem (BS7)

Let f dissipative and hyperbolic, and let J be connected. Then J^- is essentially a complex solenoid. The complex solenoid gives external rays, which land, and give J as a quotient of the (real) solenoid Σ_0.

Problem

What sorts of identifications can arise when we take the quotient of the real solenoid: $J \cong \Sigma_0 / \sim$?

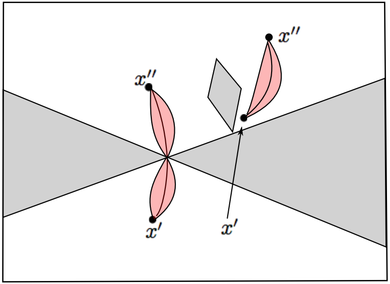
What can unstable slice pictures show? Hyperbolicity

Theorem (BS7)

If f is hyperbolic, then the unstable slices have the John "bow tie" condition: there is a uniform $\epsilon > 0$ such that x' and x'' can be connected by an ϵ-"cigar" or ϵ-"bow tie" with

$$Length(\gamma) \leq C\, dist(x', x'')$$

Unstable slice:

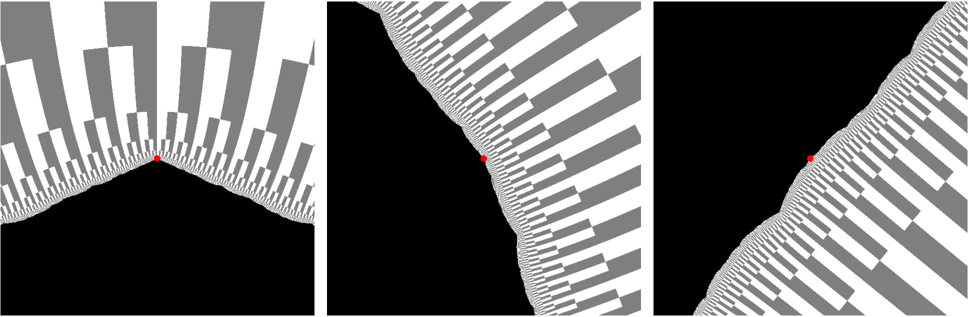


Problem

Does a John condition hold in the quasi-hyperbolic case?

Unstable slice pictures for the map $f(x,y) = (y, y^2 - .1 - .15x)$ This time, J^+ is a topological 3-manifold.

Gray/white regions give binary coding for G^+/ harmonic conjugate;
Black $= K^+$ (basin of fixed point); boundary of black $= J^+$.



Unstable slices with centers (red) at fixed point, 2-cycle and a 3-cycle:
Multipliers are ≈ 2.4, ≈ 4.6, and 3-cycle with multiplier $\approx -9.2 + 4.7i$

Example 1. Power map $\quad p: \ z \mapsto z^2$
Julia set is the circle $\{|z| = 1\}$.
If $z_0 \neq 0$ has period n, then $(p^n)'(z_0) = 2^n$

Example 2. Chebyshev map $\quad p: \ z \mapsto z^2 - 2$
Julia set is the interval $[-2, 2]$.
$0 \to -2 \to 2 \to 2 \quad p'(2) = 4$
If $z_0 \neq 2$ has period n, then $(p^n)'(z_0) = \pm 2^n$

There are Chebyshev maps in higher dimension. Some of these Julia sets have been described in detail by S. Nakane and K. Uchimura. The corresponding Julia sets are semi-algebraic.

Are there Hénon maps that are special? Are there Hénon maps with smooth Julia sets?

Julia sets for Hénon maps are never smooth

Theorem (B-Kyounghee Kim)

For any composition $f = f_n \circ \cdots \circ f_1$ *of generalized Hénon maps, the Julia set* J^+ *is not* C^1 *smooth, as a manifold-with-boundary.*

Definition of *manifold-with-boundary:*
At an *interior point*, J^+ is given locally as $\{r = 0\}$, where r is class C^1, and $dr \neq 0$.
At a *boundary point*, there are r and s of class C^1 with $dr \wedge ds \neq 0$, and $J^+ = \{r = 0, s \geq 0\}$, and the boundary is $\{r = s = 0\}$.

Remark. Replacing f by f^{-1}, we conclude that J^- is never smooth.

Lemma

$\partial J^+ = \emptyset$.

Proof.

J^+ is Levi-flat. That is, the 1-form ∂r generates a foliation of J^+ by Riemann surfaces.

The boundary $M := \partial J^+$ is a Riemann surface, which is a closed submanifold of $\mathbb{C}^2$. The restriction $g := G^-|_M$ is a subharmonic exhaustion. Further, g is harmonic on $M - K = \{g > 0\}$. By the Maximum Principle, each connected component M_0 of M must intersect $K = \{g = 0\}$. Since K is compact, M can have only finitely many components. Passing to an iterate of f, we may assume that M_0 is invariant.

Since $g \circ f = g/\deg$, it follows that f is an automorphism of the Remann surface with an attracting fixed point q. We conclude that the restriction of $G^-|_{M_0}$ is continuous, $G^-|_{M_0} \geq 0$, and harmonic on $M_0 - \{q\}$ and $G^-(q) = 0$. Harmonic functions cannot have such isolated singularities, so we conclude $M = \emptyset$. □

If J^+ is smooth, then f is dissipative.

Lemma

K^+ has nonempty interior. Further, $|\delta| < 1$, i.e., f decreases volume.

Proof.

J^+ is orientable and divides $\mathbb{C}^2$ into at least 2 components. U^+ is a component of $\mathbb{C}^2 - J^+$. Further, for fixed x_0, the slice $U^+ \cap \{x = x_0\}$ is connected and contains a neighborhood of infinity. If the slice $\{x = x_0\} \cap \mathrm{int}(K^+)$ is empty, then $\{x = x_0\} \cap J^+$ must be an arc, but this prevents J^+ from being smooth. Thus each slice must intersect interior points of K^+.

If $|\delta| \geq 1$, then by Friedland-Milnor, $K^+ \cap \{|x| > R\}$ has no interior. Thus we must have $|\delta| < 1$. □

(Almost) all fixed points belong to J^+.

Lemma

There is at most one fixed point in $int(K^+)$.

Theorem (BS2)

If $q \in int(K^+)$ *is a fixed point, then let* $\Omega \subset int(K^+)$ *denote the component containing it. It follows that* Ω_q *is a recurrent Fatou component and is the basin of a point or an invariant (Siegel) disk. In both cases, the boundary is* $\partial\Omega_q = J^+$.

Proof of Lemma.

If J^+ is smooth, the one side of the complement is U^+, and the other side is given by Ω_q. Thus there can be at most one fixed point q. □

All fixed points in J^+ are saddles.

Lemma

If $q \in J^+$ is a fixed point, then q is a saddle.

Proof.

Let $T_q(J^+)$ denote the tangent space, and let H_q denote its $\mathbb{C}$ invariant subspace. Then H_q is invariant under Df, so we let α_q be the associated eigenvalue. Since J^+ is Levi-flat, it follows that $|\alpha_q| \leq 1$. Further, it can be shown that $|\alpha_q| < 1$. Let β_q denote the other eigenvalue of $D_q f$. Thus $|\delta| = |\alpha_q \beta_q| < 1$. We conclude that since q cannot be attracting, $|\beta_q| \geq 1$. Since the real tangent space $T_q(J^+)$ is invariant, and U^+ is invariant, it follows that $\beta_q > 0$ is real. Finally, we cannot have $\beta_q = 1$, or in this case we would have a semi-attracting/semi-parabolic point, so J^+ would have a cusp. Thus $\beta_q > 1$, and we have a saddle point. □

All saddles have the same multipliers

Lemma

If $q \in J^+$ is a fixed point, then its multipliers are d and δ/d.

Proof.

Let $\xi_q : \mathbb{C} \to W^u(q)$ be the uniformization of the unstable manifold with $\xi(0) = q$. It follows that

$$f \circ \xi_q(\zeta) = \xi_q(\beta_q \zeta)$$

and

$$G^+(\xi_q(\beta_q \zeta)) = \deg(f) \cdot G^+(\xi_q(\zeta))$$

We conclude that if $J_q := \xi_q^{-1}(J^+) \subset \mathbb{C}$ is the pre-image under ξ_q, then ξ_q is self-similar under multiplication by β_q. Since J_q is C^1 smooth and self-similar, it follows that it is actually linear. Rotating coordinates, we may assume it is the imaginary axis, and $G^+ \circ \xi_q(\zeta)$ is a multiple of $Re(\zeta)$ for $Re(\zeta) > 0$ and 0 for $Re(\zeta) < 0$. Since G^+ multiplies by deg when we compose with f, we conclude that $\beta_q = \deg(f)$. □

Remark

It will turn out that there is nothing special about the multiplier d. The important point will be that the fixed points have the *same* multipliers. From this point forward, we will forget the condition that J^+ is smooth, and we replace it by the condition:

With at most one exception, the multipliers of all the fixed points are the same.

We will now show by algebra that this is not possible.

Defining equations for fixed points: unfolding dynamical space.

If $q = (x, y)$ is a fixed point for $f = f_n \circ \cdots \circ f_1$, then we may represent it as a finite sequence (x_j, y_j) with $j \in \mathbb{Z}/n\mathbb{Z}$, subject to the conditions $(x, y) = (x_1, y_1) = (x_{n+1}, y_{n+1})$ and

$$f_j(x_j, y_j) = (x_{j+1}, y_{j+1})$$

Use the notation $(x_0, y_0) = (x, y)$ and $(x_{j+1}, y_{j+1}) = f_j(x_j, y_j)$.
Fixed point:
$(x_n, y_n) = f(x, y) = f_n(\cdots(f_1(x, y)\cdots) = (x, y) = (x_0, y_0)$,

$$\mathbb{C}^2_{x_1,y_1} \xrightarrow{f_1} \mathbb{C}^2_{x_2,y_2} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \mathbb{C}^2_{x_n,y_n} \xrightarrow{f_n} \mathbb{C}^2_{x_1,y_1}$$

Defining equations for the fixed point set

Given the form of $f_j(x,y) = (y, p_j(y) - \delta_j x)$, we have $x_{j+1} = y_j$, so we may drop the x_j's from our notation and write $q = (y_n, y_1)$. Thus we can simplify our space from $\mathbb{C}^2_{x_1,y_1} \times \cdots \times \mathbb{C}^2_{x_n,y_n}$ to $\mathbb{C}^n_{y_1,\ldots,y_n}$. We identify this point with the sequence $\hat{q} = (y_1, \ldots, y_n) \in \mathbb{C}^n$, and we define the polynomials

$$\varphi_1 := p_1(y_1) - \delta_1 y_n - y_2$$

$$\varphi_2 := p_2(y_2) - \delta_2 y_1 - y_3$$

$$\cdots\cdots\cdots$$

$$\varphi_n := p_n(y_n) - \delta_n y_{n-1} - y_1$$

The condition to be a fixed point is that $\hat{q} = (y_1, \ldots, y_n)$ belongs to the zero locus $Z(\varphi_1, \ldots, \varphi_n)$ of the φ_i's.

Differential of f; condition for multiplier λ

By the Chain Rule, the differential of f at $q = (y_n, y_1)$ is given by

$$Df(q) = \begin{pmatrix} 0 & 1 \\ -\delta_n & p_n'(y_n) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\delta_1 & p_1'(y_1) \end{pmatrix}$$

The condition for Df to have a multiplier λ at q is $\Phi(\hat{q}) = 0$, where

$$\Phi = \det\left(Df - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

Lemma

$$\Phi = p_1'(y_1) \cdots p_n'(y_n) + \sum c_{i_1,\ldots,i_m} \prod_{i_1 < \cdots < i_m} p_{i_j}'(y_{i_j})$$

where the summation is taken over terms $m \leq n - 2$.

Reformulation as a problem in algebraic geometry

Heuristically, our Theorem will follow if we show:

Theorem (Simplified)

For all choices of $p_1, \ldots, p_n$ *and* $\delta_1, \ldots, \delta_n$*, and for any multiplier* λ*,* Φ *does not vanish on the entire zero set* $Z(\varphi_1, \ldots, \varphi_n) \subset \mathbb{C}^n_{y_1,\ldots,y_n}$*, i.e.*

$$Z(\varphi_1, \ldots, \varphi_n) \not\subset \{\Phi = 0\}$$

Equivalently, Φ *does not belong to the ideal* $\langle \varphi_1, \ldots \varphi_n \rangle$*.*
Equivalently, there are polynomials $A_j(y_1, \ldots, y_n)$*,* $1 \leq j \leq n$ *such that*

$$\Phi = A_1\varphi_1 + \cdots + A_n\varphi_n$$

If we look at the definitions of φ_j and Φ, this Theorem seems clear.

In fact, one of the fixed points is not a saddle, so if we let α denote its y-coordinate, we must show that there are no $A_1, \ldots A_n$ such that

$$(y_1 - \alpha)\Phi = A_1\varphi_1 + \cdots + A_n\varphi_n$$

Multivariate Division Algorithm

We want to determine whether a polynomial f belongs to the ideal $\langle \varphi_1, \ldots, \varphi_n \rangle$. We choose an ordering on the set of monomials, and we let $LT(\varphi_j)$ denote the leading term of φ_i. Let M be a monomial term in f which is divisible by some $LT(\varphi_{i_1})$. We define the *reduction* f_1 by φ_{i_1}:

$$f = q_1 \varphi_{i_1} + f_1$$

where $q_1 := M / LT(\varphi_{i_1})$. We continue by reducing f_1 if some monomial term is divisible by some leading term $LT(\varphi_j)$. We continue as far as possible to reach

$$f = q_1 \varphi_{i_1} + \cdots + q_m \varphi_{i_m} + r$$

Note that the remainder r obtained by this Algorithm depends on the choice of monomial ordering, as well as choices of φ_{i_j}, so may not be unique.

However, we have uniqueness if we use a Gröbner basis. In particular, with a Gröbner basis, we will have $r = 0$ if and only if f belongs to the ideal $\langle \varphi_1, \ldots, \varphi_n \rangle$.

Let $\mathcal{I} = \mathcal{I}(G)$ denote the ideal generated by the basis G. Choose a monomial ordering.

Theorem (Equivalent properties that define/characterize a Gröbner basis with respect to a given monomial ordering)

(i) The ideal given by the leading terms of polynomials in $\mathcal{I}$ is itself generated by the leading terms of the basis G;

(ii) The leading term of any polynomial in $\mathcal{I}$ is divisible by the leading term of some polynomial in the basis G;

(iii) The multivariate division of any polynomial in the polynomial ring R by G gives a unique remainder;

(iv) The multivariate division by G of any polynomial in the ideal $\mathcal{I}$ gives the remainder 0.

The *degree* of a monomial $y^a := y_1^{a_1} \cdots y_n^{a_n}$ is $\deg(y^a) = a_1 + \cdots + a_n$. We will use the *graded lexicographical order* on the monomials in $\{y_1, \ldots, y_n\}$. That is, $y^a > y^b$ if either $\deg(y^a) > \deg(y^b)$, or if $\deg(y^a) = \deg(y^b)$ and $a_i > b_i$, where $i = \min\{1 \leq j \leq n : a_j \neq b_j\}$.

Lemma

With the graded lexicographical order, $G := \{\varphi_1, \ldots, \varphi_n\}$ is a Gröbner basis.

Theorem

Suppose that $f = f_n \circ \cdots \circ f_1$ with $n \geq 3$. Then $(y_1 - \alpha)\Phi \neq A_1\varphi_1 + \cdots + A_n\varphi_n$.

Outline of proof.

We divide L.H.S. first by φ_1, then φ_2, then φ_n. The remainder is now

$$(d_1 d_2 \delta_2 y_1 y_n^{d_n - 1} + d_1 d_n \delta_1 y_1 y_2^{d_2 - 1}) \prod_{i=3}^{n-1} y_i^{d_i - 1} + \text{l.o.t}$$

which cannot be removed by any φ_j for $3 \leq j \leq n-1$, since no further division is possible.