DYNAMICS OF FAMILIES OF MAPS TANGENT TO THE IDENTITY

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Parameter problems in analytic dynamics

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INTRODUCTION

A holomorphic germ $f\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ is tangent to the identity if $df_O = \mathrm{id}$, that is if it can be written as

$$f(z) = z + P_{\nu+1}(z) + \cdots$$

where $\nu + 1 \geq 2$ is the order of f, and $P_{\nu+1} \not\equiv O$ is a n-uple of homogeneous polynomials of degree $\nu + 1 \geq 2$.

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Goal: to describe (at least topologically) the dynamics in a full neighborhood of the origin.

INTRODUCTION ($n = 1$)

Leau-Fatou flower theorem (1920).

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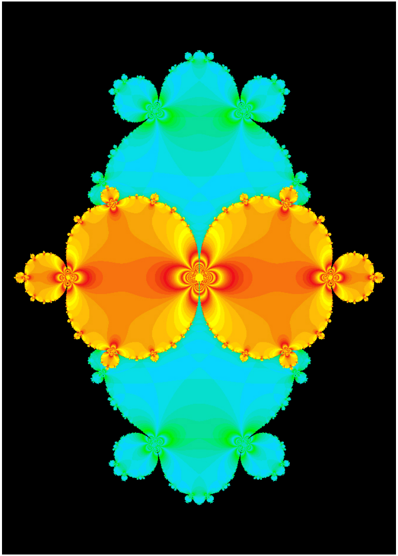

$$f(z) = z - z^3$$

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Remark: the number of (attracting or repelling) petals is equal to ν.

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Thus in dimension one *the topological local dynamics is completely determined by the order, and time-1 maps of homogeneous vector fields provide a complete list of models.*

INTRODUCTION ($n \geq 2$)

Aim of this talk is to advertise a *geometric* approach that in principle might lead to a description of the local topological dynamics in a full neighborhood of the origin for generic germs — and that surely works for time-1 maps of (even non-generic) homogeneous vector fields.

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- two meromorphic connections defined along the leaves of the foliation;
- the real geodesic flow along the leaves induced by the connections.

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Joint work with F. Tovena (Roma Tor Vergata) and F. Bianchi (Pisa-Toulouse).

INTRODUCTION ($n \geq 2$)

A parabolic curve for a germ f tangent to the identity is a injective holomorphic curve $\varphi\colon \Omega \to U \setminus \{O\}$ such that:

- $\Omega \subset \mathbb{C}$ is a simply connected domain with $0 \in \partial\Omega$;
- φ is continuous at 0 and $\varphi(0) = O$;
- $\varphi(\Omega)$ is f-invariant;
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Let $[\cdot]\colon \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ be the canonical projection.

A parabolic curve φ is tangent to $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ if $[\varphi(\zeta)] \to [v]$ as $\zeta \to 0$.

A Fatou flower is a set of ν disjoint parabolic curves tangent to the same direction $[v]$, where $\nu + 1$ is the order of f.

Introduction ($n \geq 2$)

Let $f(z) = z + P_{\nu+1}(z) + \cdots$.

A direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ is characteristic if $P_{\nu+1}(v) = \lambda v$ for some $\lambda \in \mathbb{C}$; it is degenerate if $\lambda = 0$, non-degenerate otherwise.

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Remark: f is dicritical if all directions are characteristic.

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THEOREM (ÉCALLE, 1985; HAKIM, 1998)

Let $f\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ be tangent to the identity at $O \in \mathbb{C}^n$, and $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ a non-degenerate characteristic direction. Then f admits a Fatou flower tangent to $[v]$.

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But even when $k = n$ these techniques are not enough for describing the dynamics in a full neighborhood of the origin; new techniques are needed.

BLOWING UP

Let $\pi\colon (M,S) \to (\mathbb{C}^n, O)$ be the blow-up of the origin in $\mathbb{C}^n$. The exceptional divisor $S = \pi^{-1}(O)$ can be identified with $\mathbb{P}^{n-1}(\mathbb{C})$.

Any germ $f_o\colon (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ tangent to the identity can be lifted to a holomorphic self-map $f\colon (M,S) \to (M,S)$ fixing pointwise the exceptional divisor.

To study the dynamics of f_o in a neighborhood of the origin is equivalent to study the dynamics of f in a neighborhood of S; e.g., (characteristic) directions for f_o becomes (special) points in S.

ORDER OF CONTACT

Let $f: M \to M$ be a holomorphic self-map of a complex n-dimensional manifold M leaving a complex smooth hypersurface $S \subset M$ pointwise fixed (actually, it suffices having f defined in a neighborhood of S).

We denote by $\mathcal{O}_M$ the sheaf of germs of of holomorphic functions on M, and by $\mathcal{I}_S$ the ideal subsheaf of germs of holomorphic functions vanishing on S.

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$$\nu_f(h;p) = \max\left\{\mu \in \mathbb{N} \;\middle|\; h \circ f - h \in \mathcal{I}^{\mu}_{S,p}\right\} .$$

The order of contact of f with S is

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Order of contact

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It is independent of p.

Remark

If f_o has order $\nu + 1$ then

$$\nu_f = \begin{cases} \nu & \text{if } f_o \text{ is non-dicritical,} \\ \nu + 1 & \text{if } f_o \text{ is dicritical.} \end{cases}$$

CANONICAL MORPHISM

In coordinates (U,z) adapted to S, that is such that $S \cap U = \{z^1 = 0\}$, setting $f^j = z^j \circ f$ we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z) \ ,$$

where z^1 does not divide at least one g^j, for $j = 1, \ldots, n$.

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where z^1 does not divide at least one g^j, for $j = 1, \ldots, n$.

The g^j's depend on the local coordinates. However, if we set

$$\tilde{X}_f = \sum_{j=1}^{n} g^j \frac{\partial}{\partial z^j} \otimes (dz^1)^{\otimes \nu_f}$$

then $X_f = \tilde{X}_f|_S$ is independent of the local coordinates, and defines a *global* canonical section of the bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$, where N_S is the normal bundle of S in M, and thus a canonical morphism $X_f \colon N_S^{\otimes \nu_f} \to TM|_S$.

CANONICAL FOLIATION

We say that f is tangential if the image of X_f is contained in TS. In coordinates adapted to S, this is equivalent to requiring $g^1|_S \equiv 0$, that is to $z^1|g^1$.

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Remark

f_o is non-dicritical if and only if f is tangential. So the tangential case is the most interesting one.

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We say that $p \in S$ is singular for f if it is a zero of X_f, and we write $p \in \mathrm{Sing}(f)$. We set $S^o = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$.

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Remark

$[v] \in S = \mathbb{P}^{n-1}(\mathbb{C})$ is singular for f if and only if it is a characteristic direction of f_o.

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PROPOSITION

If f is tangential and $p \in S^o$ is not singular, then no infinite orbit of f can stay close to p, that is there is a neighborhood $U \subset M$ of p such that for every $z \in U$ there exists $k_0 > 0$ such that $f^{k_0}(z) \notin U$ or $f^{k_0}(z) \in S$.

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Since S is a hypersurface, $N_S^{\otimes \nu_f}$ has rank one; therefore if f is tangential then the image of X_f yields a canonical foliation $\mathcal{F}_f$, which is a *singular holomorphic foliation* of S in Riemann surfaces.

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Remark

When $n = 2$, S is a Riemann surface; so the canonical foliation reduces to the data of its singular points. This is the reason why (as we'll see) the dynamics in dimension 2 is substantially simpler to study than the dynamics in dimension $n \geq 3$.

PARTIAL MEROMORPHIC CONNECTIONS

Assume we have a complex vector bundle F on a complex manifold S, and a morphism $X\colon F \to TS$. Let E be another complex vector bundle on S, and denote by $\mathcal{E}$ (respectively, $\mathcal{F}$) the sheaf of germs of holomorphic sections of E (respectively, F).

A partial meromorphic connection on E along X is a $\mathbb{C}$-linear map $\nabla\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ satisfying the Leibniz condition

$$\nabla(hs) = (dh \circ X) \otimes s + h\nabla s$$

for every $h \in \mathcal{O}_S$ and $s \in \mathcal{E}$. In other words, we can differentiate the sections of E only along directions in $X(F)$. The poles of the connection are the points where X is not injective.

PARTIAL MEROMORPHIC CONNECTIONS

In the tangential case, we can take $F = N_S^{\otimes \nu_f}$ and $X = X_f$. Then we get:

- a partial meromorphic connection ∇ on $E = N_S$ along X_f by setting

$$\nabla_u(s) = \pi\big([\tilde{X}_f(\tilde{u}), \tilde{s}]|_S\big)$$

where: $s \in \mathcal{N}_S$; $u \in \mathcal{N}_S^{\otimes \nu_f}$; $\pi\colon \mathcal{T}_{M,S} \to \mathcal{N}_S$ is the canonical projection; $\tilde{s}$ is any element in $\mathcal{T}_{M,S}$ such that $\pi(\tilde{s}) = s$; and $\tilde{u}$ is any element of $\mathcal{T}_{M,S}^{\otimes \nu_f}$ such that $\pi(\tilde{u}) = u$. Small miracle: ∇ is independent of all the choices.

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- a partial meromorphic connection, still denoted by ∇, on $N_S^{\otimes \nu_f}$ along X_f;
- a partial meromorphic connection ∇^o on the tangent bundle to the foliation $\mathcal{F}_f$ along the identity by setting

$$\nabla^o_v s = X_f\big(\nabla_{X_f^{-1}(v)} X_f^{-1}(s)\big) \ .$$

Notice that ∇^o induces a (classical) meromorphic connection on each leaf of the canonical foliation.

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In local coordinates (U, z) adapted to S (that is, $U \cap S = \{z^1 = 0\}$) and to $\mathcal{F}_f$ (that is a leaf is given by $\{z^3 = \text{cst.}, \ldots, z^n = \text{cst.}\}$), ∇ is represented by the meromorphic 1-form

$$\eta = -\, \nu_f \frac{1}{g^2} \frac{\partial g^1}{\partial z^1}\bigg|_S dz^2 \ ,$$

while ∇^o is represented by the meromorphic 1-form

$$\eta^o = \eta - \frac{1}{g^2} \frac{\partial g^2}{\partial z^2}\bigg|_S dz^2 \ .$$

GEODESICS

A geodesic is a smooth curve $\sigma\colon I \to S^o$, with $I \subseteq \mathbb{R}$, such that the image of σ is contained in a leaf of $\mathcal{F}_f$ and

$$\nabla^o_{\sigma'}\sigma' \equiv O \; .$$

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If $\eta^o = k\,dz^2$ is the form representing ∇^o in suitable coordinates then σ is a geodesic if and only if

$$\sigma'' + (k \circ \sigma)(\sigma')^2 = 0 \,.$$

Notice that k is meromorphic.

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If $\eta^o = k\,dz^2$ is the form representing ∇^o in suitable coordinates then σ is a geodesic if and only if

$$\sigma'' + (k \circ \sigma)(\sigma')^2 = 0\ .$$

The geodesic field G on the total space of $N_S^{\otimes \nu_f}$ is given by

$$G = \sum_{p=2}^{n} g^p|_S\, v \frac{\partial}{\partial z^p} + \nu_f \left.\frac{\partial g^1}{\partial z^1}\right|_S v^2 \frac{\partial}{\partial v}\ ,$$

where $(z^2, \ldots, z^n; v)$ are local coordinates on $N_E^{\otimes \nu_f}$. It is globally defined!

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PROPOSITION

σ is a geodesic for ∇^o if and only if $X^{-1}(\sigma')$ is an integral curve of G.

HEURISTIC PRINCIPLE

Heuristic guiding principle: *the dynamics of the geodesic flow represents the dynamics of f in a neighborhood of S*, at least in generic cases.

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When f comes from a f_o tangent to the identity, “generic" means “when f_o only has non-degenerate characteristic directions."

HEURISTIC PRINCIPLE

Heuristic guiding principle: *the dynamics of the geodesic flow represents the dynamics of f in a neighborhood of S*, at least in generic cases.

This becomes a rigorous statement, valid even in non-generic situations, when f comes from the time-1 map of a homogeneous vector field.

HOMOGENEOUS VECTOR FIELDS

A homogeneous vector field of degree $\nu+1\geq 2$ on $\mathbb{C}^n$ is given by

$$Q = Q^1\frac{\partial}{\partial z^1} + \cdots + Q^n\frac{\partial}{\partial z^n}$$

where $Q^1,\ldots,Q^n$ are homogeneous polynomials in $z^1,\ldots,z^n$ of degree $\nu+1$. We say that Q is non-dicritical if it is not a multiple of the radial vector field.

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The time-1 map of a homogeneous vector field of degree $\nu + 1$ is a holomorphic self-map of $\mathbb{C}^n$ tangent to the identity at the origin of order $\nu + 1$, dicritical if and only if Q is dicritical.

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The time-1 map of a homogeneous vector field of degree $\nu+1$ is a holomorphic self-map of $\mathbb{C}^n$ tangent to the identity at the origin of order $\nu+1$, dicritical if and only if Q is dicritical.

A characteristic leaf is a Q-invariant line $L_v=\mathbb{C}v\subset\mathbb{C}^n$. A line L_v is a characteristic leaf if and only if $[v]$ is a characteristic direction of the time-1 map of Q. The dynamics of Q inside a characteristic leaf is 1-dimensional and easy to study.

HOMOGENEOUS VECTOR FIELDS

THEOREM (A.-TOVENA, 2011)

Let Q be a homogeneous vector field in $\mathbb{C}^n$ of degree $\nu+1\geq 2$. Let S be the exceptional set in the blow-up of the origin in $\mathbb{C}^n$, and denote by $\pi\colon N_S^{\otimes\nu}\to S$ and by $[\cdot]\colon \mathbb{C}^n\setminus\{O\}\to\mathbb{P}^{n-1}(\mathbb{C})$ the canonical projections. Then there exists a ν-to-1 holomorphic covering map $\chi_\nu\colon \mathbb{C}^n\setminus\{O\}\to N_S^{\otimes\nu}\setminus S$ such that $\pi\circ\chi_\nu=[\cdot]$ and $d\chi_\nu(Q)=G$.

HOMOGENEOUS VECTOR FIELDS

THEOREM (A.-TOVENA, 2011)

Let Q be a homogeneous vector field in $\mathbb{C}^n$ of degree $\nu + 1 \geq 2$. Let S be the exceptional set in the blow-up of the origin in $\mathbb{C}^n$, and denote by $\pi\colon N_S^{\otimes\nu} \to S$ and by $[\cdot]\colon \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ the canonical projections. Then there exists a ν-to-1 holomorphic covering map $\chi_\nu\colon \mathbb{C}^n \setminus \{O\} \to N_S^{\otimes\nu} \setminus S$ such that $\pi \circ \chi_\nu = [\cdot]$ and $d\chi_\nu(Q) = G$. Therefore:

(I) *γ is a real integral curve of G (outside the characteristic leaves) if and only if $\chi_\nu \circ \gamma$ is an integral curve of G;*

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- (III) *every geodesic in $\mathbb{P}^{n-1}(\mathbb{C})$ is covered by exactly ν integral curves of Q.*

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Thus the study of integral curves of homogeneous vector fields is equivalent to the study of geodesics for partial meromorphic connections on $\mathbb{P}^{n-1}(\mathbb{C})$.

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- (III) *every geodesic in $\mathbb{P}^{n-1}(\mathbb{C})$ is covered by exactly ν integral curves of Q.*

The geodesic $\sigma(t)=[\gamma(t)]$ gives the complex line containing $\gamma(t)$; the "speed" $X_f^{-1}\big(\sigma'(t)\big)$ gives the position of $\gamma(t)$ in that line. In particular, $\gamma(t)\to O$ if and only if $X^{-1}\big(\sigma'(t)\big)\to O$.

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2. study of the global recurrence properties of the geodesics: it depends on the residues of (the local meromorphic 1-form representing) ∇^o.
3. study of the local behavior of the geodesics near the poles: it depends on the residues of (the local meromorphic 1-form representing) ∇.

A POINCARÉ-BENDIXSON THEOREM

THEOREM (A.-TOVENA, 2011, $R = \mathbb{P}^1(\mathbb{C})$; A.-BIANCHI, 2016, ANY R)

Let $\sigma\colon [0,T) \to R \setminus \{poles\}$ be a maximal geodesic for a meromorphic connection ∇^o on a compact Riemann surface R. Then:

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A recurring geodesic is closed, dense or self-intersects infinitely many times.

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Closed does not mean periodic.

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A saddle connection is a geodesic connecting two poles.

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Case (4) cannot happen when $R = \mathbb{P}^1(\mathbb{C})$. We do not have examples of cases (3) or (4).

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We have examples of case (5) when R is a torus, and examples of case (6) when $R = \mathbb{P}^1(\mathbb{C})$. We do not know whether (6) implies (5). If $R = \mathbb{P}^1(\mathbb{C})$ then (5) might happen only in case (6).

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Case (1) is generic; cases (2), (3), (4) and (6) can happen only if the poles of the connection satisfy some necessary conditions expressed in terms of the residues of ∇^o.

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If $R = \mathbb{P}^1(\mathbb{C})$, closed geodesics or boundary graphs of saddle connections can appear only if the real part of the sum of some residues is -1; a similar condition holds for R generic.

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If $R = \mathbb{P}^1(\mathbb{C})$ geodesics self-intersecting infinitely many times can appear only if the real part of the sum of some residues belongs to $(-3/2, -1) \cup (-1, -1/2)$; a similar condition holds for R generic.

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We have a less precise statement for non-compact Riemann surfaces.

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Main tools for the proof:

- ∇^o is flat;
- Gauss-Bonnet theorem relating geodesics and residues;
- a Poincaré-Bendixson theorem for smooth flows.

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COROLLARY

If γ is a recurrent integral curve of a homogeneous vector field then γ is periodic or $[\gamma]$ intersects itself infinitely many times.

LOCAL BEHAVIOR NEAR THE POLES $(n = 2)$

In dimension 2

$$G = g^2|_S\, v \frac{\partial}{\partial z^2} + \nu_f \left.\frac{\partial g^1}{\partial z^1}\right|_S v^2 \frac{\partial}{\partial v} \,.$$

Three classes of singularities:

- apparent if $1 \le \mathrm{ord}_p(g^2|S) \le \mathrm{ord}_p\left(\left.\frac{\partial g^1}{\partial z^1}\right|_S\right)$, that is p is not a pole of ∇;
- Fuchsian if $\mathrm{ord}_p(g^2|S) = \mathrm{ord}_p\left(\left.\frac{\partial g^1}{\partial z^1}\right|_S\right) + 1$, that is p is a pole of order 1;
- irregular if $\mathrm{ord}_p(g^2|S) > \mathrm{ord}_p\left(\left.\frac{\partial g^1}{\partial z^1}\right|_S\right) + 1$, that is p is a pole of order larger than 1.

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- Fuchsian if $\operatorname{ord}_p(g^2|S) = \operatorname{ord}_p \left(\left. \frac{\partial g^1}{\partial z^1} \right|_S \right) + 1$, that is p is a pole of order 1;
- irregular if $\operatorname{ord}_p(g^2|S) > \operatorname{ord}_p \left(\left. \frac{\partial g^1}{\partial z^1} \right|_S \right) + 1$, that is p is a pole of order larger than 1.

THEOREM (A.-TOVENA, 2011)

Local holomorphic classification of apparent and Fuchsian singularities, and formal classification of irregular singularities.

LOCAL BEHAVIOR NEAR THE POLES: APPARENT SINGULARITIES ($n = 2$)

Let $p_0 \in S$ an apparent singularity, and $\mu = \mathrm{ord}_{p_0}(g^2|_S) \geq 1$. Assume $\mu = 1$ (we have a complete statement for $\mu > 1$ too). Take $p \in S^o$ close enough to p_0. Then:

- for an open half-plane of initial directions the geodesic issuing from p tends to p_0;
- for the complementary open half-plane of initial directions the geodesic issuing from p escapes;
- for a line of initial directions the geodesic issuing from p is periodic.

LOCAL BEHAVIOR NEAR THE POLES: APPARENT SINGULARITIES $(n = 2)$

Furthermore, if Q is a homogeneous vector field having a characteristic leaf L_v such that $[v]$ is an apparent singularity with $\mu = 1$:

- no integral curve of Q tends to the origin tangent to $[v]$;
- there is an open set of initial conditions whose integral curves tend to a non-zero point of L_v;
- Q admits periodic integral curves of arbitrarily long periods accumulating at the origin.

Local behavior near the poles: Fuchsian singularities $(n = 2)$

Let $p_0 \in S$ a Fuchsian singularity, and $\mu = \mathrm{ord}_{p_0}(g^2|_S) \geq 1$. Assume $\mu = 1$ (we have an almost complete statement for $\mu > 1$ too: resonances appear). Let $\rho = \mathrm{Res}_{p_0}(\nabla)$ (necessarily $\rho \neq 0$ since $\mu = 1$). Take $p \in S^o$ close enough to p_0. Then:

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- if $\mathrm{Re}\,\rho < 0$ then p_0 is attracting, that is all geodesics σ issuing from p except one tends to p_0 with $X^{-1}\big(\sigma'(t)\big) \to O$; the only exceptional geodesic escapes;

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- if $\mathrm{Re}\,\rho < 0$ then p_0 is attracting, that is all geodesics σ issuing from p except one tends to p_0 with $X^{-1}\big(\sigma'(t)\big) \to O$; the only exceptional geodesic escapes;
- if $\mathrm{Re}\,\rho > 0$ then p_0 is repelling, that is all geodesics σ issuing from p except one escape, and the only exceptional geodesic tends to p_0 in finite time with $\big|X^{-1}\big(\sigma'(t)\big)\big| \to +\infty$;

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- if $\mathrm{Re}\,\rho > 0$ then p_0 is repelling, that is all geodesics σ issuing from p except one escape, and the only exceptional geodesic tends to p_0 in finite time with $\left|X^{-1}\big(\sigma'(t)\big)\right| \to +\infty$;
- if $\mathrm{Re}\,\rho = 0$ then issuing from p there are closed geodesics (with "speed" converging either to 0 or to $+\infty$), geodesics accumulating the support of a closed geodesic, and escaping geodesics.

LOCAL BEHAVIOR NEAR THE POLES: FUCHSIAN SINGULARITIES $(n = 2)$

Furthermore, if Q is a homogeneous vector field having a characteristic leaf L_v such that $[v]$ is a Fuchsian singularity with $\mu = 1$ and residue $\rho \neq 0$:

- if $\operatorname{Re} \rho < 0$ there is an open set of initial conditions whose integral curves tend to the origin tangent to $[v]$;
- if $\operatorname{Re} \rho > 0$ then no integral curve outside of L_v tends to O tangent to $[v]$;
- if $\operatorname{Re} \rho = 0$ then there are integral curves converging to O without being tangent to any direction.

LOCAL BEHAVIOR NEAR THE POLES: IRREGULAR SINGULARITIES $(n = 2)$

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Results by Vivas (2012) on the existence of parabolic domains.

LOCAL BEHAVIOR NEAR THE POLES: IRREGULAR SINGULARITIES $(n = 2)$

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Results by Vivas (2012) on the existence of parabolic domains.
Possibly Stokes phenomena.

Families of homogeneous vector fields $(n = 2)$

Interesting families of homogenous vector fields of fixed degree $\nu + 1$ can be obtained by fixing the number and (whenever possible) the location of distinct characteristic directions, and then using the residues at the characteristic directions as parameters.

FAMILIES OF HOMOGENEOUS VECTOR FIELDS $(n = 2)$

Interesting families of homogenous vector fields of fixed degree $\nu + 1$ can be obtained by fixing the number and (whenever possible) the location of distinct characteristic directions, and then using the residues at the characteristic directions as parameters.

Non-dicritical quadratic ($\nu = 1$) homogeneous vector fields can have at most 3 distinct characteristic directions. Up to holomorphic conjugation there are:

1. 3 distinct quadratic fields with exactly one characteristic direction;
2. 2 distinct families of quadratic fields with exactly two characteristic directions, parametrized by the residue at (any) one of them;
3. 1 family of quadratic fields with three distinct characteristic directions, parametrized by the residues at (any) two of them.

TWO DISTINCT CHARACTERISTIC DIRECTIONS

Given $\rho \in \mathbb{C}$ take

$$Q_\rho(z,w) = -\rho z^2 \frac{\partial}{\partial z} + (1-\rho) z w \frac{\partial}{\partial w} \ .$$

Two characteristic directions:

- $[1:0]$: Fuchsian singularity of order $\mu = 1$ and residue ρ (unless $\rho = 0$, when it is an apparent singularity of order 1);
- $[0:1]$: Fuchsian singularity of order $\mu = 2$ and residue $1-\rho$.

Two distinct characteristic directions

$$Q_\rho(z,w) = -\rho z^2 \frac{\partial}{\partial z} + (1-\rho) z w \frac{\partial}{\partial w} \; .$$

- If $\operatorname{Re}\rho < 0$ then almost all integral curves converge to the origin tangent to $[1:0]$; each L_v contains exactly one line of exceptional initial values of integral curves diverging to infinity tangent to $L_{[0:1]}$.
- If $\operatorname{Re}\rho > 0$ the roles of $[1:0]$ and $[0:1]$ are reversed.
- If $\operatorname{Re}\rho = 0$ but $\rho \neq 0$ then almost all integral curves converge to the origin without being tangent to any direction; each L_v contains exactly one line of exceptional initial values of integral curves diverging to infinity without being tangent to any direction.
- If $\rho = 0$ then almost all integral curves go from one point in $L_{[1:0]}$ to infinity toward $L_{[0:1]}$; each L_v contains exactly one real curve of exceptional initial values of periodic integral curves, and these periodic integral curves accumulate at the origin.

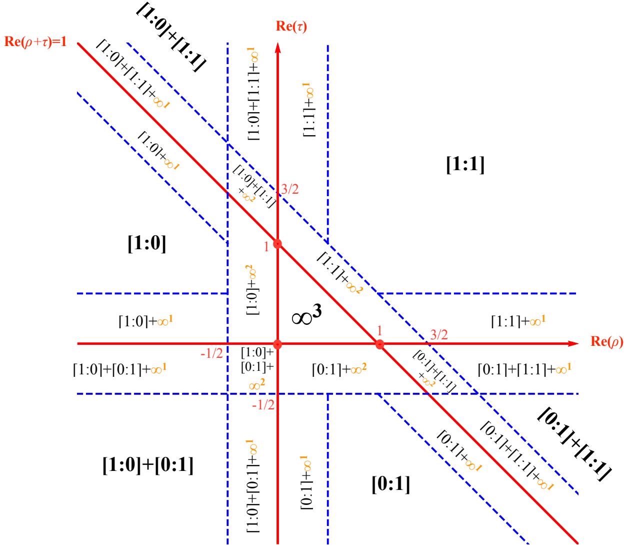
THREE DISTINCT CHARACTERISTIC DIRECTIONS

$$Q_{\rho,\tau}(z,w) = \big(-\rho z^2 + (1-\tau)zw\big)\frac{\partial}{\partial z} + \big((1-\rho)zw - \tau w^2\big)\frac{\partial}{\partial w}\;.$$

Three characteristic directions:

- $[1:0]$: Fuchsian singularity of order $\mu = 1$ and residue ρ (unless $\rho = 0$, when it is an apparent singularity of order 1);
- $[0:1]$: Fuchsian singularity of order $\mu = 1$ and residue τ (unless $\tau = 0$, when it is an apparent singularity of order 1);
- $[1:1]$: Fuchsian singularity of order $\mu = 1$ and residue $1-\rho-\tau$ (unless $\rho+\tau = 1$, when it is an apparent singularity of order 1).

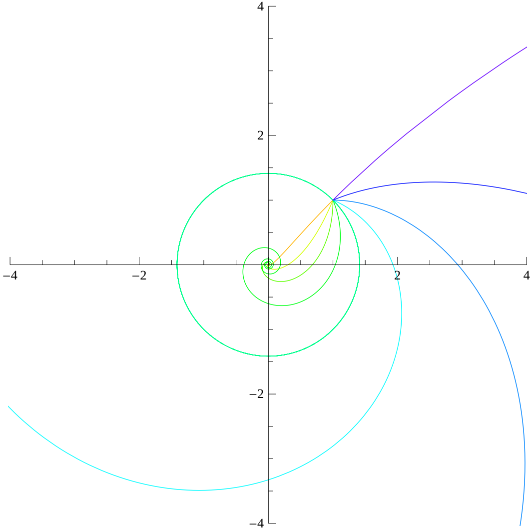
THREE DISTINCT CHARACTERISTIC DIRECTIONS



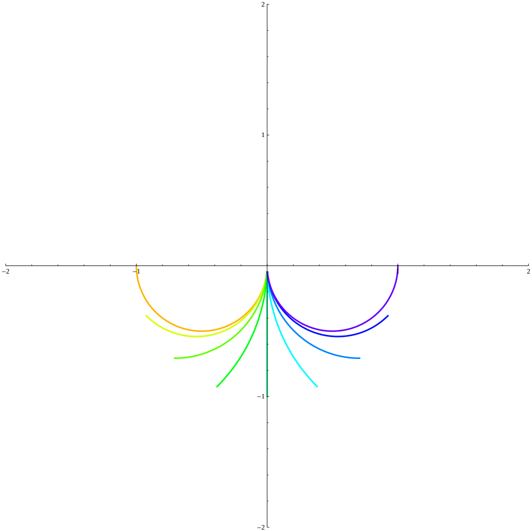
MOVIES

Movies!
(If there is time. . .)

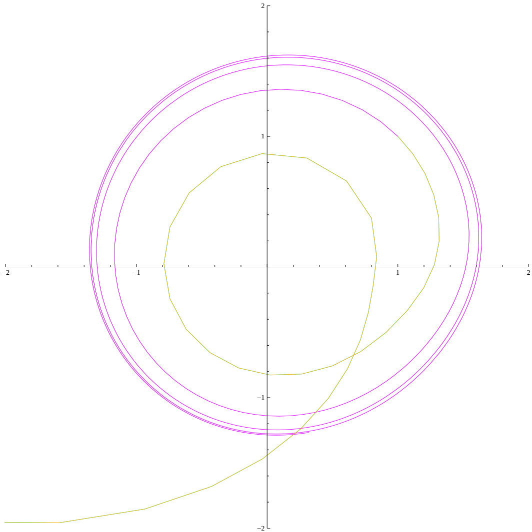
$Q(z,w) = -0.1iz^2\frac{\partial}{\partial z} + (1+0.1i)zw\frac{\partial}{\partial w}$



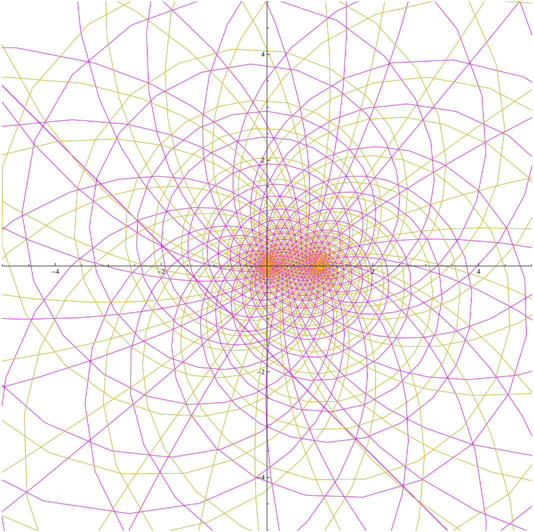
$Q(z,w) = -0.1iz^2\frac{\partial}{\partial z} + (1+0.1i)zw\frac{\partial}{\partial w}$



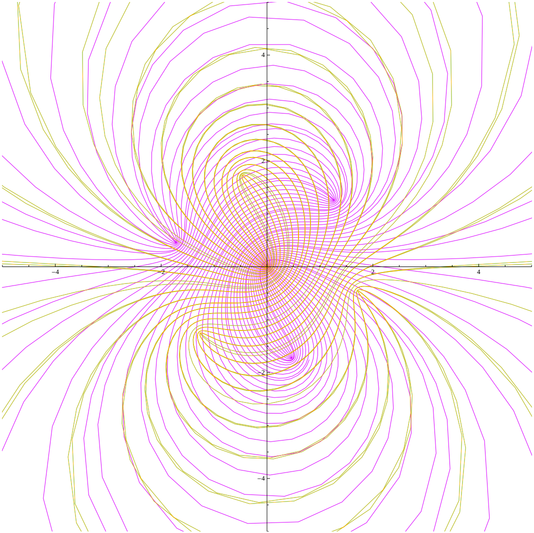
$$Q(z,w) = (-0.1z^2 + (1-0.2i)zw)\frac{\partial}{\partial z} + (1.1zw - 0.2iw^2)\frac{\partial}{\partial w}$$



$$Q(z,w) = \left(-\frac{1}{3}z^2 + \frac{2}{3}zw\right)\frac{\partial}{\partial z} + \left(\frac{2}{3}zw - \frac{1}{3}w^2\right)\frac{\partial}{\partial w}$$



$$Q(z,w) = \left(-\tfrac{1}{3}z^2 + \tfrac{2}{3}zw\right)\frac{\partial}{\partial z} + \left(\tfrac{2}{3}zw - \tfrac{1}{3}w^2\right)\frac{\partial}{\partial w}$$



THANKS!