An exact formula for default swaptions' pricing in the SSRJD stochastic intensity model

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ABSTRACT

We develop and test a fast and accurate semi-analytical formula for single-name default swaptions in the context of the shifted square root jump diffusion (SSRJD) default intensity model. The formula consists of a decomposition of an option on a summation of survival probabilities in a summation of options on the underlying survival probabilities, where the strike for each option is adjusted[1].

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1 INTRODUCTION

We develop and test a semi-analytical formula for single-name default swaptions in the context of an affine jump diffusion default intensity model. We specifically consider the shifted square root jump diffusion (SSRJD) model discussed in Brigo and El-Bachir (2006) and its restricted version, the diffusion-only SSRD model introduced earlier by Brigo and Alfonsi (2005) whose properties were further analyzed in Brigo and Cousot (2006). The semi-analytical formula is based on the celebrated decomposition due to Jamshidian (1989) for the valuation of options on coupon bonds in one-factor affine models. The formula for default swaptions has been first discussed in an unpublished version of Brigo and Alfonsi (2005) for the SSRD model. We extend it here to the SSRJD model, and give a more detailed and complete proof. Finally, we address its practical implementation and confirm its accuracy.

2 THE SSRJD DEFAULT INTENSITY MODEL

We denote the market filtration by $\mathbb{G} = (\mathcal{G}_t)_{(t\geq 0)}$ and let $\mathbb{Q}$ be a risk-neutral probability measure. We follow the intensity based approach to default risk modeling and introduce the default time as a totally inaccessible $\mathbb{G}-$stopping time τ. We further assume the usual structure for $\mathbb{G}$, namely that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{(t\geq 0)}$ is the filtration generated by the stochastic market variables (interest rates, default intensities, etc) except default events and $\mathbb{H} = (\mathcal{H}_t)_{(t\geq 0)}$ is the filtration generated by the default process: $\mathcal{H}_t = \sigma\left(\mathbf{1}_{\{\tau<u\}}, u \leq t\right)$. It is also assumed that there exists a strictly positive $\mathbb{F}-$adapted process $(\lambda_t)_{(t\geq 0)}$ such that the process $(M_t)_{(t\geq 0)}$ given by

$$M_t = \mathbf{1}_{\{\tau\leq t\}} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds = \mathbf{1}_{\{\tau\leq t\}} - \int_0^{t\wedge\tau} \lambda_s ds \quad (1)$$

is a uniformly integrable $\mathbb{G}-$martingale under $\mathbb{Q}$. The process $(\lambda_t)_{(t\geq 0)}$ is referred to as the $\mathbb{G}$ marginal intensity of the stopping time τ under $\mathbb{Q}$ or risk-neutral pre-default intensity. This setup is commonly referred to as a doubly stochastic Poisson default process or the Cox process framework. In the SSRJD model, the intensity λ_t is written as the sum of a positive deterministic function $\psi(t)$ and of a positive stochastic process y_t:

$$\lambda_t = y_t + \psi(t),\ t \geq 0 \quad (2)$$

where ψ is a deterministic function of time, and is integrable on closed intervals. The dynamics of $(y_t)_{(t\geq 0)}$ satisfy

$$\begin{aligned} dy_t &= \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_t + dJ_t \\ y(0) &= y_0 \end{aligned} \quad (3)$$

with the following condition to ensure the process cannot reach zero:

$$2\kappa\mu > \nu^2 \quad (4)$$

where $h = \sqrt{\kappa^2 + 2\nu^2}$. $(W_t)_{(t\geq 0)}$ is a Wiener process and $(J_t)_{(t\geq 0)}$ is a pure jump process with jumps arrival rate α and exponentially distributed jump sizes with mean γ. All the parameters $y_0, \kappa, \mu, \nu, \alpha, \gamma$ are also constrained to nonnegative values. Since this model belongs to the tractable

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Affine Jump Diffusion (AJD) class of models, the survival probability $\overline{\mathbb{S}}$ has the typical "log-affine" shape before default

$$\begin{aligned}\overline{\mathbb{S}}(t,T) &= \mathbf{1}_{\{\tau>t\}}\mathbb{S}(t,T) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_s ds\right)|\mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T [\psi(s)+y_s]ds\right)|\mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}A(t,T)\exp\left(-\int_t^T \psi(s)ds - B(t,T)y_t\right) \qquad (5)\end{aligned}$$

where:

$$A(t,T) = \xi(t,T)\zeta(t,T) \qquad (6)$$

with

$$\zeta(t,T) = \left(\frac{2h\exp\left(\frac{h+\kappa+2\gamma}{2}(T-t)\right)}{2h+(\kappa+h+2\gamma)(e^{h(T-t)}-1)}\right)^{\frac{2\alpha\gamma}{\nu^2-2\kappa\gamma-2\gamma^2}} \qquad (7)$$

Note that when $\gamma = \frac{h-\kappa}{2}$, the denominator in the exponent of $\zeta(t,T)$ goes to zero, i.e. $\nu^2-2\kappa\gamma-2\gamma^2 = 0$, leading to potential numerical instabilities due to division by zero. However, one can check in this case that the base of $\zeta(t,T)$ is then equal to one. Thus for robustness of the implementation, it is necessary to set $\zeta(t,T)=1$ when $\gamma = \frac{h-\kappa}{2}$. Finally, $\xi(t,T)$, $B(t,T)$ are given by:

$$\begin{aligned}\xi(t,T) &= \left(\frac{2h\exp\left(\frac{h+\kappa}{2}(T-t)\right)}{2h+(\kappa+h)(e^{h(T-t)}-1)}\right)^{\frac{2\kappa\mu}{\nu^2}} \qquad (8)\\ B(t,T) &= \frac{2(e^{h(T-t)}-1)}{2h+(\kappa+h)(e^{h(T-t)}-1)} \qquad (9)\end{aligned}$$

The SSRD model is a diffusion-only restriction of the SSRJD model obtained by setting the jump intensity α to zero, also resulting in $\zeta(t,T)=1$ in the survival probability formula.

For default swap computations we also make use of the formula for the following transform:

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_s ds\right)\lambda_T|\mathcal{F}_t\right] = -\partial_T\mathbb{S}(t,T) \qquad (10)$$

which can be expressed after differentiation as:

$$\begin{aligned}\partial_T\mathbb{S}(t,T) &= \mathbb{S}(t,T)\left[\frac{1}{\xi(t,T)}\partial_T\xi(t,T) - y_t\partial_T B(t,T) + \frac{1}{\zeta(t,T)}\partial_T\zeta(t,T) - \psi(T)\right] \qquad (11)\\ \partial_T\xi(t,T) &= \frac{-2\kappa\mu\left(e^{h(T-t)}-1\right)}{2h+(\kappa+h)\left(e^{h(T-t)}-1\right)}\xi(t,T) \qquad (12)\\ \partial_T B(t,T) &= \frac{4h^2e^{h(T-t)}}{\left[2h+(\kappa+h)\left(e^{h(T-t)}-1\right)\right]^2} \qquad (13)\\ \partial_T\zeta(t,T) &= \frac{-2\alpha\gamma\left(e^{h(T-t)}-1\right)}{2h+(\kappa+h+2\gamma)\left(e^{h(T-t)}-1\right)}\zeta(t,T) \qquad (14)\end{aligned}$$

Again for the SSRD model, the corresponding formulae can be obtained by simply using the fact that $\zeta(t,T)=1$ and $\partial_T\zeta(t,T)=0$.

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3 Pricing equations for default swaps and swaptions

3.1 Credit Default swaps

In this section, we briefly review default swaps pricing and refer to Brigo and Alfonsi (2005) for further details. A (credit) default swap is a financial instrument used by two counterparties to buy or sell protection against the default risk of a reference credit name. In a default swap signed at time t starting at time T_a with maturity T_b, the protection buyer pays a periodic fee or spread $R^{a,b}(t)$ at the payment dates $T_{a+1}, \ldots, T_b$ (typically quarterly) as long as the reference entity does not default. In case of a default occurring at time τ with $T_a < \tau \leq T_b$, the protection seller compensates the protection buyer for his loss given default that we assume to be a known constant L_{GD}. In addition, the protection seller receives from the protection buyer the spread accrued since the last payment date before default. In the case where $t < T_a$, the contract is a forward default swap, while if $t = T_a$ we are dealing with a spot default swap.

Default swaps have been shown in Brigo and Alfonsi (2005) to be relatively insensitive to the correlation between brownians driving the intensity and interest rate processes when both are modeled as SSRD processes, while Brigo and Cousot (2006) confirm that it is also relatively insignificant for default swaptions. Furthermore, Brigo and Cousot (2006) find that the short rate volatility has relatively little impact on the valuation of typically traded default swaptions characterized by short maturities, thus concluding that the randomness of the short rate adds little value to stochastic intensity models for default swaptions. Therefore, we assume a deterministic term structure of interest rates, and denote the price at time t of the default-free discount factor for maturity T or risk-free T-zero coupon bond by $D(t, T) = \exp\left(-\int_t^T r_s ds\right)$.

From the perspective of a protection buyer, the value at time t denoted by $CDS(t, \Upsilon, R, L_{GD})$ of a default swap with a payment schedule $\Upsilon = \{T_{a+1}, \ldots, T_b\}$, a spread R and a loss given default L_{GD} is given by the following expression:

$$CDS(t, \Upsilon, R, L_{GD}) = -\mathbf{1}_{\{\tau>t\}} \left[R\overline{C}_{a,b}(t) + L_{GD} \int_{T_a}^{T_b} D(t,u)\partial_u \mathbb{S}(t,u)du \right] \tag{15}$$

where

$$\overline{C}_{a,b}(t) = \left[\sum_{i=a+1}^{b} \alpha_i D(t, T_i)\mathbb{S}(t, T_i) - \int_{T_a}^{T_b} (u - T_{(\beta(u)-1)}) D(t,u)\partial_u \mathbb{S}(t,u)du \right] \tag{16}$$

and $T_{\beta(t)}$ is the first date in the set $\{T_a, \ldots, T_b\}$ that follows t and $\alpha_i = T_i - T_{i-1}$ is the year fraction between T_{i-1} and T_i.

Hence, the fair spread $R_{a,b}(t)$ as long as default has not occurred can be computed as the value of R that equates the default swap value to zero:

$$\mathbf{1}_{\{\tau>t\}} R_{a,b}(t) = -\mathbf{1}_{\{\tau>t\}} \frac{L_{GD} \int_{T_a}^{T_b} D(t,u)\partial_u \mathbb{S}(t,u)du}{\overline{C}_{a,b}(t)} \tag{17}$$

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3.2 Credit Default swaptions

A default swaption is an option written on a default swap. In the sequel, we will restrict the analysis to European payer default swaptions. A payer default swaption entitles its holder the right but not the obligation to become a protection buyer in the underlying default swap at the expiration of the option, paying a protection fee equal to the strike spread. Most traded single name default swaptions are canceled (or knocked out) at default of the underlying reference name if this occurs before the option's maturity. The maturity of the option will typically be equal to the starting date of the underlying default swap T_a. That is, the default swaption holder enters a spot default swap if she chooses to exercise the option at maturity.

For the pricing of a default swaption at a valuation date t, the underlying reference is thus the T_a maturity forward default swap with payment dates $T_{a+1}, \ldots, T_b$. The strike K specified in the contract is the periodic fixed rate that is to be paid in exchange for the default protection of the default swap in case of exercise, instead of the fair market spread $R_{a,b}(T_a)$ that will be available at time T_a only. The T_a-defaultable payoff can be valued at time t by taking the risk-neutral expectation of its discounted value. Hence, the payer default swaption can be valued as in Brigo and Alfonsi (2005):

$$PSO(t, T_a, \Upsilon, K) = D(t, T_a)\mathbb{E}_{\mathbb{Q}}\left[(CDS(T_a, \Upsilon, K, L_{GD}))^+ |\mathcal{G}_t\right] \tag{18}$$

Brigo and Alfonsi (2005) proposed a formula for solving this pricing equation in the case of the SSRD model. The formula is based on the insightful decomposition of Jamshidian (1989), where in a 1-factor yield curve model, an option on a portfolio of cash flows is decomposed in a portfolio of options on each cash flow, where the strike for each option is judiciously adjusted. In the next section, we prove and extend this formula for the SSRJD model.

4 ANALYTICAL FORMULA FOR DEFAULT SWAPTIONS PRICING

The derivation of the formula follows three main steps. In proposition 1, we rewrite the pricing equation (18) in a suitable form for the application of the decomposition, i.e. as an option on an integral of multiples of survival probabilities. Then we use our decomposition in corollary 1, resulting in the appearance of an integral of terms that are akin to options on survival probabilities. And lastly, we give an explicit formula for these options in proposition 2.

Proposition 1. *The default swaption price satisfies the following formula:*

$$PSO(t, T_a, \Upsilon, K) = \mathbf{1}_{\{\tau>t\}}D(t, T_a)\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_a}\lambda_s ds}\left(L_{GD} - \int_{T_a}^{T_b} h(u)\mathbb{S}(T_a, u)du\right)^+ |\mathcal{F}_t\right] \tag{19}$$

where h is defined as:

$$h(u) = D(T_a, u)\left[L_{GD}\left(r_u + \delta_{T_b}(u)\right) + K\left(1 - (u - T_{\beta(u)-1})r_u\right)\right] \tag{20}$$

with $\delta_{T_b}(u)$ the Dirac delta function centered at T_b.

Proof. Starting from equation (18), we substitute the default swap T_a-value from equations (15) and (16) resulting in the following formula:

$$\begin{aligned} PSO(t, T_a, \Upsilon, K) &= D(t, T_a)\mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\{\tau > T_a\}}\left(K\int_{T_a}^{T_b} D(T_a, u)(u - T_{\beta(u)-1})\partial_u \mathbb{S}(T_a, u)du\right.\right. \\ &\left.\left. - \; K\sum_{i=a+1}^{b} \alpha_i D(T_a, T_i)\mathbb{S}(T_a, T_i) - L_{GD}\int_{T_a}^{T_b} D(T_a, u)\partial_u \mathbb{S}(T_a, u)du\right)^+ |\mathcal{G}_t\right] \end{aligned}$$

We can integrate by parts the last integral of the above expression:

$$\begin{aligned} \int_{T_a}^{T_b} D(T_a, u)\partial_u \mathbb{S}(T_a, u)du &= \Big[D(T_a, u)\mathbb{S}(T_a, u)\Big]_{T_a}^{T_b} - \int_{T_a}^{T_b} \mathbb{S}(T_a, u)\partial_u D(T_a, u)du \\ &= D(T_a, T_b)\mathbb{S}(T_a, T_b) - 1 - \int_{T_a}^{T_b} \mathbb{S}(T_a, u)\partial_u D(T_a, u)du \end{aligned}$$

For the other integral appearing in the default swaption price, we first decompose it in a sum of integrals where the limits of integration are the default swap payment dates:

$$\int_{T_a}^{T_b} D(T_a, u)(u - T_{\beta(u)-1})\partial_u \mathbb{S}(T_a, u)du = \sum_{i=a}^{b-1}\int_{T_i}^{T_{i+1}} D(T_a, u)(u - T_i)\partial_u \mathbb{S}(T_a, u)du$$

where we used the fact that for $T_i < u < T_{i+1}$, $T_{\beta(u)-1} = T_i$. And we can now integrate by parts these integrals:

$$\begin{aligned} \int_{T_i}^{T_{i+1}} D(T_a, u)(u - T_i)\partial_u \mathbb{S}(T_a, u)du &= \Big[D(T_a, u)(u - T_i)\mathbb{S}(T_a, u)\Big]_{T_i}^{T_{i+1}} \\ &- \int_{T_i}^{T_{i+1}} D(T_a, u)\mathbb{S}(T_a, u)du \\ &- \int_{T_i}^{T_{i+1}} \mathbb{S}(T_a, u)(u - T_i)\partial_u D(T_a, u)du \end{aligned}$$

Using the fact that $T_i - T_{i-1} = \alpha_i$, we obtain after summation:

$$\begin{aligned} \int_{T_a}^{T_b} D(T_a, u)(u - T_{\beta(u)-1})\partial_u \mathbb{S}(T_a, u)du &= \sum_{i=a+1}^{b} \alpha_i D(T_a, T_i)\mathbb{S}(T_a, T_i) \\ &- \int_{T_a}^{T_b} D(T_a, u)\mathbb{S}(T_a, u)du \\ &- \int_{T_a}^{T_b} \mathbb{S}(T_a, u)(u - T_{\beta(u)-1})\partial_u D(T_a, u)du \end{aligned}$$

Note that $\partial_u D(T_a, u) = -r_u D(T_a, u)$, substitute the expressions obtained for the integrals back in the original formula, using

$$D(T_a, T_b)\mathbb{S}(T_a, T_b) = \int_{T_a}^{T_b} D(T_a, u)\mathbb{S}(T_a, u)\delta_{T_b}(u)du$$

and finally, using the formula

$$\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\tau > T_a\}} Y_{T_a}|\mathcal{G}_t] = \mathbf{1}_{\{\tau > t\}}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^{T_a}\lambda_s ds\right) Y_{T_a}|\mathcal{F}_t\right]$$

(see for example Bielecki and Rutkowski (2001), Corollary 5.1.1 p.145), we obtain the result of the proposition after rearranging. □

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Jamshidian (1989) decomposes an option on a portfolio of zero-coupon bonds in a portfolio of options on the zero-coupon bonds. The rewriting of the pricing problem as in equation (19) will now allow us to achieve a similar result. Indeed, the term $\int_{T_a}^{T_b} h(u)\mathbb{S}(T_a,u)du$ is akin to a portfolio of survival probabilities of infinitely many maturities. We also note that survival probabilities satisfy the same formulas as zero-coupon bonds where the default intensity plays the role of the short rate. Hence, the expectation in equation (19) can be seen as a put option on a portfolio of zero-coupon bonds (although with infinitely many) where the strike is L_{GD} and the interest rate is given by the default intensity λ_t. Therefore, it is only natural that we are able to decompose it as a portfolio of infinitely many options on survival probabilities.

Corollary 1. *If the following integral is positive*

$$\int_{T_a}^{T_b}\left[L_{GD}D(T_a,u)\partial_u\mathbb{S}(T_a,u;0)+K\mathbb{S}(T_a,u;0)D(T_a,u)\left(1-(u-T_{\beta(u)-1})r_u\right)\right]du \tag{21}$$

then the default swaption price is the solution to the following formula:

$$\mathbf{1}_{\{\tau>t\}}D(t,T_a)\int_{T_a}^{T_b}h(u)\mathbb{E}\left[\exp\left(-\int_t^{T_a}\lambda_s ds\right)\left(\mathbb{S}(T_a,u;y^*)-\mathbb{S}(T_a,u;y_{T_a})\right)^+|\mathcal{F}_t\right]du \tag{22}$$

where $y^*\geq 0$ *satisfies:*

$$\int_{T_a}^{T_b}h(u)\mathbb{S}(T_a,u;y^*)du = L_{GD} \tag{23}$$

Otherwise, the default swaption price is simply given by the corresponding forward default swap value:

$$CDS(t,\Upsilon,K,L_{GD})$$

Proof. Recall the definition of $h(u)$:

$$h(u)=D(T_a,u)\left[L_{GD}\big(r_u+\delta_{T_b}(u)\big)+K\big(1-(u-T_{\beta(u)-1})r_u\big)\right]$$

Assume the short rate r_u is nonnegative and bounded by 100%, that is $0\leq r_u\leq 1$. Also, suppose that the spread payments occur at least once a year (usually spreads are paid quarterly) such that $0\leq u-T_{\beta(u)-1}\leq 1$. It follows that

$$h(u)\geq 0,\ \text{for all u}$$

Also note that $h(u)$ is a deterministic function that does not depend on y, while the survival probability $\mathbb{S}(T_a,u;y)$ given by equation (5) is clearly monotonically decreasing in y for all T_a and u. Hence,

$$\int_{T_a}^{T_b}h(u)\mathbb{S}(T_a,u;y)du$$

is a monotonically decreasing function of y. Furthermore, it is easy to see from equation (5) that

$$\lim_{y\to\infty}\int_{T_a}^{T_b}h(u)\mathbb{S}(T_a,u;y)du = 0 < L_{GD}$$

or just recall that $\mathbb{S}(T_a,u;y)$ is a survival probability and y is the initial value of the stochastic process driving the default intensity.

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We are interested in finding if there exists $y^* \geq 0$ satisfying equation (23). Now, recall that

$$h(u) = L_{GD} r_u D(T_a, u) - K(u - T_{\beta(u)-1}) r_u D(T_a, u) + L_{GD} \delta_{T_b}(u) D(T_a, u) + K D(T_a, u)$$

and note that (integrating by parts):

$$\int_{T_a}^{T_b} r_u D(T_a, u) \mathbb{S}(T_a, u) du = 1 - D(T_a, T_b) \mathbb{S}(T_a, T_b) + \int_{T_a}^{T_b} D(T_a, u) \partial_u \mathbb{S}(T_a, u) du$$

Hence, substituting back in the original integral, we obtain the following:

$$\begin{aligned} \int_{T_a}^{T_b} h(u) \mathbb{S}(T_a, u) du &= L_{GD} + \int_{T_a}^{T_b} \Bigg[L_{GD} D(T_a, u) \partial_u \mathbb{S}(T_a, u) \\ &+ K \mathbb{S}(T_a, u) D(T_a, u) \left(1 - (u - T_{\beta(u)-1}) r_u\right) \Bigg] du \end{aligned}$$

So that:

$$\begin{aligned} \lim_{y \to 0^+} \int_{T_a}^{T_b} h(u) \mathbb{S}(T_a, u; y) du &= L_{GD} + \int_{T_a}^{T_b} \Bigg[L_{GD} D(T_a, u) \partial_u \mathbb{S}(T_a, u; 0) \\ &+ K \mathbb{S}(T_a, u; 0) D(T_a, u) \left(1 - (u - T_{\beta(u)-1}) r_u\right) \Bigg] du \end{aligned}$$

As we already observed, $\left(1 - (u - T_{\beta(u)-1}) r_u\right)$ should normally be positive since r_u is the short rate at time u and $(u - T_{\beta(u)-1})$ is smaller than the period between two spread payment dates which is typically a quarter of a year. And since $S(T_a, u; 0)$ and $D(T_a, u)$ are both nonnegative being respectively the survival probability and the discount factor at time T_a for maturity u, it follows that:

$$K \mathbb{S}(T_a, u; 0) D(T_a, u) \left(1 - (u - T_{\beta(u)-1}) r_u\right) \geq 0 \text{ for all u } \geq T_a$$

On the other hand, $\partial_u \mathbb{S}(T_a, u; 0) \leq 0$ since the survival probability is decreasing with maturity.

We can then consider two cases depending on whether the integral

$$\int_{T_a}^{T_b} \left[L_{GD} D(T_a, u) \partial_u \mathbb{S}(T_a, u; 0) + K \mathbb{S}(T_a, u; 0) D(T_a, u) \left(1 - (u - T_{\beta(u)-1}) r_u\right) \right] du$$

is negative or not.

In the first case, i.e. when the integral is negative:

$$\lim_{y \to 0^+} \int_{T_a}^{T_b} h(u) \mathbb{S}(T_a, u; y) du < L_{GD}$$

and then the equation (23) does not admit a solution in y. However, in this case the payoff of the option is $\mathbb{Q} - a.s.$ strictly positive and hence the payoff of the option simplifies to a forward default swap payoff.

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In the other case (i.e. when the integral is nonnegative), by the intermediate value theorem the equation (23) admits a unique solution y^* by continuity and monotonicity, and we can replace L_{GD} by $\int_{T_a}^{T_b} h(u)\mathbb{S}(T_a,u;y^*)du$ in (19). Since $\mathbb{S}(T_a,u;y)$ is a monotonic function in y, then the terms $\mathbb{S}(T_a,u;y^*) - \mathbb{S}(T_a,u;y_{T_a})$ will be all of the same sign for all values of u, and therefore:

$$\left(\int_{T_a}^{T_b} h(u)\big(\mathbb{S}(T_a,u;y^*) - \mathbb{S}(T_a,u;y_{T_a})\big)du\right)^+ = \int_{T_a}^{T_b} h(u)\big(\mathbb{S}(T_a,u;y^*) - \mathbb{S}(T_a,u;y_{T_a})\big)^+ du$$

which we can substitute back in the expression (19) for the default swaption value, and use Fubini's theorem to change the order of the integrations, resulting in equation (22), thus completing the proof. □

Having decomposed the default swaption price in terms of options on survival probabilities, we are left with the task of computing these option values. Indeed, to further compute the quantity given in equation (22), recall that:

$$\int_t^T \lambda_s ds = \int_t^T \psi(s)ds + \int_t^T y_s ds$$

and that the survival probabilities $\mathbb{S}(t,T)$ satisfy equation (5). Substituting these in formula (22) result in the following expression for the default swaption:

$$\begin{aligned}&\mathbf{1}_{\{\tau>t\}} D(t,T_a)\exp\left(-\int_t^{T_a}\psi(s)ds\right) * \\ &\int_{T_a}^{T_b} h(u)A(T_a,u)e^{-\int_{T_a}^{u}\psi(s)ds}\mathbb{E}\left[\exp\left(-\int_t^{T_a} y_s ds\right)\left(e^{-B(T_a,u)y^*} - e^{-B(T_a,u)y_{T_a}}\right)^+ |\mathcal{F}_t\right]du\end{aligned} \tag{24}$$

The above expression is analytic up to an integral if we are able to find a formula for the expectation involved. We take up that task in the next proposition where:

$$\Psi(t,T,y_t,\varsigma,\varrho) := \mathbb{E}\left[\exp\left(-\int_t^T y_s ds\right)\left(e^{-\varrho\varsigma} - e^{-\varrho y_T}\right)^+ |\mathcal{F}_t\right]$$

where ς and ϱ are positive values.

Proposition 2.

$$\Psi(t,T,y_t,\varsigma,\varrho) = e^{-\varrho\varsigma}\Pi(T-t,y_t,\varsigma,0) - \Pi(T-t,y_t,\varsigma,\varrho) \tag{25}$$

where

$$\Pi(T,y_0,\varsigma,\varrho) = \frac{1}{2}\alpha_\psi(T)e^{-\beta_\psi(T)y_0} - \frac{1}{\pi}\int_0^\infty \frac{e^{Uy_0}[S\cos(Wy_0+v\varsigma) + R\sin(Wy_0+v\varsigma)]}{v}dv \tag{26}$$

with

$$\beta_\psi(T) = \frac{2\varrho h + (2+\varrho(h-\kappa))\left(e^{hT}-1\right)}{2h + (h+\kappa+\varrho\nu^2)(e^{hT}-1)} \tag{27}$$

$$\begin{aligned}\alpha_\psi(T) &= \left[\frac{2h\exp\left(\frac{\kappa+h}{2}T\right)}{2h+(h+\kappa+\varrho\nu^2)(e^{hT}-1)}\right]^{\frac{2\kappa\mu}{\nu^2}} \\ &* \left[\frac{2h(1+\varrho\gamma)\exp\left(\frac{(h^2-(\kappa+2\gamma)^2)(1-\frac{\varrho}{2}(h+\kappa))}{2(h-\kappa-2\gamma+\varrho(\gamma(h+\kappa)-\nu^2))}T\right)}{2h(1+\varrho\gamma) + [h+\kappa+\varrho\nu^2+\gamma(2+\varrho(h-\kappa))]\left(e^{hT}-1\right)}\right]^{\frac{2\alpha\gamma}{\nu^2-2\kappa\gamma-2\gamma^2}}\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
R &= (J^2+K^2)^{\frac{D}{2}} e^G [E\cos(H + D\arctan(K/J)) - F\sin(H + D\arctan(K/J))] \\
S &= (J^2+K^2)^{\frac{D}{2}} e^G [F\cos(H + D\arctan(K/J)) + E\sin(H + D\arctan(K/J))] \\
U &= \frac{\delta + \varepsilon e^{hT} + \phi e^{2hT}}{N} \\
W &= -\frac{4v h^2 e^{hT}}{N} \\
E &= (\widetilde{x}^2 + \widetilde{y}^2)^{\frac{\kappa\mu}{\nu^2}} \cos\left(\frac{2\kappa\mu}{\nu^2}\arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right)\right) \\
F &= (\widetilde{x}^2 + \widetilde{y}^2)^{\frac{\kappa\mu}{\nu^2}} \sin\left(\frac{2\kappa\mu}{\nu^2}\arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right)\right) \\
\widetilde{x} &= \frac{2he^{(h+\kappa)\frac{T}{2}}[2h + (h+\kappa+\varrho\nu^2)(e^{hT}-1)]}{N} \\
\widetilde{y} &= -\frac{2he^{(h+\kappa)\frac{T}{2}} v\nu^2 [e^{hT}-1]}{N} \\
D &= \frac{-2\gamma\alpha}{\nu^2 - 2\gamma\kappa - 2\gamma^2} \\
G &= \frac{\alpha\gamma T[(2-\varrho(h+\kappa))(h-\kappa-2\gamma-\varrho[\nu^2-\gamma(h+\kappa)]) + v^2(h+\kappa)[\nu^2-\gamma(h+\kappa)]]}{(h-\kappa-2\gamma-\varrho[\nu^2-\gamma(h+\kappa)])^2 + v^2[\nu^2-\gamma(h+\kappa)]^2} \\
H &= \frac{\alpha\gamma T v[(2-\varrho(h+\kappa))[\nu^2-\gamma(h+\kappa)] - (h+\kappa)(h-\kappa-2\gamma-\varrho[\nu^2-\gamma(h+\kappa)])]}{(h-\kappa-2\gamma-\varrho[\nu^2-\gamma(h+\kappa)])^2 + v^2[\nu^2-\gamma(h+\kappa)]^2} \\
J &= 1 + \frac{(e^{hT}-1)[(h+\kappa+2\gamma)(1+\varrho\gamma) + (\nu^2+\gamma(h-\kappa))[\varrho(\varrho\gamma+1)+v^2\gamma]]}{2h(1+\varrho\gamma)^2 + 2hv^2\gamma^2} \\
K &= -\frac{(e^{hT}-1)v[2\gamma\kappa + 2\gamma^2 - \nu^2]}{2h(1+\varrho\gamma)^2 + 2hv^2\gamma^2} \\
N &= (2h + (h+\kappa+\varrho\nu^2)[e^{hT}-1])^2 + v^2\nu^4[e^{hT}-1]^2 \\
\delta &= 2(h-\kappa) - 4\nu^2\varrho + \varrho^2\nu^2(h+\kappa) + v^2\nu^2(h+\kappa) \\
\varepsilon &= 4\kappa - 4\kappa^2\varrho - 2\kappa\varrho^2\nu^2 - 2v^2\nu^2\kappa \\
\phi &= -2(h+\kappa) - 4\nu^2\varrho - \varrho^2\nu^2(h-\kappa) - v^2\nu^2(h-\kappa)
\end{aligned}
$$

Proof. First note the equivalence between the following events:

$$\{e^{-\varrho\varsigma} \geq e^{-\varrho y_T}\} \Leftrightarrow \{y_T \geq \varsigma\}$$

Hence:

$$
\begin{aligned}
\Psi(t,T,y_t,\varsigma,\varrho) &= e^{-\varrho\varsigma}\mathbb{E}\left[\exp\left(-\int_t^T y_s ds\right)\mathbf{1}_{\{y_T\geq\varsigma\}}|\mathcal{F}_t\right] \\
&- \mathbb{E}\left[\exp\left(-\varrho y_T - \int_t^T y_s ds\right)\mathbf{1}_{\{y_T\geq\varsigma\}}|\mathcal{F}_t\right]
\end{aligned}
$$

We define Π as follows:

$$\Pi(T,y_0,\varsigma,\varrho) := \mathbb{E}\left[\exp\left(-\varrho y_T - \int_0^T y_s ds\right)\mathbf{1}_{\{y_T\geq\varsigma\}}\right]$$

Christensen (2002) derived a formula for Π that is analytic up to an integral. His formula is also reported in Lando (2004) Appendix E. We recall it below:

$$\Pi(T, y_0, \varsigma, \varrho) = \frac{1}{2}\psi(T, y_0, \varrho) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[e^{iv\varsigma}\psi(T, y_0, -\varrho - iv)\right]}{v} dv$$

with

$$\psi(T, y_0, \varrho) = \alpha_\psi(T)e^{-\beta_\psi(T)y_0}$$

where α_ψ and β_ψ satisfy formulae (28) and (27) respectively. The imaginary part appearing above admits an explicit expression as given in the statement of the proposition.

Since the process y_t is a homogenous and markovian jump-diffusion

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\varrho y_T - \int_t^T y_s ds\right)\mathbf{1}_{\{y_T \geq \varsigma\}}|\mathcal{F}_t\right] &= \mathbb{E}_{y_t}\left[\exp\left(-\varrho y_{T-t} - \int_0^{T-t} y_s ds\right)\mathbf{1}_{\{y_{T-t} \geq \varsigma\}}\right] \\ &= \Pi(T - t, y_t, \varsigma, \varrho)\end{aligned}$$

□

In summary, if

$$\int_{T_a}^{T_b}\left[L_{GD}D(T_a, u)\partial_u \mathbb{S}(T_a, u; 0) + K\mathbb{S}(T_a, u; 0)D(T_a, u)\left(1 - (u - T_{\beta(u)-1})r_u\right)\right]du > 0$$

then it is possible to solve for a positive y^* satisfying $\int_{T_a}^{T_b} h(u)\mathbb{S}(T_a, u; y^*)du = L_{GD}$, and such that the default swaption price is given by:

$$\mathbf{1}_{\{\tau>t\}}D(t, T_a)e^{-\int_t^{T_a}\psi(s)ds}\int_{T_a}^{T_b} h(u)A(T_a, u)e^{-\int_{T_a}^{u}\psi(s)ds}\Psi(t, T_a, y_t, y^*, B(T_a, u))du$$

On the other hand, if

$$\int_{T_a}^{T_b}\left[L_{GD}D(T_a, u)\partial_u \mathbb{S}(T_a, u; 0) + K\mathbb{S}(T_a, u; 0)D(T_a, u)\left(1 - (u - T_{\beta(u)-1})r_u\right)\right]du < 0$$

the default swaption is so deeply in the money that the probability of it moving out of the money is null. Therefore, in this case the default swaption is equivalent to a forward default swap, hence it can be valued by computing the price of the equivalent forward default swap.

5 Implementation and numerical results

To implement the formula presented in the previous section, we need to compute the relevant integrals numerically. We first focus on deriving a quadrature formula for computing the integral appearing in the formula for Π:

$$\int_0^\infty \frac{e^{Uy_0}[S\cos(Wy_0 + v\varsigma) + R\sin(Wy_0 + v\varsigma)]}{v} dv \tag{29}$$

Define $f(v) := S\cos(Wy_0 + v\varsigma) + R\sin(Wy_0 + v\varsigma)$. In the following lemma, we simplify the equation for the function and prove that our integrand is continuous and bounded on the interval $(0, \infty)$ with finite limits on both ends of the interval.

Lemma 1. *The function* $\frac{e^{Uy_0}}{v} f(v)$ *is continuous and bounded for* $v \in (0, \infty)$*. Moreover,*

$$\lim_{v\to\infty} \frac{e^{Uy_0}}{v} f(v) = 0 \tag{30}$$

$$\lim_{v\to 0} \frac{e^{Uy_0}}{v} f(v) = C \tag{31}$$

where C is a constant depending on the model parameters.

Proof. Since the function $\frac{e^{Uy_0}}{v} f(v)$ is obtained by combinations and compositions of continuous functions on $(0, \infty)$, it is therefore continuous.

Let ϕ be the angle such that[2]:

$$\begin{aligned}
\sin\phi &= \frac{S}{\sqrt{S^2+R^2}},\ S = \sqrt{S^2+R^2}\sin\phi \\
\cos\phi &= \frac{R}{\sqrt{S^2+R^2}},\ R = \sqrt{S^2+R^2}\cos\phi \\
\tan\phi &= \frac{S}{R},\ \text{and } \phi = \arctan\frac{S}{R}
\end{aligned}$$

then

$$\begin{aligned}
f(v) &= \sqrt{S^2+R^2}\left[\sin\phi\cos(Wy_0 + v\varsigma) + \cos\phi\sin(Wy_0 + v\varsigma)\right] \\
&= \sqrt{S^2+R^2}\sin\left(Wy_0 + v\varsigma + \arctan\frac{S}{R}\right)
\end{aligned}$$

Also, note that

$$\sqrt{S^2+R^2} = (J^2+K^2)^{\frac{D}{2}} e^G \sqrt{E^2+F^2} = (J^2+K^2)^{\frac{D}{2}} e^G \left(\frac{4h^2 e^{(h+\kappa)T}}{N}\right)^{\frac{\kappa\mu}{\nu^2}}$$

since

$$\begin{aligned}
E^2+F^2 &= (\tilde{x}^2+\tilde{y}^2)^{2\frac{\kappa\mu}{\nu^2}}\left[\cos^2\left(2\frac{\kappa\mu}{\nu^2}\arctan\frac{\tilde{y}}{\tilde{x}}\right) + \sin^2\left(2\frac{\kappa\mu}{\nu^2}\arctan\frac{\tilde{y}}{\tilde{x}}\right)\right] \\
&= (\tilde{x}^2+\tilde{y}^2)^{2\frac{\kappa\mu}{\nu^2}} = \left(\frac{4h^2 e^{(h+\kappa)T}}{N}\right)^{2\frac{\kappa\mu}{\nu^2}}
\end{aligned}$$

In summary, the integrand can be written as follows:

$$\frac{e^{Uy_0}}{v} f(v) = e^{Uy_0+G}(J^2+K^2)^{\frac{D}{2}}\left(\frac{4h^2 e^{(h+\kappa)T}}{N}\right)^{\frac{\kappa\mu}{\nu^2}} \frac{\sin\left(Wy_0 + v\varsigma + \arctan\frac{S}{R}\right)}{v}$$

[2] We thank Aanand Venkatramanan for pointing this out.

To compute the limit for $v \to \infty$, one can easily verify that

$$\begin{aligned}
\left|\sin\left(Wy_0 + v\varsigma + \arctan\frac{S}{R}\right)\right| &\leq 1 \\
\left(\frac{4h^2 e^{(h+\kappa)T}}{N}\right)^{\frac{\kappa\mu}{\nu^2}} &\longrightarrow 0 \\
K &\longrightarrow 0 \\
J &\longrightarrow 1 + \frac{(\nu^2 + \gamma(h-\kappa))(e^{hT}-1)}{2h\gamma} \\
G &\longrightarrow \frac{\alpha\gamma T(h+\kappa)}{\nu^2 - \gamma(h+\kappa)} \\
U &\longrightarrow \frac{\kappa}{\nu^2} + \frac{h(1-e^{2hT})}{\nu^2(e^{hT}-1)^2}
\end{aligned}$$

We can therefore conclude that

$$\lim_{v\to\infty} \frac{e^{Uy_0}}{v} f(v) = 0$$

We now consider the limit at zero. First, we can further write:

$$\begin{aligned}
\frac{S}{R} &= \frac{F\cos(H + D\arctan\frac{K}{J}) + E\sin(H + D\arctan\frac{K}{J})}{E\cos(H + D\arctan\frac{K}{J}) - F\sin(H + D\arctan\frac{K}{J})} \\
&= \frac{\sin(H + D\arctan\frac{K}{J} + \arctan\frac{F}{E})}{\cos(H + D\arctan\frac{K}{J} + \arctan\frac{F}{E})} \\
\arctan\frac{S}{R} &= H + D\arctan\frac{K}{J} + \arctan\frac{F}{E}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{F}{E} &= \frac{\sin\left(\frac{2\kappa\mu}{\nu^2}\arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right)\right)}{\cos\left(\frac{2\kappa\mu}{\nu^2}\arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right)\right)} \\
\arctan\frac{F}{E} &= \frac{2\kappa\mu}{\nu^2}\arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right)
\end{aligned}$$

Hence,

$$\arctan\frac{S}{R} = H + D\arctan\frac{K}{J} + \frac{2\kappa\mu}{\nu^2}\arctan\frac{\widetilde{y}}{\widetilde{x}}$$

Therefore,

$$\lim_{v\to 0}\frac{\sin\left(Wy_0 + v\varsigma + \arctan\frac{S}{R}\right)}{v} = \lim_{v\to 0}\frac{\sin\left(Wy_0 + v\varsigma + H + D\arctan\frac{K}{J} + \frac{2\kappa\mu}{\nu^2}\arctan\frac{\widetilde{y}}{\widetilde{x}}\right)}{v}$$

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Note from the formulae for $W, H, K, J, \widetilde{x}, \widetilde{y}$, that we can write:

$$\begin{aligned} H &= \frac{c_1 v}{c_2 + c_3 v^2} \\ \frac{K}{J} &= \frac{c_4 v^2 + c_5 v^3}{c_6 + c_7 v^2 + c_8 v^4} \\ \frac{\widetilde{y}}{\widetilde{x}} &= c_9 v \\ W y_0 + v\varsigma &= \left(\frac{c_{10}}{c_{11} + c_{12} v^2} + \varsigma \right) v \end{aligned}$$

where the constants $c_1, \ldots, c_{12}$ depend on the model parameters but not on v. This yields the following limit:

$$\lim_{v \to 0} \frac{\sin\left(W y_0 + v\varsigma + \arctan \frac{S}{R} \right)}{v} = \frac{c_1}{c_2} + \frac{2\kappa\mu}{\nu^2} c_9 + \varsigma + \frac{c_{10}}{c_{11}} + D \frac{c_4}{c_6}$$

Furthermore, since $\frac{e^{U y_0}}{v} f(v) = e^{U y_0 + G} (J^2 + K^2)^{\frac{D}{2}} \left(\frac{4h^2 e^{(h+\kappa)T}}{N} \right)^{\frac{\kappa\mu}{\nu^2}}$ is defined and equals some constant, say c_0, for $v = 0$, we can conclude that for some constant C depending on the model parameters:

$$\lim_{v \to 0} \frac{e^{U y_0}}{v} f(v) = C$$

Finally, we show that the function $\frac{e^{U y_0}}{v} f(v)$ is bounded since it is a product of bounded functions. Indeed recall that

$$U = \frac{\delta + \varepsilon e^{hT} + \phi e^{2hT}}{N}$$

In addition, note that N is a positive and increasing function of v, and

$$\delta + \varepsilon e^{hT} + \phi e^{2hT} = \cdots + v^2 \nu^2 (1 - e^{hT}) \left[h + \kappa + e^{hT} (h - \kappa) \right]$$

Since $hT > 0$, and $h = \sqrt{\kappa^2 + 2\nu^2} > \kappa$ then $1 - e^{hT} < 0$ and the term multiplied by v^2 in the expression above is negative. Hence U is decreasing in v. As a consequence, $e^{U y_0}$ is a positive and decreasing function of v attaining its maximum $e^{U(0) y_0}$ for $v = 0$.

□

We have thus shown that the infinite integral is well defined. For a visual view of the integrand $\frac{e^{U y_0}}{v} f(v)$, we plot it for a given set of parameter values in figure (1). For the numerical computation of the integral we use the four-point adaptive Gauss-Lobatto quadrature with seven point Kronrod refinement provided by Matlab's "quadl" routine based on Gander and Gautschi (2000). Numerical convergence can be verified in table (1). Experiments -not reported here- against a mid-point trapezoidal and Simpson's quadratures confirmed the accuracy of the faster and more convenient adaptive Gauss-Lobatto algorithm. For the outer integral appearing in the formula (22), some experimentation shows that Simpson's rule with at worst two quadrature points per

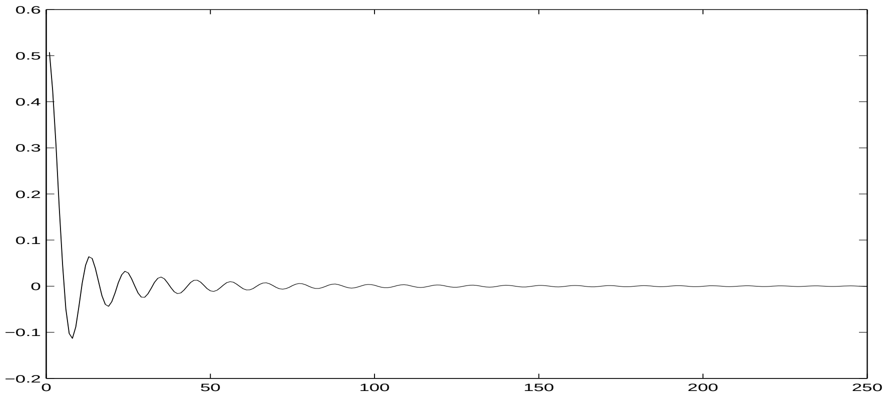

FIGURE 1: *Plot of* $\frac{e^{Uy_0}}{v} f(v)$ *when* $v \in (0, 250]$ *for* $y_0 = 0.005$, $\kappa = 0.196$, $\mu = 0.065$, $\nu = 0.1594$, $\alpha = 0.5$, $\gamma = 0.025$, $T - t = 1$

Integral bound: N	10^2	10^3	10^4	10^5	10^6	10^7
Numerical integral	-0.75859	-0.76983	-0.77173	-0.77178	-0.77178	-0.77178

TABLE 1: *Numerical approximation of* $\int_0^\infty \frac{e^{Uy_0}}{v} f(v)dv$ *by* $\int_0^N \frac{e^{Uy_0}}{v} f(v)dv$ *using adaptive Gauss-Lobatto quadrature for* $y_0 = 0.005$, $\kappa = 0.196$, $\mu = 0.065$, $\nu = 0.1594$, $\alpha = 0.5$, $\gamma = 0.025$, $T - t = 1$, $\varrho = B(0,3)$, $\varsigma = 0.0062$

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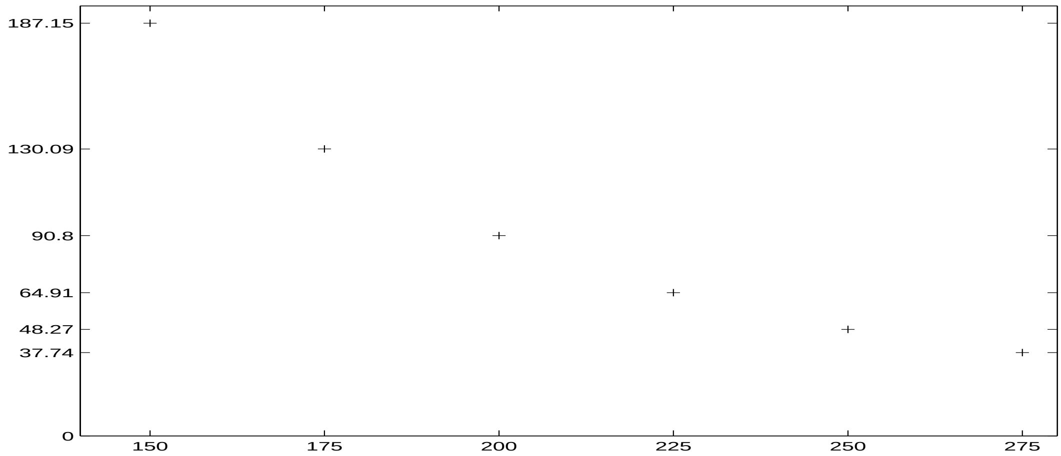

FIGURE 2: *Payer default swaption prices (bps) for different strike values (bps) in SRJD model with no deterministic shifts and parameters:* $T_a = 1y$, $T_b = 5y$, $r = 0.03$, $L_{GD} = 0.7$, $y_0 = 0.005$, $\kappa = 0.229$, $\mu = 0.0134$, $\nu = 0.078$, $\alpha = 1.5$, $\gamma = 0.0067$. *The fair value of the underlying forward default swap rate is 204 bps.*

quarterly spread payment period is usually enough for convergence of the numerical approximation, while using the spread payment dates as the only quadrature points in most cases leads to a good accuracy.

In figure (2) we present some numerical results for payer default swaption prices for different strikes, obtained using the quasi-analytic formula developed[3]. These are for a homogenous non-shifted version of the model with constant short rate. The set of parameters used are reported on the figure. We refer the interested reader to Brigo and El-Bachir (2006) for a detailed analysis of the SSRJD model, its implied volatility smile patterns and calibration procedures.

6 CONCLUSION

The SSRJD model can fit the current default swap term structure while being consistent with some dynamic future deformations and implying a volatility smile for default swaptions. The quasi-analytic formula presented in this paper permits fast and accurate pricing of default swaptions. Hence, the model could be calibrated to the CDS term structure and a few default swaptions, to price and hedge other credit derivatives consistently.

[3]Notice that the jumps in the intensity process can only take positive values. If one thinks in terms of zero-mean shocks, the long term mean reversion level of the process including jumps is no longer the purely diffusive long term mean μ but the larger $\mu + \frac{\alpha\gamma}{\kappa}$ as summarized in the following equivalent way of writing the dynamics of the process y_t:

$$dy_t = \kappa(\mu + \frac{\alpha\gamma}{\kappa} - y_t)dt + \nu\sqrt{y_t}dW_t + (dJ_t - \alpha\gamma dt)$$

where the jump process J_t is centered. This is why, in particular, we find a fair value of the underlying forward default swap rate is 204 bps when both the initial condition y_0 and the basic long term mean μ are much smaller.

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