A Comparison between the stochastic intensity SSRD Model and the Market Model for CDS Options Pricing *

Damiano Brigo
Credit Models
Banca IMI
Corso Matteotti 6
20121 Milano, Italy
damiano.brigo@bancaimi.it
http://www.damianobrigo.it

Laurent Cousot
Courant Institute
New York University
251 Mercer Street
New York, NY 10012, USA
laurent.cousot@polytechnique.org

First Version: August 1, 2003. This version: September 12, 2004

Abstract

In this paper we investigate implied volatility patterns in the Shifted Square Root Diffusion (SSRD) model as functions of the model parameters. We begin by recalling the Credit Default Swap (CDS) options market model that is consistent with a market Black-like formula, thus introducing a notion of implied volatility for CDS options. We examine implied volatilies coming from SSRD prices and characterize the qualitative behavior of implied volatilities as functions of the SSRD model parameters. We introduce an analytical approximation for the SSRD implied volatility that follows the same patterns in the model parameters and that can be used to have a first rough estimate of the implied volatility following a calibration. We compute numerically the CDS-rate volatility smile for the adopted SSRD model. We find a decreasing pattern of SSRD implied volatilities in the interest-rate/intensity correlation. We check whether it is possible to assume zero correlation after the option maturity in computing the option price and provide an upper bound for the Monte Carlo standard error in cases where this is not possible.

JEL classification code: G13.
AMS classification codes: 60H10, 60J60, 60J75, 91B70

*Presented at the third Bachelier Conference, Chicago, July 21-24, 2004. We are grateful to Aurélien Alfonsi for helpful comments and suggestions

1 Introduction

In the present paper we consider the issue of credit default swap (CDS) option pricing. We briefly summarize the shifted square-root diffusion (SSRD) model for interest rate derivatives and single-name credit derivatives introduced in Brigo and Alfonsi (2003), by recalling that the SSRD is the unique known stochastic (positive-) intensity and interest-rate model allowing for an analytical automatic calibration of the term structure of interest rates and of credit default swaps (CDS's). We consider the market model for CDS options introduced in Brigo (2004), similar in spirit to the defaultable LIBOR and swap models introduced in Schönbucher (2000) and perfected in Jamshidian (2002), and after pricing CDS options under the SSRD model we back out the implied volatility for the CDS-rate underlying the CDS option market model. We analyze numerically the dependence between dynamics parameters in the intensity process of the SSRD model and the implied CDS volatility in the market model. We also analyze the impact of correlation between stochastic intensities and interest rates on implied volatilities obtained from the SSRD model. We analyze an approximated formula providing the CDS implied volatility in term of SSRD dynamics parameters. This formula can be useful in quickly characterizing implied volatility patterns in the model dynamics parameters but is of very limited precision.

We also discuss the impact of interest-rate and default-intensity correlation ρ on SSRD CDS option implied volatilities, analogously to what was done earlier in Brigo and Alfonsi (2003) for simple CDS's, and test it by means of Monte Carlo simulation. In particular, the possibility to set this correlation to zero from the option maturity on, during the life of the underlying CDS, is investigated. This possibility would allow us to value the underlying CDS at option maturity analytically in each intensity and interest rate scenario, whereas if correlation had to be kept different from zero we would have to go on with the simulation up to the underlying CDS final maturity.

The paper is structured as follows: Section 2 introduces notation, CDS options, and recalls the notion of forward CDS rate and of "defaultable present value per basis point" numeraire. Section 3 recalls briefly the market model formula for CDS options as from Brigo (2004), where the market model is developed in detail. Section 4 recalls briefly the SSRD model introduced in Brigo and Alfonsi (2003) and hints at how CDS options can be priced within such model. Section 5 derives a formula that, under the assumption of zero correlation between stochastic interest rates and stochastic intensities, provides an approximation linking the SSRD model to the market model for CDS options, and explains how this approximation leads to an analytical formula for pricing CDS options with the SSRD model. Section 6 presents numerical investigations of the proposed formula and also of the way the exact CDS-rate volatility implied by Monte Carlo CDS-option prices under the SSRD model changes as a function of the SSRD model parameters. The SSRD volatility smile and the possibility to set $\rho = 0$ from the option maturity on are also investigated. Section 7 concludes the paper summarizing the main findings.

2 Credit Default Swaps Options

We recall briefly some basic definitions and then introduce CDS options.

Consider a CDS where we agree to receive protection payment rates R from a protection buyer at times $T_{a+1}, \ldots, T_b$ in exchange for a single protection payment $\mathrm{L{\scriptsize GD}}$ (loss given default) at the default time τ of a reference entity, provided that $T_a < \tau \leq T_b$ (receiver CDS). The CDS seen from the point of view of the protection buyer is a payer CDS, and the related discounted payoff is exactly the opposite of the receiver version.

Formally, we may write the receiver CDS discounted value at time t as

$$D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\ \mathrm{L{\scriptsize GD}} \quad (1)$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t, and $\alpha_i = T_i - T_{i-1}$ or, more generally, α_i is the year fraction between T_{i-1} and T_i. The stochastic discount factor at time t for maturity T is denoted by $D(t,T) = B(t)/B(T)$, where $B(t) = \exp(\int_0^t r_u du)$ denotes the bank-account numeraire, r being the instantaneous short interest rate.

Sometimes a slightly different payoff is considered for CDS contracts. Instead of considering the exact default time τ, the protection payment $\mathrm{L{\scriptsize GD}}$ is postponed to the first time T_i following default, i.e. to $T_{\beta(\tau)}$. If the grid is three-or six months spaced, this postponement consists in a few months at worst. With this formulation, the CDS discounted payoff could be rewritten in a way that avoids the accrued-interest term in $(\tau - T_{\beta(\tau)-1})$ and brings in equivalence with approximated defaultable floaters, see Brigo (2004) for the details.

We denote by $\mathrm{CDS}(t, [T_{a+1}, \ldots, T_b], T_a, T_b, R, \mathrm{L{\scriptsize GD}})$ the price at time t of the above CDS. At times some terms are omitted, such as for example the list of payment dates $[T_{a+1}, \ldots, T_b]$. The pricing formula for this product depends on the assumptions on interest-rate dynamics and on the default time τ.

In general, we can compute the CDS price according to risk-neutral valuation (see for example Bielecki and Rutkowski (2001)):

$$\begin{aligned} \mathrm{CDS}(t, T_a, T_b, R, \mathrm{L{\scriptsize GD}}) = \mathbb{E}\Big\{ & D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} \\ & + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\ \mathrm{L{\scriptsize GD}} \Big| \mathcal{G}_t \Big\} \end{aligned} \quad (2)$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$, $\mathcal{F}_t$ denoting the basic filtration without default, typically representing the information flow of interest rates, intensities and possibly other default-free market quantities, and $\mathbb{E}$ denotes the risk-neutral expectation in the enlarged probability space supporting τ.

This expected value can also be written as

$$\mathrm{CDS}(t, T_a, T_b, R, \mathrm{L_{GD}}) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)} \mathbb{E}\left\{ D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\le T_b\}} D(t,\tau)\, \mathrm{L_{GD}} \right| \mathcal{F}_t \Bigg\} \tag{3}$$

(see again Bielecki and Rutkowski (2001) formula (5.1) p. 143).

Now we explain shortly how the market quotes CDS prices. Usually at time t, provided default has not yet occurred, the market sets R to a value $R^{\mathrm{MID}}_{a,b}(t)$ that makes the CDS fair at time t, i.e. such that $\mathrm{CDS}(t, T_a, T_b, R^{\mathrm{MID}}_{a,b}(t), \mathrm{L_{GD}}) = 0$. In fact, in the market CDS's are quoted at a time t through a bid and an ask value for this "fair" $R^{\mathrm{MID}}_{a,b}(t)$, for CDS's with $T_a = t$ and with T_b spanning a set of canonical final maturities, $T_b = t + 1y$ up to $T_b = t + 10y$. Recently the quoting mechanism has slightly changed and a periodic maturities roll-over has been adopted, similarly to what happens in some futures markets, see Brigo (2004). Brigo and Alfonsi (2003) illustrate in detail the notion of implied deterministic intensity (hazard function), satisfying

$$\mathbb{Q}\{s < \tau \le t\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)).$$

The market Γ's are obtained by inverting a pricing formula based on the assumption that τ is the first jump time of a Poisson process with intensity $\gamma(t) = d\Gamma(t)/dt$. In this case one can derive a formula for CDS prices based on integrals of γ, and on the initial interest-rate curve, resulting from the above expectation. One then can extract the γ's corresponding to CDS market quotes and obtain market implied γ^{mkt} and Γ^{mkt}'s. It is important to point out that usually the actual model one assumes for τ is more complex and may involve stochastic intensity. Even so, the γ^{mkt}'s are retained as a mere quoting mechanism for CDS rate market quotes, and are taken as inputs in the calibration of more complex models, as we shall see in Section 4.

We finally introduce CDS options. A payer CDS option is the right to enter into a payer CDS at its first reset time $T_a > t$ at a pre-specified strike rate $R = K$. Clearly this right will be exercised only if the payoff is positive at T_a, so that the discounted CDS option payoff reads, at time t,

$$D(t, T_a)[\mathrm{CDS}(T_a, T_b, R_{a,b}(T_a), \mathrm{L_{GD}}) - \mathrm{CDS}(T_a, T_b, K, \mathrm{L_{GD}})]^+. \tag{4}$$

We explicitly point out that we are assuming the offered protection amount $\mathrm{L_{GD}}$ not to depend on the CDS rate but only on the reference entity. By recalling that the fair CDS rate R makes the CDS value equal to zero, we have that in general

$$\mathrm{CDS}(t, T_a, T_b, R_{a,b}(t), \mathrm{L_{GD}}) = 0.$$

The idea is then solving this equation in $R_{a,b}(t)$. We resort to expression (3), equate it to zero and derive R correspondingly. Strictly speaking, the resulting R would be defined on $\{\tau > t\}$ only, since elsewhere we obtain zero thanks to the indicator in front of the expression, regardless of R. Since the value of R does not matter when $\{\tau < t\}$, the equation being satisfied automatically, we extend the value of R we find also to $\{\tau < t\}$.

To find R we equate to zero the part of the right hand side of expression (3) after the indicator. We thus find what we may call the "forward CDS rate"

$$R_{a,b}(t) = \frac{\mathrm{L}_{\mathrm{GD}}\ \mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\leq T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_i)+\mathbb{E}\left\{D(t,\tau)(\tau-T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t\right\}}, \quad (5)$$

where $\bar{P}(t,T) := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau>t|\mathcal{F}_t)$ and

$$\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau>t|\mathcal{F}_t) = \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T)$$

is the price at time t of a defaultable bond maturing at time T. In particular, from the above formula we can compute $R_{a,b}(T_a)$. Notice that $R_{a,b}(t)$ is $(\mathcal{F}_t)_t$ adapted. The above option payoff (4) can be rewritten in two different ways through some basic algebra and the definition of CDS. We may write it either as

$$\frac{1_{\{\tau>T_a\}}}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}D(t,T_a)\left[\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})\bar{P}(T_a,T_i)+\right. \quad (6)$$
$$\left.+\mathbb{E}\left\{D(T_a,\tau)(\tau-T_{\beta(\tau)-1})\mathbf{1}_{\{\tau<T_b\}}|\mathcal{F}_{T_a}\right\}\right](R_{a,b}(T_a)-K)^+$$

or, by remembering that by definition $\mathrm{CDS}(T_a,T_a,T_b,R_{a,b}(T_a),\mathrm{L}_{\mathrm{GD}})=0$, as

$$D(t,T_a)[-\mathrm{CDS}(T_a,T_a,T_b,K,\mathrm{L}_{\mathrm{GD}})]^+. \quad (7)$$

The quantity inside square brackets in (6) will play a key role in the following. We will often neglect the accrued interest term in $(\tau-T_{\beta(\tau)-1})$ and consider the related simplified payoff: in such a case the quantity between square brackets is denoted by $\widehat{C}_{a,b}(T_a)$ and is called "defaultable present value per basis point numeraire". More generally, at time t, we set

$$\widehat{C}_{a,b}(t) := \mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{C}_{a,b}(t), \quad \bar{C}_{a,b}(t) := \sum_{i=a+1}^{b}\alpha_i\bar{P}(t,T_i).$$

When including a survival indicator this quantity can be seen as a present value per basis point numeraire in the defaultable bonds. Neglecting the accrued interest term, the option discounted payoff simplifies to

$$1_{\{\tau>T_a\}}D(t,T_a)\left[\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)\right](R_{a,b}(T_a)-K)^+ \quad (8)$$

The same payoff is obtained as *exact* payoff when using postponed CDS formulations. For the details and for parallels with the LIBOR/SWAP market models see Brigo (2004).

3 A market model for CDS options

As usual, one would like to quote CDS options through the implied volatility of their underlying CDS rates R. In order to do so rigorously, one has to come up with an appropriate dynamics for $R_{a,b}$ directly, rather than modeling instantaneous default intensities explicitly. This somehow parallels what we find in the default-free interest rate market when we resort to the swap market model as opposed for example to a one-factor short-rate model for pricing swaptions. In a one-factor short-rate model the dynamics of the forward swap rate is a byproduct of the short-rate dynamics itself, through Ito's formula. On the contrary, the market model for swaptions directly postulates, under the relevant numeraire a (lognormal) dynamics for the forward swap rate. In the case of CDS options formulated in the context of this paper, the market model is derived in Brigo (2004). We do not repeat the derivation here, but present instead the resulting Black-like formula:

$$\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\bar{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+|\mathcal{G}_t\} = 1_{\{\tau>t\}}\bar{C}_{a,b}(t)[R_{a,b}(t)N(d_1(t))-KN(d_2(t))] \tag{9}$$

$$d_{1,2} = \left(\ln(R_{a,b}(t)/K) \pm (T_a-t)\sigma^2_{a,b}/2\right)/(\sigma_{a,b}\sqrt{T_a-t}).$$

This formula follows from assuming a dynamics

$$dR_{a,b}(t) = \sigma_{a,b}R_{a,b}(t)dW^{a,b}(t), \tag{10}$$

where $W^{a,b}$ is a Brownian motion under $\widehat{\mathbb{Q}}^{a,b}$, the measure associated with the numeraire $\widehat{C}_{a,b}$. As happens in most markets, this formula may be used as a quoting mechanism rather than as a real model formula. That is, the market price is converted into its implied volatility matching the given price when substituted in the above formula, and the market might quote CDS options through this implied volatility.

If we have no direct quote for the initial condition of the dynamics of $R_{a,b}$, we may compute its approximation from the market implied γ^{mkt} according to

$$R_{a,b}(0) = \frac{-\mathrm{L}_{\mathrm{GD}}\int_{T_a}^{T_b} P(0,u)d(e^{-\Gamma^{\text{mkt}}(u)})}{\sum_{i=a+1}^{b}\alpha_i P(0,T_i)e^{-\Gamma^{\text{mkt}}(T_i)}}$$

4 The SSRD model for CDS options

We recall briefly the SSRD model introduced in Brigo and Alfonsi (2003).

We write the short-rate r_t as a CIR++ process, i.e. as the sum of a deterministic function φ and of a Markovian process x_t^α:

$$r_t = x_t^\alpha + \varphi(t;\alpha)\ , \quad t \geq 0, \tag{11}$$

where φ depends on the parameter vector α (which includes x_0^α) and is integrable on closed intervals.

We take as reference model for x the Cox-Ingersoll-Ross (1985) process:

$$dx_t^\alpha = k(\theta - x_t^\alpha)dt + \sigma\sqrt{x_t^\alpha}dW_t,$$

where the parameter vector is $\alpha = (k, \theta, \sigma, x_0^\alpha)$, with k, θ, σ, x_0^α positive deterministic constants. The condition

$$2k\theta > \sigma^2$$

ensures that the origin is inaccessible to the reference model, so that the process x^α remains strictly positive. We may input the initial market interest rate curve into φ automatically, so as to calibrate the market curve exactly. We can then find the dynamic parameters α by fitting some cap prices. We set $\Phi(t, \alpha) := \int_0^t \varphi(s, \alpha)ds$.

For the intensity model we adopt a similar CIR++ model, in that we set

$$\lambda_t = y_t^\beta + \psi(t;\beta) \ , \quad t \geq 0, \tag{12}$$

where ψ is a deterministic function, depending on the parameter vector β (which includes y_0^β), that is integrable on closed intervals.

We take y again of the form:

$$dy_t^\beta = \kappa(\mu - y_t^\beta)dt + \nu\sqrt{y_t^\beta}dZ_t,$$

where the parameter vector is $\beta = (\kappa, \mu, \nu, y_0^\beta)$, with κ, μ, ν, y_0^β positive deterministic constants. Again we assume the origin to be inaccessible, i.e.

$$2\kappa\mu > \nu^2.$$

For restrictions on the β's that keep λ positive, as is required in intensity models, see Brigo and Mercurio (2001, 2001b). We will often use the integrated process, that is $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y^\beta(t) = \int_0^t y_s^\beta ds$ and $\Psi(t, \beta) = \int_0^t \psi(s, \beta)ds$.

The function ψ can take as inputs the market curve γ^{mkt} automatically, so as to calibrate CDS quotes exactly. The remaining dynamic parameters β are those who have impact on CDS options pricing. For the explicit formulae and automatic calibration of φ and ψ see Brigo and Alfonsi (2003). Here we only say that automatic calibration follows when computing φ and ψ from

$$\Phi(T, \beta) = \ln P^{\text{CIR}}(0, T; x_0, \alpha) - \ln P^{\text{Mkt}}(0, T), \quad \Psi(T, \beta) = \ln P^{\text{CIR}}(0, T; y_0, \beta) + \Gamma^{\text{mkt}}(T),$$

at all relevant T, where P^{CIR} is the bond price formula in the CIR standard model, $P^{\text{CIR}}(t, T; y_t, \beta) = A(t, T; \beta)\exp(-B(t, T; \beta)y_t)$ (and similarly for x), with A and B the classical expressions for the CIR model bond price (see for example Formula (3.25) in Brigo Mercurio (2001b)).

We take the short interest-rate and the intensity processes to be correlated, by assuming the driving Brownian motions W and Z to be instantaneously correlated according to

$$dW_t \, dZ_t = \rho \, dt.$$

This could in principle destroy the separated calibration paradigm summarized above. However, in Brigo and Alfonsi (2003) the issue is discussed at length and it is shown that, in practice, one can calibrate as above even in presence of nonzero correlation. It is shown that the parameter ρ has an impact on CDS valuation that is typically a fraction of the bid-ask spread, so that one may safely set $\rho = 0$ when pricing (or calibrating) CDS's.

Let us now consider the CDS option price under the SSRD model.

Valuing this contract with the CIR++ model when $\rho \neq 0$ can be a problem, since we have no closed form formula for $\bar{P}$ or the other terms at time T_a. We would thus be forced, in principle, to sub-simulate paths from T_a on just to be able to obtain the underlying asset of the option at T_a. This is computationally undesirable and we need to find alternatives. One way out is assuming zero correlation between interest rate and intensity from T_a on. Indeed, we have already seen that said correlation has almost no impact on CDS's, so that we may expect no real impact on the two CDS terms concurring to the payoff at T_a. Then, with zero correlation from T_a on, we have analytical expressions for the terms in the payoff and we may avoid simulations from T_a on. Compute

$$\begin{aligned}
\mathrm{CDS}(T_a, T_a, T_b, K, \mathrm{L{\scriptsize GD}}) = 1_{\{\tau > T_a\}} \mathbb{E}\Big\{ & D(T_a, \tau)(\tau - T_{\beta(\tau)-1}) K \mathbf{1}_{\{\tau < T_b\}} \\
& + \sum_{i=a+1}^{b} D(T_a, T_i) \alpha_i K \mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{\tau < T_b\}} D(T_a, \tau) \ \mathrm{L{\scriptsize GD}} | \mathcal{G}_{T_a} \Big\} \\
& = 1_{\{\tau > T_a\}} \Big\{ K \int_{T_a}^{T_b} \mathbb{E}\left[\exp\left(- \int_{T_a}^{u} (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_{T_a} \right] (u - T_{\beta(u)-1}) du \\
& + K \sum_{i=a+1}^{b} \alpha_i \mathbb{E}\left[\exp\left(- \int_{T_a}^{T_i} (r_s + \lambda_s) ds \right) | \mathcal{F}_{T_a} \right] \\
& - \mathrm{L{\scriptsize GD}} \int_{T_a}^{T_b} \mathbb{E}\left[\exp\left(- \int_{T_a}^{u} (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_{T_a} \right] du \Big\} \\
& := 1_{\{\tau > T_a\}} \mathrm{CDS}_{\mathcal{F}}(T_a, T_a, T_b, K, \mathrm{L{\scriptsize GD}}; x_{T_a}, y_{T_a}).
\end{aligned}$$

Assuming $\rho = 0$ from T_a on, the expectations appearing in the above expression can be computed as follows:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(- \int_{T_a}^{T_i} (r_s + \lambda_s) ds \right) | \mathcal{F}_{T_a} \right] = & \exp(\Psi(T_a, \beta) - \Psi(T_i, \beta)) P^{\mathrm{CIR}}(T_a, T_i; y_{T_a}, \beta) \times \\
& \times \exp(\Phi(T_a, \alpha) - \Phi(T_i, \alpha)) P^{\mathrm{CIR}}(T_a, T_i; x_{T_a}, \alpha). \quad (13)
\end{aligned}$$

Further, we may compute

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}(r_s+\lambda_s)ds\right)\lambda_u|\mathcal{F}_{T_a}\right] &= \quad (14)\\ = \mathbb{E}\left[\exp\left(-\int_{T_a}^{u}r_s ds\right)|\mathcal{F}_{T_a}\right]\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s ds\right)\lambda_u|\mathcal{F}_{T_a}\right] &= \\ = \mathbb{E}\left[\exp\left(-\int_{T_a}^{u}r_s ds\right)|\mathcal{F}_{T_a}\right]\left(-\frac{d}{du}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s ds\right)|\mathcal{F}_{T_a}\right]\right) &= \\ = -\exp(\Phi(T_a,\alpha)-\Phi(u,\alpha))P^{\text{CIR}}(T_a,u;x_{T_a},\alpha) &\times \\ \times\frac{d}{du}\left[\exp(\Psi(T_a,\beta)-\Psi(u,\beta))P^{\text{CIR}}(T_a,u;y_{T_a},\beta)\right]&\end{aligned}$$

so that all terms are known analytically given the simulated paths of x_{T_a} and y_{T_a}, which are to be simulated with nonzero ρ from time 0 to time T_a. Putting all pieces together, without forgetting the indicator $1_{\{\tau>T_a\}}$, we may value the CDS option payoff (7) by simulation.

$$\begin{aligned}&\mathbb{E}\left[D(t,T_a)[-\text{CDS}(T_a,T_a,T_b,K,\text{L}_{\text{GD}})]^+|\mathcal{G}_t\right]\\ &= \mathbb{E}\left[D(t,T_a)1_{\{\tau>T_a\}}[-\text{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\text{L}_{\text{GD}};x_{T_a},y_{T_a})]^+|\mathcal{G}_t\right]\\ &= \frac{1_{\{\tau>t\}}}{\exp(-\Lambda(t))}\mathbb{E}\left[D(t,T_a)1_{\{\tau>T_a\}}[-\text{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\text{L}_{\text{GD}};x_{T_a},y_{T_a})]^+|\mathcal{F}_t\right]\\ &= 1_{\{\tau>t\}}\mathbb{E}\left[D(t,T_a)\exp(-\Lambda(T_a)+\Lambda(t))[-\text{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\text{L}_{\text{GD}};x_{T_a},y_{T_a})]^+|\mathcal{F}_t\right]\\ &= \boxed{1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^{T_a}(r_s+\lambda_s)ds\right)[-\text{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\text{L}_{\text{GD}};x_{T_a},y_{T_a})]^+|\mathcal{F}_t\right]} \quad (15)\end{aligned}$$

The assumption above that $\rho=0$ from T_a on allows us to compute the $\mathcal{F}$-measurable part of the CDS payoff, i.e. $\text{CDS}_{\mathcal{F}}$, as a function of the simulated x_{T_a} and y_{T_a} without further simulation from T_a to T_b. It suffices to use formulas (13) and (14). However, we have to check that we can set $\rho=0$ from T_a on. We know from Brigo and Alfonsi (2003) that ρ has little impact on "at the money" CDS contracts valued at time 0. We plan to check whether this is the case with the option payoff from T_a on. We will thus compute the option price both by taking $\rho=0$ from T_a on and by keeping the nonzero ρ also in $[T_a,T_b]$. In the latter case we can resort to the "sub-path" method. We simulate n paths of λ and r from 0 to T_a, and then for each T_a realization we subsimulate m paths up to T_b to compute the inner discounted payoff at T_a conditional on the T_a scenario. We need a way to compute the standard error of the Monte Carlo method. In our tests below $n=50000$ and $m=5000$.

4.1 SSRD standard error upper bound under nonzero correlation

Write the above option payoff at maturity T_a by collecting the expected values as

$$\mathrm{CDS}(T_a, T_a, T_b, K, \mathrm{L}_{\mathrm{GD}}) = 1_{\{\tau > T_a\}} \Big\{ \mathbb{E} \Big[-K \int_{T_a}^{T_b} \exp\left(-\int_{T_a}^{u} (r_s + \lambda_s) ds\right) \lambda_u (u - T_{\beta(u)-1}) du$$
$$-K \sum_{i=a+1}^{b} \alpha_i \exp\left(-\int_{T_a}^{T_i} (r_s + \lambda_s) ds\right) + \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} \exp\left(-\int_{T_a}^{u} (r_s + \lambda_s) ds\right) \lambda_u du \Big| \mathcal{F}_{T_a} \Big] \Big\}^+$$

Call X the part of this expression inside the expectation after the indicator, i.e. the part inside the expectation inside the curly brackets. The CDS option price can be written as

$$\mathbb{E}\left[\exp\left(-\int_0^{T_a} (r_s + \lambda_s) ds\right) (E_{T_a} X)^+\right] = \mathbb{E}\left[Y \cdot (E_{T_a} X)^+\right]$$

where Y denotes the exponential term. The method we use is generate some scenarios ω_i, $i = 1, \ldots, n$ for r and λ up to T_a. Then, conditional on each such ω_i, we generate m subpaths $\omega_{i,j}$, $j = 1, \ldots, m$ for r and λ from T_a to T_b. We call Y^i the realization of Y corresponding to ω_i and $X^{i,j}$ the realization of X corresponding to $\omega_{i,j}$.

Our Monte Carlo estimate for the above price will then be

$$\Pi_{MC} = \frac{1}{n} \sum_{i=1}^{n} Y^i \left(\frac{1}{m} \sum_{j=1}^{m} X^{i,j}\right)^+$$

Under a large number of scenarios, the central limit theorem tells us that this Π_{MC} is approximately normal. Thus, if we find an upper bound for its standard deviation we may find conservative windows for the Monte Carlo error and conservative confidence intervals around the true mean, i.e. around the price we seek.

Compute then said variance.

$$var(\Pi_{MC}) = var\left(\frac{1}{n} \sum_{i=1}^{n} Y^i \left(\frac{1}{m} \sum_{j=1}^{m} X^{i,j}\right)^+\right) = \ldots$$

Since the paths ω_i are independent, we may add variances with respect to different ω_i's,

$$\ldots = \frac{1}{n^2} \sum_{i=1}^{n} var\left(Y^i \left(\frac{1}{m} \sum_{j=1}^{m} X^{i,j}\right)^+\right) = \ldots$$

Now we use a first bound. In general, it is easy to show that, given a random variable Z, we have $var(Z^+) < var(Z)$ if $E(Z^+) + E(Z) > 0$. Assuming the condition to hold (one may check it on the simulated sample, more on this later) we may then write

$$\ldots \le \frac{1}{n^2} \sum_{i=1}^{n} var\left(Y^i \left(\frac{1}{m} \sum_{j=1}^{m} X^{i,j}\right)\right) = \frac{1}{n^2 m^2} \sum_{i=1}^{n} var\left(\sum_{j=1}^{m} (Y^i X^{i,j})\right) = \ldots$$

Now we are computing the variance of a summation of correlated variables. This has as upper bound the case where all correlations are one, corresponding to adding up standard deviations and squaring:

$$... \leq \frac{1}{n^2m^2}\sum_{i=1}^{n}\left(\sum_{j=1}^{m} stdv(Y^iX^{i,j})\right)^2 = ...$$

Since the "i" samples are i.i.d., all the above standard deviations are equal to each other. Thus, if we call "stdvxy" such standard deviation, we obtain

$$... = \frac{1}{n^2m^2}\sum_{i=1}^{n}(m \cdot stdvxy)^2 = \frac{1}{n} stdvxy^2$$

The sample stdvxy above may be computed from the simulated sample: indeed, take the simulated realizations and compute the standard deviation of the discrete random variable taking the following values, each with $1/(n\ m)$ probability:

$$Y^1X^{1,1},\ Y^1X^{1,2},\ Y^1X^{1,3},\ldots,\ Y^1X^{1,m}$$

$$Y^2X^{2,1},\ Y^2X^{2,2},\ Y^2X^{2,3},\ldots,\ Y^2X^{2,m}$$

$$....$$

$$Y^iX^{i,1},\ Y^iX^{i,2},\ Y^iX^{i,3},\ldots,\ Y^iX^{i,m}$$

$$...$$

$$Y^nX^{n,1},\ Y^nX^{n,2},\ Y^nX^{n,3},\ldots,\ Y^nX^{n,m}$$

This standard deviation can be computed through cumulated quantities, so that it is not necessary to store all the paths.

We get

$$MCerr = \frac{stdvxy}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{(x^{i,j}\ y^i)^2}{nm} - \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{x^{i,j}\ y^i}{nm}\right)^2}$$

where x and y are the simulated realizations of X and Y (not to be confused with the processes of the interest rate and intensity). As one simulates paths and subpaths, it is best to keep a cumulated variable updating the sum of terms $x^{i,j}\ y^i$ and a cumulated variable also for the sum of $(x^{i,j}\ y^i)^2$.

The last thing one has to check, to make sure things work, is that our assumption $E(Z^+) + E(Z) > 0$ applies. We need to check that

$$E\left[\left(Y_i\frac{1}{m}\sum_{j=1}^{m}X^{i,j}\right)^+\right] + E\left[\left(Y_i\frac{1}{m}\sum_{j=1}^{m}X^{i,j}\right)\right] > 0$$

Again, we may decide to test this condition on the simulated sample itself. For each i we use the simulated sampled subpaths to compute the means $\sum_{j=1}^{m} x^{i,j}/m = \mu_i$, and then check that

$$\frac{1}{n}\sum_{i=1}^{n} y_i(\mu_i)^+ + \frac{1}{n}\sum_{i=1}^{n} y_i(\mu_i) > 0$$

This roughly amounts to say that the estimated CDS option price plus the opposite of the corresponding forward-start CDS price (with a CDS rate set to K) gives a positive value. Note that if the μ_i's are strongly negative (as may happen for large K) then our assumption may not hold. We used this MC error bounds successfully in our subsequent tests.

5 A Formula linking the Market and SSRD models

We develop an approximated formula based on the assumption of null instantaneous correlation ρ between stochastic interest rates r and intensities λ. A particular case is given by deterministic rates r.

First we derive an approximated formula for the volatility of $R_{a,b}(t)$ under the CIR++ model in case of zero correlation $\rho = 0$. In case of the CIR++ model for λ independent of r, Formula (5) (with postponed payoff or by ignoring the accruing term in the denominator) reads

$$R_{a,b}(t) = \frac{\text{L}_{\text{GD}} \int_{T_a}^{T_b} \mathbb{E}[\lambda_u \exp(-\int_t^u r_s ds - \int_0^u \lambda_s ds)|\mathcal{F}_t] du}{\sum_{i=a+1}^{b} \alpha_i \exp(-\int_0^t \lambda_s ds)\mathbb{E}[\exp(-\int_t^{T_i}(r_s+\lambda_s)ds)|\mathcal{F}_t]} \qquad (16)$$
$$= \frac{\text{L}_{\text{GD}} \int_{T_a}^{T_b} \mathbb{E}[\lambda_u \exp(-\int_t^u (r_s+\lambda_s) ds)|\mathcal{F}_t] du}{\sum_{i=a+1}^{b} \alpha_i \mathbb{E}[\exp(-\int_t^{T_i}(r_s+\lambda_s)ds)|\mathcal{F}_t]}$$

Under the SSRD assumptions for λ and r this simplifies to:

$$R_{a,b}(t) = \frac{\text{L}_{\text{GD}} \int_{T_a}^{T_b} P(t,u)\mathbb{E}[\lambda_u \exp(-\int_t^u \lambda_s ds)|\mathcal{F}_t] du}{\sum_{i=a+1}^{b} \alpha_i P(t,T_i)\mathbb{E}[\exp(-\int_t^{T_i}\lambda_s ds)|\mathcal{F}_t]} \qquad (17)$$
$$= -\frac{\text{L}_{\text{GD}} \int_{T_a}^{T_b} \exp(\Phi(t,\alpha)-\Phi(u,\alpha))P^{\text{CIR}}(t,u;x_t,\alpha)\frac{d}{du}\Big[\exp(\Psi(t,\beta)-\Psi(u,\beta))P^{\text{CIR}}(t,u;y_t,\beta)\Big]du}{\sum_{i=a+1}^{b} \alpha_i \exp(\Phi(t,\alpha)-\Phi(T_i,\alpha))P^{\text{CIR}}(t,T_i;x_t,\alpha)\exp(\Psi(t,\beta)-\Psi(T_i,\beta))P^{\text{CIR}}(t,T_i;y_t,\beta)}$$
$$:= R_{a,b}(t;x_t,y_t,\alpha,\beta). \qquad (18)$$

At times we omit α and β as arguments of $R_{a,b}$. Consider the instantaneous return variance of R, i.e. the quadratic covariation

$$
\begin{aligned}
d\langle \ln R_{a,b}(\cdot; x_\cdot, y_\cdot)\rangle_t &= \left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial x}\right)^2 d\langle x, x\rangle_t + \left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial y}\right)^2 d\langle y, y\rangle_t \\
&+ 2\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial x}\right)\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial y}\right) d\langle x, y\rangle_t \\
&= \left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial x}\right)^2 \sigma^2 x_t \, dt + \left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial y}\right)^2 \nu^2 y_t \, dt \\
&+ 2\rho\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial x}\right)\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t)}{\partial y}\right) \sigma\nu\sqrt{x_t y_t} \, dt
\end{aligned}
$$

Since in the market model (10) we have

$$
d\langle \ln R_{a,b}(\cdot)\rangle_t = \sigma_{a,b}^2 dt,
$$

we may consider

$$
\begin{aligned}
\sigma_{a,b}^{\text{CIR++}}(t)^2 := &\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t, \alpha, \beta)}{\partial x}\right)^2 \sigma^2 x_t + \left(\frac{\partial \ln R_{a,b}(t; x_t, y_t, \alpha, \beta)}{\partial y}\right)^2 \nu^2 y_t \\
&+2\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t, \alpha, \beta)}{\partial x}\right)\left(\frac{\partial \ln R_{a,b}(t; x_t, y_t, \alpha, \beta)}{\partial y}\right) \rho\, \sigma\nu\sqrt{x_t y_t}
\end{aligned}
\tag{19}
$$

as a proxy, in the CIR++ model, for the market model volatility. Of course, while in the market model this volatility is deterministic, here it is a random variable due to the presence of x and y. Notice also that the above approximated formula for R in the SSRD model holds only for $\rho = 0$, since our formula for $R_{a,b}$ in the SSRD model holds only under $\rho = 0$. We have to set $\rho = 0$ in (19). However, we might "cheat" and use the above approximation even when $\rho \neq 0$, although this may lead to a worsening of the approximation. *We are not doing so in the present paper and, when applying the above formula, we take always $\rho = 0$ even if the Monte Carlo prices are computed with $\rho \neq 0$.*

The average return-volatility of R in the CIR++ model would thus be a random variable given by the square root of $(1/T_a)\int_0^{T_a} \sigma_{a,b}^{\text{CIR++}}(t)^2 dt$. However, we aim at a fast approximation which can be computed without simulation. To obtain such an approximation, we replace any occurrence of x_t and y_t in (19) by the respective expectations at time 0. We compute then the volatility $v_{a,b}$ according to

$$
\begin{aligned}
v_{a,b}^{CIR++}(\alpha, \beta)^2 := \frac{1}{T_a}\Bigg[&\int_0^{T_a} \left(\frac{\partial \ln R_{a,b}(t; \mathbb{E}_0(x_t), \mathbb{E}_0(y_t))}{\partial x}\right)^2 \sigma^2 \mathbb{E}_0(x_t) dt \\
&+ \int_0^{T_a} \left(\frac{\partial \ln R_{a,b}(t; \mathbb{E}_0(x_t), \mathbb{E}_0(y_t))}{\partial y}\right)^2 \nu^2 \mathbb{E}_0(y_t) dt \Bigg],
\end{aligned}
\tag{20}
$$

where for example $\mathbb{E}_0(y_t) = y_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$. One may wonder about which term in the above approximation is larger. In case we consider also deterministic interest rates, the first integral has to be omitted and the formula simplifies. We may anticipate that in all our subsequent tests, the first integral gives a much smaller contribution than the second one. Typically the difference is smaller that 0.1%, so that when the total formula gives us a volatility 22.3%, the formula with only the second term would give us 22.2%. This points out that what contributes to the volatility of the CDS rate is the intensity stochasticity, while the interest-rate stochasticity has almost no impact on it. This, however, might change in presence of nonzero ρ so that we keep stochastic r in our tests. Below we give the results for the stochastic r case.

Again in the CIR++ model, in general, we have what we may call "SSRD-implied CDS-rate volatility", resulting from backing out the volatility from the market formula (9) for CDS option prices in correspondence of the SSRD model option price (15).

In other terms, at time 0 we solve the equation

$$E_0\left[e^{-\int_0^{T_a}(r_s+\lambda_s)ds}[-\mathrm{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\mathrm{L{\scriptstyle GD}};x_{T_a},y_{T_a})]^+\right] =$$
$$= \bar{C}_{a,b}(0)[R_{a,b}(0)N(d_1(v^{imp}_{a,b}(\alpha,\beta,\rho))) - KN(d_2(v^{imp}_{a,b}(\alpha,\beta,\rho)))]$$

in $v^{imp}_{a,b}(\alpha,\beta,\rho)$. We do so by valuing the left hand side (depending on α,β and ρ in general) through Monte Carlo simulation.

The first investigation we are interested in is understanding the qualitative dependence of the implied volatility as a function of the model parameters κ,μ,ν,y_0,ρ and also of strike K. We assess this dependence via Monte Carlo simulation. The simulation is however much easier, as explained earlier, if we are allowed to set $\rho=0$ from T_a on. We check this a posteriori and find that this is possible but mostly for negative ρ. En passant, we test how close $v^{CIR++}_{a,b}(\alpha,\beta)$ is to $v^{imp}_{a,b}(\alpha,\beta,\rho)$. The approximation can be helpful for a number of reasons. First, in all situations where the two quantities are close, we may have a first quick analytical valuation of a CDS option in the SSRD model with no need for Monte Carlo simulation. Secondly, the formula provides us with a market quantity linked to the CIR++ dynamical parameters β. Model parameters are not too useful to practitioners, unless they can be translated into views on market quantities. In this sense, it is also useful to have an idea of the impact of each single model parameter onto market quantities. The correct market quantity associated to the SSRD model would be $v^{imp}_{a,b}$, but checking the impact of changing say κ onto this quantity can be time-consuming, given the need to perform a Monte Carlo simulation. However, if we know the approximated quantity $v^{CIR++}_{a,b}(\alpha,\beta)$ patterns to be reliable, we may use it to check the impact of the model parameters, since in this case we may re-value this quantity for different parameter values analytically. The formula for $v^{CIR++}_{a,b}(\alpha,\beta)$ may thus provide us a quick means to translate the CIR++ parameters changes in market patterns.

6 Numerical tests and Results

6.1 Market data and Simulation Setup

Below we report the points in the (time,intensity) dimension determining the deterministic piecewise linear hazard rates $(t, \gamma^{\text{mkt}}(t))$ implied by CDS quotes for Parmalat on June 26^{th}, 2003:

Time	γ^{mkt} of Parmalat
26-Jun-03	0.0374016
28-Jun-04	0.0413386
27-Jun-06	0.0442196
27-Jun-08	0.0446496

The corresponding survival probabilities $(t, e^{-\Gamma^{\text{mkt}}(t)})$ are:

Time	Survival Probability of Parmalat
26-Jun-03	1
28-Jun-04	0.96108375
27-Jun-06	0.882378007
27-Jun-08	0.807246935

The deterministic piecewise linear hazard rates implied by CDS quotes for Peugeot on June 26^{th}, 2003:

Time	γ^{mkt} of Peugeot
26-Jun-03	0.00208589
28-Jun-04	0.00486707
27-Jun-06	0.0070899
27-Jun-08	0.00948182

The corresponding survival probabilities:

Time	Survival Probability of Peugeot
26-Jun-03	1
28-Jun-04	0.996501081
27-Jun-06	0.98467302
27-Jun-08	0.968467763

We take the Euro default free interest rate curve of the same day $(t, P(0, t))$:

Maturity	Default Free Zero Coupon Bond Price
26-Jun-03	1
27-Jun-03	0.999940837
1-Jul-03	0.999704219
7-Jul-03	0.999340342
30-Jul-03	0.997963707
29-Aug-03	0.996180439
30-Sep-03	0.994325513
17-Dec-03	0.990175175
17-Mar-04	0.985423722
17-Jun-04	0.980568545
16-Sep-04	0.975429577
15-Dec-04	0.969927081
15-Mar-05	0.963980526
16-Jun-05	0.957470682
15-Sep-05	0.950727077
30-Jun-06	0.927244335
29-Jun-07	0.894148369
30-Jun-08	0.858510843

As concerns the Monte Carlo method, all the following simulations are obtained by means of 50,000 paths, under variance reduction techniques, for the relevant stochastic processes x and y. In all simulations, the α parameters of the EURO interest-rate curve are set to

$$k = 0.4, \theta = 0.026, \sigma = 14\%, x_0 = 0.0165,$$

reflecting a possible calibration to Cap volatilities.

6.2 CDS option with maturity 1y on a CDS lasting 4y

6.2.1 Calibrating ψ to Parmalat CDS Data

The At-the-money CDS option we consider here has the following features:

T_a	1 year
T_b	5 years
$T_{a+i+1} - T_{a+i}$	6 months
K	311 bp
L_{GD}	70 %

With $\mu = 0.045$, $\nu = 15\%$, $y_0 = 0.035$, $\rho = 0$ being fixed, we change κ:

κ	$v_{a,b}^{imp}(\kappa)$	$v_{a,b}^{CIR++}(\kappa)$	$v_{a,b}^{imp}(\kappa) - v_{a,b}^{CIR++}(\kappa)$
0.35	29.6 (29.4 ; 29.9) %	30.1 %	-0.5 (-0.7 ; -0.2) %
0.45	25.0 (24.8 ; 25.2) %	25.4 %	-0.4 (-0.6 ; -0.2) %
0.55	21.3 (21.1 ; 21.5) %	21.7 %	-0.4 (-0.6 ; -0.2) %
0.65	18.4 (18.2 ; 18.5) %	18.7 %	-0.3 (-0.5 ; -0.2) %
0.75	16.0 (15.8 ; 16.1) %	16.3 %	-0.3 (-0.5 ; -0.2) %

With $\kappa = 0.5$, $\nu = 15\%$, $y_0 = 0.037$, $\rho = 0$ being fixed, we change μ:

μ	$v_{a,b}^{imp}(\mu)$	$v_{a,b}^{CIR++}(\mu)$	$v_{a,b}^{imp}(\mu) - v_{a,b}^{CIR++}(\mu)$
0.025	21.9 (21.7 ; 22.1) %	22.4 %	-0.5 (-0.7 ; -0.3) %
0.03	22.3 (22.1 ; 22.5) %	22.8 %	-0.5 (-0.7 ; -0.3) %
0.035	22.7 (22.5 ; 22.9) %	23.1 %	-0.4 (-0.6 ; -0.2) %
0.04	23.1 (22.9 ; 23.3) %	23.5 %	-0.4 (-0.6 ; -0.2) %
0.045	23.5 (23.3 ; 23.7) %	23.9 %	-0.4 (-0.6 ; -0.2) %

With $\kappa = 0.5$, $\mu = 0.046$, $y_0 = 0.036$, $\rho = 0$ being fixed, we change ν:

ν	$v_{a,b}^{imp}(\nu)$	$v_{a,b}^{CIR++}(\nu)$	$v_{a,b}^{imp}(\nu) - v_{a,b}^{CIR++}(\nu)$
11 %	17.5 (17.4 ; 17.7) %	17.7 %	-0.2 (-0.4 ; 0.0) %
13 %	20.5 (20.3 ; 20.7) %	20.8 %	-0.3 (-0.5 ; -0.1) %
15 %	23.4 (23.2 ; 23.6) %	23.7 %	-0.3 (-0.5 ; -0.1) %
17 %	26.1 (25.9 ; 26.3) %	26.7 %	-0.6 (-0.8 ; -0.4) %
19 %	28.7 (28.5 ; 28.9) %	29.5 %	-0.8 (-1.0 ; -0.6) %
21 %	31.1 (31.0 ; 31.4) %	32.3 %	-1.2 (-1.3 ; -0.9) %

With $\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $\rho = 0$ being fixed, we change y_0:

y_0	$v_{a,b}^{imp}(y_0)$	$v_{a,b}^{CIR++}(y_0)$	$v_{a,b}^{imp}(y_0) - v_{a,b}^{CIR++}(y_0)$
0.012	21.5 (21.4 ; 21.7) %	22.8 %	-1.3 (-1.4 ; -1.1) %
0.017	23.5 (23.3 ; 23.7) %	24.7 %	-1.2 (-1.4 ; -1.0) %
0.022	25.4 (25.2 ; 25.6) %	26.5 %	-1.1 (-1.3 ; -0.9) %
0.027	27.2 (27.0 ; 27.4) %	28.2 %	-1.0 (-1.2 ; -0.8) %
0.032	28.8 (28.6 ; 29.0) %	29.8 %	-1.0 (-1.2 ; -0.8) %
0.037	30.4 (30.2 ; 30.6) %	31.3 %	-0.9 (-1.1 ; -0.7) %

With $\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $y_0 = 0.037$, $\rho = 0$ being fixed, we change K:

K	$v_{a,b}^{imp}(K)$	$v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K) - v_{a,b}^{CIR++}$
251 bps	27.6 (27.4 ; 27.8) %	31.3 %	-3.7 (-3.9 ; -3.5) %
311 bps (ATM)	30.4 (30.2 ; 30.6) %	31.3 %	-0.9 (-1.1 ; - 0.7) %
371 bps	31.8 (31.5 ; 32.1) %	31.3 %	0.5 (0.2 ; 0.8) %
431 bps	32.6 (32.2 ; 32.9) %	31.3 %	1.3 (0.9 ; 1.6) %

With $\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $y_0 = 0.037$ being fixed we change ρ and K. Here $v_{a,b}^{CIR++} = 31.3\%$ and does not depend either on K or on ρ.

	$v_{a,b}^{imp}(K, -1)$	$v_{a,b}^{imp}(K, \rho = 0)$	$v_{a,b}^{imp}(K, 1)$
$K = 251$ bps	29.2 (29.0 ; 29.3) %	27.6 (27.4 ; 27.8) %	25.5 (25.3 ; 25.7) %
$K = 311$ bps (ATM)	31.0 (30.9 ; 31.2) %	30.4 (30.2 ; 30.6) %	29.4 (29.2 ; 29.5) %
$K = 371$ bps	32.0 (31.8 ; 32.2) %	31.8 (31.5 ; 32.1) %	31.0 (30.8 ; 31.2) %

	$v_{a,b}^{imp}(K, -1) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K, 0) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K, 1) - v_{a,b}^{CIR++}$
$K = 251$ bps	-2.1 (-2.3 ; -2.0) %	-3.7 (-3.9; -3.5) %	-5.8 (-6.0 ; -5.6) %
$K = 311$ bps (ATM)	-0.3 (-0.4 ; -0.1) %	-0.9 (-1.1 ; -0.7) %	-1.9 (-2.1 ; -1.8) %
$K = 371$ bps	0.7 (0.5 ; 0.9) %	0.5 (0.2 ; 0.8) %	-0.3 (-0.5 ; -0.1) %

In the next table, we give the coupon strikes, $K_{equiv}(K, 1)$ (resp. $K_{equiv}(K, -1)$) that match, in a 0-correlation world, the option prices obtained for a strike K and a correlation of 1 (resp. -1). In other terms, we solve
$\text{CDSOption}(K_{equiv}(K, 1), \rho = 0) = \text{CDSOption}(K, \rho = 1)$ and
$\text{CDSOption}(K_{equiv}(K, -1), \rho = 0) = \text{CDSOption}(K, \rho = -1)$ respectively.

	$K_{equiv}(K, -1)$	$K_{equiv}(K, 1)$	$K_{equiv}(K, 1) - K_{equiv}(K, -1)$
$K = 251$ bps	(248 ; 250) bps	(253 ; 254) bps	(3 ; 6) bps
$K = 311$ bps (ATM)	(308 ; 311) bps	(313 ; 316) bps	(2 ; 8) bps
$K = 371$ bps	(368 ; 371) bps	(374 ; 378) bps	(3 ; 10) bps

With the same parameters, we investigate in the following table the impact of non-zero correlation from T_a on for ATM options. The prices are obtained with 50,000 paths and the CDS prices at T_a are approximated with 5,000 paths when the correlation is not zero after T_a.

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	30.7 (29.2 ; 32.2) %	31.0 (30.9 ; 31.2) %	—
$\rho_{0,T_a} = 0$	31.7 (30.2 ; 33.2) %	30.4 (30.2 ; 30.6) %	30.0 (28.5 ; 31.4) %
$\rho_{0,T_a} = 1$	—	29.4 (29.2 ; 29.5) %	24.8 (23.3 ; 26.2) %

In the next table, we give the above defined K_{equiv}'s:

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	(305 ; 316) bps	(308 ; 311) bps %	—
$\rho_{0,T_a} = 0$	(302 ; 312) bps	311 bps	(307 ; 318) bps
$\rho_{0,T_a} = 1$	—	(313 ; 316) bps	(324 ; 337) bps

6.2.2 Calibrating ψ to Peugeot CDS Data

The At-the-money CDS option we consider here has the following features:

T_a	1 year
T_b	5 years
$T_{a+i+1} - T_{a+i}$	6 months
K	50 bp
LGD	70 %

With $\mu = 0.0071$, $\nu = 7\%$, $y_0 = 0.0019$, $\rho = 0$ being fixed, we change κ:

κ	$v_{a,b}^{imp}(\kappa)$	$v_{a,b}^{CIR++}(\kappa)$	$v_{a,b}^{imp}(\kappa) - v_{a,b}^{CIR++}(\kappa)$
0.35	23.0 (22.8 ; 23.2) %	24.1 %	-1.1 (-1.3 ; -0.9) %
0.4	21.5 (21.3 ; 21.7) %	22.4 %	-0.9 (-1.1 ; -0.7) %
0.5	18.8 (18.6 ; 18.9) %	19.5 %	-0.7 (-0.9 ; -0.6) %
0.6	16.5 (16.4 ; 16.7) %	17.1 %	-0.6 (-0.7 ; -0.4) %
0.7	14.7 (14.5 ; 14.8) %	15.1 %	-0.4 (-0.6 ; -0.3) %

With $\kappa = 0.5$, $\nu = 7\%$, $y_0 = 0.0017$, $\rho = 0$ being fixed, we change μ:

μ	$v_{a,b}^{imp}(\mu)$	$v_{a,b}^{CIR++}(\mu)$	$v_{a,b}^{imp}(\mu) - v_{a,b}^{CIR++}(\mu)$
0.005	16.5 (16.3 ; 16.6) %	17.4 %	-0.9 (-1.1 ; -0.8) %
0.006	17.4 (17.2 ; 17.5) %	18.2 %	-0.8 (-1.0 ; -0.7) %
0.007	18.2 (18.0 ; 18.4) %	19.0 %	-0.8 (-1.0 ; -0.6) %
0.0083	19.3 (19.1 ; 19.4) %	20.0 %	-0.7 (-0.9 ; -0.6) %

With $\kappa = 0.8$, $\mu = 0.007$, $y_0 = 0.0012$, $\rho = 0$ being fixed, we change ν:

ν	$v_{a,b}^{imp}(\nu)$	$v_{a,b}^{CIR++}(\nu)$	$v_{a,b}^{imp}(\nu) - v_{a,b}^{CIR++}(\nu)$
6 %	10.6 (10.5 ; 10.7) %	10.8 %	-0.2 (-0.3 ; -0.1) %
7 %	12.2 (12.1 ; 12.4) %	12.6 %	-0.4 (-0.5 ; -0.2) %
8 %	13.8 (13.7 ; 14.0) %	14.4 %	-0.6 (-0.7 ; -0.4) %
9 %	15.4 (15.2 ; 15.6) %	16.2 %	-0.8 (-1.0 ; -0.6) %
10 %	16.9 (16.8 ; 17.0) %	18.0 %	-1.1 (-1.2 ; -1.0) %

With $\kappa = 0.5$, $\mu = 0.0083$, $\nu = 9\%$, $\rho = 0$ being fixed, we change y_0:

y_0	$v_{a,b}^{imp}(y_0)$	$v_{a,b}^{CIR++}(y_0)$	$v_{a,b}^{imp}(y_0) - v_{a,b}^{CIR++}(y_0)$
0.0004	20.0 (19.8 ; 20.2) %	21.4 %	-1.4 (-1.6 ; -1.2) %
0.0009	21.6 (21.4 ; 21.8) %	23.1 %	-1.5 (-1.7 ; -1.3) %
0.0014	23.2 (23.0 ; 23.4) %	24.7 %	-1.5 (-1.7 ; -1.3) %
0.0019	24.6 (24.4 ; 24.8) %	26.1 %	-1.5 (-1.7 ; -1.3) %

With $\kappa = 0.5$, $\mu = 0.0083$, $\nu = 9\%$, $y_0 = 0.0019$, $\rho = 0$ being fixed, we change K:

K	$v_{a,b}^{imp}(K)$	$v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K) - v_{a,b}^{CIR++}$
40 bps	18.1 (18.0 ; 18.3) %	26.1 %	-8.0 (-8.1 ; -7.8) %
50 bps (ATM)	24.6 (24.4 ; 24.8) %	26.1 %	-1.5 (-1.7 ; - 1.3) %
60 bps	27.5 (27.2 ; 27.8) %	26.1 %	1.4 (1.1 ; 1.7) %
70 bps	29.1 (28.7 ; 29.4) %	26.1 %	3.0 (2.6 ; 3.3) %
80 bps	30.0 (29.6 ; 30.4) %	26.1 %	3.9 (3.5 ; 4.3) %

With $\kappa = 0.5$, $\mu = 0.0083$, $\nu = 9\%$, $y_0 = 0.0019$ being fixed, we change ρ and K. Here $v_{a,b}^{CIR++} = 26.1\%$ and does not depend either on K or on ρ.

	$v_{a,b}^{imp}(K, -1)$	$v_{a,b}^{imp}(K, 0)$	$v_{a,b}^{imp}(K, 1)$
$K = 45$ bps	23.2 (23.1 ; 23.3) %	22.2 (22.1 ; 22.4) %	20.9 (20.7 ; 21.0) %
$K = 50$ bps (ATM)	25.2 (25.1 ; 25.4) %	24.6 (24.4 ; 24.8) %	23.7 (23.6 ; 23.9) %
$K = 55$ bps	26.7 (26.5 ; 26.9) %	26.3 (26.0 ; 26.5) %	25.5 (25.3 ; 25.7) %

	$v_{a,b}^{imp}(K, -1) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K, 0) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K, 1) - v_{a,b}^{CIR++}$
$K = 45$ bps	-2.9 (-3.0 ; -2.8) %	-3.9 (-4.0 ; -3.7) %	-5.2 (-5.4 ; -5.1) %
$K = 50$ bps (ATM)	-0.9 (-1.0 ; -0.7) %	-1.5 (- 1.7 ; -1.3) %	-2.4 (-2.5 ; -2.2) %
$K = 55$ bps	0.6 (0.4 ; 0.8) %	0.2 (-0.1 ; 0.4) %	-0.6 (-0.8 ; -0.4) %

In the next table, we give the above defined K_{equiv}'s:

	$K_{equiv}(K, -1)$	$K_{equiv}(K, 1)$	$K_{equiv}(K, 1) - K_{equiv}(K, -1)$
$K = 45$ bps	(44 ; 45) bps	(45 ; 46) bps	(0 ; 2) bps
$K = 50$ bps (ATM)	(49 ; 50) bps	(50 ; 51) bps	(0 ; 2) bps
$K = 55$ bps	(54 ; 55) bps	(55 ; 56) bps	(0 ; 2) bps

With the same parameters, we investigate in the following table the impact of non-zero correlation from T_a on for ATM options. The prices are obtained with 50,000 paths and the CDS prices at T_a are approximated with 5,000 paths when the correlation is not zero after T_a.

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	25.5 (23.9 ; 27.2) %	25.2 (25.1 ; 25.4) %	—
$\rho_{0,T_a} = 0$	25.1 (23.5 ; 26.8) %	24.6 (24.4 ; 24.8) %	24.7 (23.1 ; 26.3) %
$\rho_{0,T_a} = 1$	—	23.7 (23.6 ; 23.9) %	20.3 (18.8 ; 21.9) %

In the next table, we give the above defined K_{equiv}'s:

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	(48 ; 51) bps	(49 ; 50) bps %	- –
$\rho_{0,T_a} = 0$	(48 ; 51) bps	50 bps	(49 ; 51) bps
$\rho_{0,T_a} = 1$	—	(50 ; 51) bps	(52 ; 54) bps

6.3 CDS option with maturity 4y on a CDS lasting 1y

6.3.1 Calibrating ψ to Parmalat CDS Data

he At-the-money CDS option we consider here has the following features:

T_a	4 years
T_b	5 years
$T_{a+i+1} - T_{a+i}$	6 months
K	319 bp
LGD	70 %

With $\mu = 0.045$, $\nu = 15\%$, $y_0 = 0.035$, $\rho = 0.$ being fixed, we change κ:

κ	$v_{a,b}^{imp}(\kappa)$	$v_{a,b}^{CIR++}(\kappa)$	$v_{a,b}^{imp}(\kappa) - v_{a,b}^{CIR++}(\kappa)$
0.35	30.7 (30.5 ; 30.9) %	31.7 %	-1.0 (-1.2 ; -0.8) %
0.45	27.2 (27.0 ; 27.4) %	27.9 %	-0.7 (-0.9 ; -0.5) %
0.55	24.2 (24.0 ; 24.4) %	24.8 %	-0.6 (-0.8 ; -0.4) %
0.65	21.8 (21.6 ; 22.0) %	22.2 %	-0.4 (-0.6 ; -0.2) %
0.75	19.7 (19.5 ; 19.9) %	20.0 %	-0.3 (-0.5 ; -0.1) %

With $\kappa = 0.5$, $\nu = 15\%$, $y_0 = 0.037$, $\rho = 0.$ being fixed, we change μ:

μ	$v_{a,b}^{imp}(\mu)$	$v_{a,b}^{CIR++}(\mu)$	$v_{a,b}^{imp}(\mu) - v_{a,b}^{CIR++}(\mu)$
0.025	19.8 (19.6 ; 20.0) %	21.4 %	-1.6 (-1.8 ; -1.4) %
0.03	21.4 (21.3 ; 21.6) %	22.8 %	-1.4 (-1.5 ; -1.2) %
0.035	23.0 (22.8 ; 23.2) %	24.0 %	-1.0 (-1.2 ; -0.8) %
0.04	24.4 (24.2 ; 24.6) %	25.2 %	-0.8 (-1.0 ; -0.6) %
0.045	25.8 (25.6 ; 26.0) %	26.4 %	-0.6 (-0.8 ; -0.4) %

With $\kappa = 0.5$, $\mu = 0.046$, $y_0 = 0.036$, $\rho = 0.$ being fixed, we change ν:

ν	$v_{a,b}^{imp}(\nu)$	$v_{a,b}^{CIR++}(\nu)$	$v_{a,b}^{imp}(\nu) - v_{a,b}^{CIR++}(\nu)$
11 %	19.7 (19.5 ; 19.9) %	19.8 %	-0.1 (-0.3 ; 0.1) %
13 %	22.9 (22.7 ; 23.1) %	23.2 %	-0.3 (-0.5 ; -0.1) %
15 %	26.0 (25.8 ; 26.2) %	26.5 %	-0.5 (-0.7 ; -0.3) %
17 %	28.9 (28.6 ; 29.1) %	29.7 %	-0.8 (-1.0 ; -0.5) %
19 %	31.5 (31.3 ; 31.8) %	32.8 %	-1.3 (-1.5 ; -1.0) %
21 %	34.0 (33.8 ; 34.3) %	35.8 %	-1.8 (-2.0 ; -1.5) %

With $\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $\rho = 0$ being fixed, we change y_0:

y_0	$v_{a,b}^{imp}(y_0)$	$v_{a,b}^{CIR++}(y_0)$	$v_{a,b}^{imp}(y_0) - v_{a,b}^{CIR++}(y_0)$
0.012	31.6 (31.4 ; 31.9) %	32.6 %	-1.0 (-1.2 ; -0.7) %
0.017	32.0 (31.7 ; 32.2) %	33.1 %	-1.1 (-1.4 ; -0.9) %
0.022	32.3 (32.1 ; 32.6) %	33.5 %	-1.2 (-1.4 ; -0.9) %
0.027	32.7 (32.5 ; 33.0) %	34.0 %	-1.3 (-1.5 ; -1.0) %
0.032	33.1 (32.8 ; 33.3) %	34.4 %	-1.3 (-1.6 ; -1.1) %
0.037	33.4 (33.2 ; 33.7) %	34.9 %	-1.5 (-1.7 ; -1.2) %

With $\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $y_0 = 0.037$, $\rho = 0.$ being fixed, we change K:

K	$v_{a,b}^{imp}(K)$	$v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K) - v_{a,b}^{CIR++}$
199 bps	33.3 (33.0 ; 33.6) %	34.9 %	-1.6 (-1.9 ; -1.3) %
259 bps	33.5 (33.3 ; 33.8) %	34.9 %	-1.4 (-1.6 ; -1.1) %
319 bps (ATM)	33.4 (33.2 ; 33.7) %	34.9 %	-1.5 (-1.7 ; - 1.2) %
379 bps	33.2 (33.0 ; 33.5) %	34.9 %	-1.7 (-1.9 ; -1.4) %
439 bps	32.9 (32.7 ; 33.2) %	34.9 %	-2.0 (-2.2 ; -1.7) %

With $\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $y_0 = 0.037$ being fixed, we change ρ and K. Here $v_{a,b}^{CIR++} = 34.9\%$ and does not depend either on K or on ρ.

	$v_{a,b}^{imp}(K, -1)$	$v_{a,b}^{imp}(K, 0)$	$v_{a,b}^{imp}(K, 1)$
$K = 259$ bps	35.3 (35.1 ; 35.5) %	33.5 (33.3 ; 33.8) %	30.9 (30.7 ; 31.2) %
$K = 319$ bps (ATM)	34.7 (34.5 ; 34.9) %	33.4 (33.2 ; 33.7) %	31.4 (31.2 ; 31.7) %
$K = 379$ bps	34.3 (34.1 ; 34.5) %	33.2 (33.0 ; 33.5) %	31.5 (31.3 ; 31.8) %

	$v_{a,b}^{imp}(K, -1) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K, 0) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K, 1) - v_{a,b}^{CIR++}$
$K = 259$ bps	0.4 (0.2 ; 0.6) %	-1.4 (-1.6; -1.1) %	-4.0 (-4.2 ; -3.7) %
$K = 319$ bps (ATM)	-0.2 (-0.4 ; 0.0) %	-1.5 (- 1.7 ; -1.2) %	-3.5 (-3.7 ; -3.2) %
$K = 379$ bps	-0.6 (-0.8 ; -0.4) %	-1.7 (-1.9 ; - 1.4) %	-3.4 (-3.6 ; -3.1) %

As for K_{equiv}, we obtain

	$K_{equiv}(K, -1)$	$K_{equiv}(K, 1)$	$K_{equiv}(K, 1) - K_{equiv}(K, -1)$
$K = 259$ bps	(250 ; 254) bps	(268 ; 271) bps	(14 ; 21) bps
$K = 319$ bps (ATM)	(309 ; 314) bps	(329 ; 335) bps	(15 ; 26) bps
$K = 379$ bps	(367 ; 373) bps	(390 ; 399) bps	(17 ; 32) bps

With the same parameters, we investigate in the following table the impact of non-zero correlation from T_a on for ATM options. The prices are obtained with 50,000 paths and the CDS prices at T_a are approximated with 5,000 paths when the correlation is not zero after T_a.

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	33.4 (32.7 ; 34.2) %	34.7 (34.5 ; 34.9) %	—
$\rho_{0,T_a} = 0$	33.0 (32.3 ; 33.7) %	33.4 (33.2 ; 33.7) %	33.3 (32.6 ; 34.1) %
$\rho_{0,T_a} = 1$	—	31.4 (31.2 ; 31.7) %	30.7 (30.0 ; 31.4) %

In the next table, we give the above defined K_{equiv}'s:

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	(313 ; 327) bps	(309 ; 314) bps %	—
$\rho_{0,T_a} = 0$	(316 ; 328) bps	319 bps	(313 ; 326) bps
$\rho_{0,T_a} = 1$	—	(329 ; 335) bps	(331 ; 342) bps

6.3.2 Calibrating ψ to Peugeot CDS Data

The At-the-money CDS option we consider here has the following features:

T_a	4 years
T_b	5 years
$T_{a+i+1} - T_{a+i}$	6 months
K	63 bp
LGD	70 %

With $\mu = 0.0071$, $\nu = 7\%$, $y_0 = 0.0019$, $\rho = 0.$ being fixed, we change κ:

κ	$v_{a,b}^{imp}(\kappa)$	$v_{a,b}^{CIR++}(\kappa)$	$v_{a,b}^{imp}(\kappa) - v_{a,b}^{CIR++}(\kappa)$
0.35	25.6 (25.4 ; 25.8) %	27.0 %	-1.4 (-1.6 ; -1.2) %
0.4	24.6 (24.4 ; 24.8) %	25.7 %	-1.1 (-1.3 ; -0.9) %
0.5	22.6 (22.4 ; 22.8) %	23.2 %	-0.6 (-0.8 ; -0.4) %
0.6	20.7 (20.5 ; 20.9) %	21.0 %	-0.3 (-0.5 ; -0.1) %
0.7	18.8 (18.6 ; 19.0) %	19.1 %	-0.3 (-0.5 ; -0.1) %

With $\kappa = 0.5$, $\nu = 7\%$, $y_0 = 0.0017$, $\rho = 0.$ being fixed, we change μ:

μ	$v_{a,b}^{imp}(\mu)$	$v_{a,b}^{CIR++}(\mu)$	$v_{a,b}^{imp}(\mu) - v_{a,b}^{CIR++}(\mu)$
0.005	18.7 (18.5 ; 18.8) %	19.8 %	-1.1 (-1.3 ; -1.0) %
0.006	20.6 (20.4 ; 20.8) %	21.4 %	-0.8 (-1.0 ; -0.6) %
0.007	22.4 (22.2 ; 22.6) %	23.0 %	-0.6 (-0.8 ; -0.4) %
0.0083	24.5 (24.3 ; 24.7) %	24.9 %	-0.4 (-0.6 ; -0.2) %

With $\kappa = 0.8$, $\mu = 0.007$, $y_0 = 0.0012$, $\rho = 0.$ being fixed, we change ν:

ν	$v_{a,b}^{imp}(\nu)$	$v_{a,b}^{CIR++}(\nu)$	$v_{a,b}^{imp}(\nu) - v_{a,b}^{CIR++}(\nu)$
6 %	14.7 (14.5 ; 14.8) %	14.8 %	-0.1 (-0.3 ; 0.0) %
7 %	16.9 (16.8 ; 17.1) %	17.2 %	-0.3 (-0.4 ; -0.1) %
8 %	19.2 (19.0 ; 19.3) %	19.7 %	-0.5 (-0.7 ; -0.4) %
9 %	21.3 (21.1 ; 21.5) %	22.1 %	-0.8 (-1.0 ; -0.6) %
10 %	23.3 (23.1 ; 23.6) %	24.5 %	-1.2 (-1.4 ; -0.9) %

With $\kappa = 0.5$, $\mu = 0.0083$, $\nu = 9\%$, $\rho = 0.$ being fixed, we change y_0:

y_0	$v_{a,b}^{imp}(y_0)$	$v_{a,b}^{CIR++}(y_0)$	$v_{a,b}^{imp}(y_0) - v_{a,b}^{CIR++}(y_0)$
0.0004	29.9 (29.6 ; 30.1) %	31.1 %	-1.2 (-1.5 ; -1.0) %
0.0009	30.1 (29.9 ; 30.4) %	31.4 %	-1.3 (-1.5 ; -1.0) %
0.0014	30.4 (30.1 ; 30.6) %	31.7 %	-1.3 (-1.6 ; -1.1) %
0.0019	30.6 (30.4 ; 30.9) %	31.9 %	-1.3 (-1.5 ; -1.0) %

With $\kappa = 0.5$, $\mu = 0.0083$, $\nu = 9\%$, $y_0 = 0.0019$, $\rho = 0.$ being fixed, we change K:

K	$v_{a,b}^{imp}(K)$	$v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K) - v_{a,b}^{CIR++}$
43 bps	29.8 (29.5 ; 30.0) %	31.9 %	-2.1 (-2.4 ; -1.9) %
53 bps	30.3 (30.1 ; 30.6) %	31.9 %	-1.6 (-1.8 ; -1.3) %
63 bps (ATM)	30.6 (30.4 ; 30.9) %	31.9 %	-1.3 (-1.5 ; -1.0) %
73 bps	30.7 (30.4 ; 31.0) %	31.9 %	-1.2 (-1.5 ; -0.9) %
83 bps	30.7 (30.4 ; 31.0) %	31.9 %	-1.2 (-1.5 ; -0.9) %

With $\kappa = 0.5$, $\mu = 0.0083$, $\nu = 9\%$, $y_0 = 0.0019$ being fixed, we change ρ and K. Here $v_{a,b}^{CIR++} = 31.9\%$ and does not depend either on K or on ρ.

	$v_{a,b}^{imp}(K,-1)$	$v_{a,b}^{imp}(K,0)$	$v_{a,b}^{imp}(K,1)$
$K = 58$ bps	32.0 (31.8 ; 32.1) %	30.5 (30.3 ; 30.7) %	28.2 (28.0 ; 28.5) %
$K = 63$ bps (ATM)	31.9 (31.7 ; 32.1) %	30.6 (30.4 ; 30.9) %	28.6 (28.4 ; 28.8) %
$K = 68$ bps	31.8 (31.6 ; 32.0) %	30.7 (30.4 ; 30.9) %	28.9 (28.6 ; 29.1) %

	$v_{a,b}^{imp}(K,-1) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K,0) - v_{a,b}^{CIR++}$	$v_{a,b}^{imp}(K,1) - v_{a,b}^{CIR++}$
$K = 58$ bps	0.1 (-0.1 ; 0.2) %	-1.4 (-1.6 ; -1.2) %	-3.7 (-3.9 ; -3.4) %
$K = 63$ bps (ATM)	0.0 (-0.2 ; 0.2) %	-1.3 (- 1.5 ; -1.0) %	-3.3 (-3.5 ; -3.1) %
$K = 68$ bps	-0.1 (-0.3 ; 0.1) %	-1.2 (-1.5 ; - 1.0) %	-3.0 (-3.3 ; -2.8) %

As for K_{equiv}, we obtain:

	$K_{equiv}(K,-1)$	$K_{equiv}(K,1)$	$K_{equiv}(K,1) - K_{equiv}(K,-1)$
$K = 58$ bps	(56 ; 57) bps	(60 ; 61) bps	(3 ; 5) bps
$K = 63$ bps (ATM)	(60 ; 62) bps	(65 ; 67) bps	(3 ; 7) bps
$K = 68$ bps	(65 ; 67) bps	(70 ; 72) bps	(3 ; 7) bps

With the same parameters, we investigate in the following table the impact of non-zero correlation from T_a on for ATM options. The prices are obtained with 50,000 paths and the CDS prices at T_a are approximated with 5,000 paths when the correlation is not zero after T_a.

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	31.2 (30.5 ; 32.0) %	31.9 (31.7 ; 32.1) %	—
$\rho_{0,T_a} = 0$	30.7 (30.0 ; 31.5) %	30.7 (30.4 ; 30.9) %	31.1 (30.3 ; 31.8) %
$\rho_{0,T_a} = 1$	—	28.6 (28.4 ; 28.8) %	28.5 (27.8 ; 29.2) %

In the next table, we give the above defined K_{equiv}'s:

	$\rho_{T_a,T_b} = -1$	$\rho_{T_a,T_b} = 0$	$\rho_{T_a,T_b} = 1$
$\rho_{0,T_a} = -1$	(61 ; 64) bps	(60 ; 62) bps %	- –
$\rho_{0,T_a} = 0$	(61 ; 64) bps	63 bps	(61 ; 64) bps
$\rho_{0,T_a} = 1$	—	(65 ; 67) bps	(65 ; 68) bps

7 Comments on numerical results and conclusions

We now interpret the numerical results obtained above. A first general comment is that in the Monte Carlo method we did not resort to a huge number of paths in order to keep the computational time limited. The 99% standard error for the price in each simulation is given between brackets at the side of each Monte Carlo estimate in terms of implied volatilities. As we can see, standard errors are not always negligible with respect to the estimates, but allow anyway to deduce qualitative patterns of market quantities with respect to model parameters. In all studied cases, with the obvious exceptions of the strike K and correlation ρ patterns, we find that qualitative patterns are always respected by the approximated formula, in that the approximated volatility increases or decreases with respect to a parameter exactly in the same cases as the exact implied volatility does. Patterns are summarized in Table 1.

param	$v_{a,b}^{imp}$	$v_{a,b}^{CIR++}$
$\kappa \uparrow$	$\downarrow$	$\downarrow$
$\mu \uparrow$	$\uparrow$	$\uparrow$
$\nu \uparrow$	$\uparrow$	$\uparrow$
$y_0 \uparrow$	$\uparrow$	$\uparrow$
$K \uparrow, \rho = 0$	$\uparrow$/flat	Const
$K \uparrow, \rho = -1$	$\uparrow$ / $\downarrow$/flat	-
$K \uparrow, \rho = 1$	$\uparrow$/flat	-
$\rho \uparrow$	$\downarrow$	-

Table 1: Volatility patterns in terms of the parameters κ, μ, ν, y_0, ρ and of the strike K

The patterns in K are a particular feature, describing what we might call the "CDS volatility smile" implied by the CIR++ model.

The accuracy of the analytical formula is not satisfactory for trading purposes in general. Clearly, we expected this to happen, especially in the strike and correlation dimensions, since the formula assumes $\rho = 0$ and does not depend on K. If we exclude the K and ρ tables accordingly, the situation improves and the error is often below 1% (and always below 1.8%), especially for κ, μ, ν. This confirms the formula to be suited more to deducing patterns or first guesses for market volatilities rather than for precise relative-value trading. See also the exact analytical formula in Brigo (2004) under deterministic interest rates. Here, under stochastic rates, results are good enough to conclude that the approximated formula reflects well the market patterns implied by the model parameters.

To find said patterns, we chose two data sets representing two different default situations: Peugeot and Parmalat. At the time the paper is written, Peugeot is a company whose risk-neutral probabilities of default, stripped from CDS through a deterministic intensity model, are relatively low. The probability that Peugeot does not default before five years is 98.85%. On the contrary, Parmalat shows higher probabilities of default, in that the analogous value for the five-year risk neutral probability is 80.72%. Indeed, in the following months Parmalat will enter a crisis.

We chose also two basic options for our tests: one to enter a CDS in one year with a final payment maturity of five years (thus with a four years length, similar to the options currently proposed on the market), and a second one to enter a CDS in four years with a final payment maturity of five years (thus with a one-year length).

In all experiments we set the β parameters affecting the time-homogeneous core y^β of the stochastic intensity λ to typical values ensuring that the resulting calibrating shift ψ be positive. We then calibrate exactly CDS quotes (through their implied γ^{mkt}) via ψ in each case. We operate analogously for the interest-rate part, for which however we only consider one case, our focus being on the credit side.

We observe the following.

The influence of κ on the CDS volatility $v_{a,b}^{imp}$ is that, all things being equal (except for ψ, always chosen to calibrate CDS's exactly), increasing κ decreases $v_{a,b}^{imp}$. This is expected, since by increasing the speed of convergence to the mean reversion parameter, y flattens earlier around its mean reversion level μ, showing less stochasticity along its life. It is interesting to notice that in all cases (with a little single exception) the difference between the approximated analytical volatility and the exact implied volatility decreases in absolute value as κ increases. This is expected too, since with stochasticity collapsing earlier around the mean reversion, the "replace y by its expectation" effect on which the approximated formula is based has less impact.

The influence of μ on the CDS volatility $v_{a,b}^{imp}$ is that, all things being equal (except for ψ, always chosen to calibrate CDS's exactly), increasing μ increases $v_{a,b}^{imp}$. This is expected, since by increasing the final mean reversion level amounts to have a final higher intensity and larger intensities in general. Since the instantaneous volatility of the intensity is $\nu\sqrt{y}$, this also is typically larger for larger μ, adding stochasticity to the system. It is interesting to notice that, in the 4y-1y option case, as μ increases, the difference between the approximated analytical volatility and the exact implied volatility decreases in absolute value. This tells us that even if μ increases stochasticity, the "replace y by its expectation and integrate" approach is improved by an increase in μ, at least in the 4y-1y case.

Now we turn to examine the impact of ν on the CDS volatility $v_{a,b}^{imp}$. Since ν is the main volatility parameter in the intensity core y, and since $R_{a,b}(t)$ is a function of y_t, we expect ν to have a direct link with the volatility $v_{a,b}^{imp}$. Indeed, we can see from all cases that increasing ν amounts to increasing $v_{a,b}^{imp}$, and the relationship is quite strong. The same holds for the approximated analytical volatility. We see that in all cases the analytical formula overestimates the implied volatility of an amount that increases with ν. This is again natural, since with stochasticity increasing (with ν) the "replace y by its expectation" approximation worsens.

An increase in y_0 implies and increase in the approximated analytical volatility, and in our tests it implies also an increase in the MC implied volatility. Recall that the change on the initial intensity level implied by the change of y_0 is irrelevant, given that ψ offsets y_0 in calibrating CDS's automatically. However, at future times the impact of

y_0 cannot be neglected and increasing y_0 amounts to increase the initial level of the core y of the intensity, giving a larger time homogeneous part of the intensity process and thus a larger implied volatility.

At this point a comment is in order on the "CDS-rate volatility smile". The CIR++ dynamics for the intensity implies an increasing or constant pattern in the implied volatility when the strike increases, with the exception of the 4y-1y case (especially for Parmalat, with an almost flat pattern for Peugeot) when $\rho = -1$. Clearly the approximated analytical implied volatility formula cannot take into account the smile, and should be used only for at the money CDS options, where $K = R_{a,b}(0)$.

As a last aspect, we investigate the impact of ρ. In all our tests the implied volatility decreases as the correlation increases, and the effect is typically marked for "in the money" payer options ($K < R_{a,b}(0)$). The impact of a nonzero correlation on the CDS option price is further analyzed by means of the equivalent strike K_{equiv}, which assesses the impact of correlation on CDS option prices in the CDS rates dimension. We see that in all cases we have maximum excursions due to correlations lower than 10bps, except for the Parmalat case where we have wider excursions. However, due to higher default probability, for Parmalat at times even the sizes of the bid ask spreads for the underlying CDS rates R is comparable in size to the excursion we have found in the worst case. Also, Peugeot shows an excursion that is always comparable to the bid ask spread in the underlying CDS rate R. We thus notice that in our tests correlation has little impact on options as well, with one exception. CDS options are starting to be offered in the market even as we write, but with quite large bid ask spreads. In the future, however, spreads might narrow and in cases with high default probability a fine tuning of ρ, based on historical estimation with judgemental adjustments or on a few implied quotes can be considered. All our tests were based on the assumption that we could set $\rho = 0$ from T_a on. But is this possible? We test this by computing the option prices when ρ is not set to zero from T_a on, by resorting to the sub-paths method. The results we obtain show that the approximation is not good only when we have positive (and high) correlation, as our case with $\rho = 1$ show. Since it is known that in general this correlation is negative, see for example Longstaff and Schwartz (1995), we see that we may set $\rho = 0$ from T_a on in realistic situations.

To sum up, we have investigated implied volatility patterns in the SSRD model as functions of the model parameters. We have found an analytical approximation for the implied volatility that follows the same patterns and that can be used to have a first rough estimate of the implied volatility following a calibration. We have found an increasing or flat CDS-rate volatility smile for the adopted SSRD model, with one exception in the negative interest-rate/ intensity correlation, which can thus be considered as one of the parameters affecting the smile shape. We find a decreasing pattern in the correlation itself and, comparing with the underlying CDS bid ask spreads, we find one case out of four where correlation has a possibly relevant impact on CDS options prices. We have to keep in mind that all the tests are done under the assumption that $\rho = 0$ from T_a on, which we test a posteriori finding that this is working better in cases with negative ρ.

References

[1] Avellaneda, M., and Newman, J. (1998). Positive interest rates and nonlinear term structure models. Courant Institute of Mathematical Sciences Working Paper, New York University.

[2] Bielecki T., Rutkowski M. (2001), Credit risk: Modeling, Valuation and Hedging. *Springer Verlag*

[3] Blanchet-Scaillet, C., and Jeanblanc, M. (2001). Hazard rate for credit risk and hedging defaultable contingent claims, working paper.

[4] Brigo D. (2004), Candidate Market Models and the Calibrated CIR++ Stochastic Intensity Model for Credit Default Swap Options and Callable Floaters. *Proceedings of the 4-th ICS Conference on Statistical Finance*, Tokyo, March 18-19, 2004. Available at www.damianobrigo.it

[5] Brigo, D., and Alfonsi, A. (2003), A two-dimensional CIR++ shifted diffusion model with automatic calibration to credit defalult swaps and interest rate derivatives data, in: Proceedings of the 6-th Columbia=JAFEE International Conference, Tokyo, March 15-16, 2003, pages 563-585. *Extended internal report* available at *http://www.damianobrigo.it*. Reduced version to appear in *Finance and Stochastics*

[6] Brigo, D., and Mercurio, F. (2001), On deterministic shift extensions of short-rate models. *Internal Report* available at *http://www.damianobrigo.it*. Reduced version published in *Finance and Stochastics* 5 (2001), pp. 369–388.

[7] Brigo, D., and Mercurio, F. (2001b), Interest Rate Models: Theory and Practice. Springer-Verlag.

[8] Duffie D., Singleton K. (1997), An econometric model of the term structure of interest-rate swap yields. *J. Finance.* 52 , 1287-1321.

[9] Duffie D., Singleton K. (1999), Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12, 687-720.

[10] Lando, D. (1998), On Cox processes and credit-risky securities. *Rev. Derivatives Res.* 2, 99-120.

[11] Jamshidian, F. (2002). Valuation of Credit Default Swaps and Swaptions. NIB Capital Bank Preprint.

[12] Longstaff, F., and Schwartz, E., (1995). A Simple Approach to Valuing Risky Fixed and Floating Debt. *Journal of Finance* 50, 789-819.

[13] Prampolini A. (2002). Private communication, Credit Derivatives Desk, Banca Profilo.

[14] Schönbucher, P. (2000). A Libor market model with default risk, preprint.