The general mixture-diffusion SDE and its relationship with an uncertain-volatility option model with volatility-asset decorrelation*

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Abstract

In the present paper, given an evolving mixture of probability densities, we define a candidate diffusion process whose marginal law follows the same evolution. We derive as a particular case a stochastic differential equation (SDE) admitting a unique strong solution and whose density evolves as a mixture of Gaussian densities. We present an interesting result on the comparison between the instantaneous and the terminal correlation between the obtained process and its squared diffusion coefficient. As an application to mathematical finance, we construct diffusion processes whose marginal densities are mixtures of lognormal densities. We explain how such processes can be used to model the market smile phenomenon. We show that the lognormal mixture dynamics is the one-dimensional diffusion version of a suitable uncertain volatility model, and suitably reinterpret the earlier correlation result. We explore numerically the relationship between the future smile structures of both the diffusion and the uncertain volatility versions.

Keywords: Stochastic Differential Equations, Mixtures of Densities, Mixtures of Gaussians, Mixtures of Lognormals, Risk-Neutral Valuation, Option Pricing, Volatility-Underlying Correlation, Smile Modeling.

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1 Introduction: SDEs and mixtures

Let us consider the scalar stochastic differential equation (SDE)

$$dX_t = f_t(X_t)dt + \sigma_t(X_t)dW_t, \tag{1}$$

of diffusion type, with deterministic initial condition $X_0 = x_0$, and where $\{W_t, t \geq 0\}$ is a standard Brownian motion. We assume that

(A1) The stochastic differential equation (1) characterized by the coefficients f, σ, and by the initial condition x_0 admits a unique strong solution, whose support is assumed to be the interval (b, ∞) at all time instants. The symbol b denotes either a real number (typically 0) or $-\infty$.

Under (A1), we can analyze the distribution of our SDE's solution at all time instants. In describing the evolution of the distribution of a diffusion process, the Fokker–Planck partial differential equation is a fundamental tool. We assume that

(A2) The unique solution X_t of (1) admits a density p_t that is absolutely continuous with respect to the Lebesgue measure in (b, ∞) and that satisfies the Fokker–Planck equation:

$$\frac{\partial p_t}{\partial t} = -\frac{\partial}{\partial x}(f_t p_t) + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}(a_t p_t), \quad a_t(\cdot) = \sigma_t^2(\cdot)\ .$$

The other main ingredients in the present paper are mixtures of densities. More specifically, we will consider a basic parametric family of densities, say

$$\mathcal{D} = \{p(\cdot, \theta), \theta \in \Theta\},$$

with Θ open in $\mathbf{R}^d$, being d a suitable integer, and where all densities in the family share a common support (b, ∞). We are interested in considering a particular mixture of densities in this family. In other words, we fix a set of non-negative weights $\lambda_1, \ldots, \lambda_m$, $\lambda \geq 0$, $\sum_i \lambda_i = 1$. We take the space of all possible mixtures of densities in $\mathcal{D}$ with fixed weights λ:

$$\mathcal{M}(\mathcal{D}, \lambda) := \{\lambda_1 p(\cdot, \theta_1) + \ldots + \lambda_m p(\cdot, \theta_m), \quad \theta_1, \ldots, \theta_m \in \Theta\}.$$

We will refer to this set of densities as to the λ-mixture family for $\mathcal{D}$, or shortly as to the "mixture family" when λ is clear from the context. We are interested in finding an SDE whose solution X_t has a density p_t that follows a prescribed evolution in a given mixture family. More precisely, we require the curve $t \mapsto p_t$, in the space of all densities, to match a given curve $t \mapsto \sum_{i=1}^m \lambda_i p(\cdot, \theta_i(t))$ in a given $\mathcal{M}(\mathcal{D}, \lambda)$.

Problem 1.1. *Let be given a mixture family $\mathcal{M}(\mathcal{D}, \lambda)$ of densities with support (b, ∞), and a drift $f_t(x)$ satisfying*

$$\textit{(A3)} \quad \lim_{y \to b+} f_t(y) p(y, \theta) = 0 \ \ \textit{for all} \ \ t \geq 0, \quad \theta \in \Theta.$$

Let $\Sigma(f, x_0)$ *denote the set of all real-valued diffusion coefficients* σ^f *such that the related SDE (1) satisfies assumptions (A1) and (A2), and such that*

$$\text{(A4)} \quad \lim_{y \to b+} \sigma_t^f(y)^2 p(y,\theta) = 0, \quad \lim_{y \to b+} \frac{\partial}{\partial y}\left(\sigma_t^f(y)^2 p(y,\theta)\right) = 0 \;\; \text{for all} \;\; t \geq 0, \quad \theta \in \Theta.$$

Assume $\Sigma(f, x_0)$ *to be non-empty. Then, given the curve* $t \mapsto \sum_{i=1}^m \lambda_i p(\cdot, \theta_i(t))$ *in* $\mathcal{M}(\mathcal{D}, \lambda)$ *(where* $t \mapsto \theta_i(t)$ *are* C^1*–curves in the parameter space* Θ*), find a diffusion coefficient in* $\Sigma(f, x_0)$ *whose related SDE has a solution with density* $p_t = \sum_{i=1}^m \lambda_i p(\cdot, \theta_i(t))$.

This problem is the analogous of the drift search problem described and solved in Brigo and Mercurio (1998) and Brigo (2000). The solution of this problem is given by the following.

Proposition 1.2. (Solution of Problem 1.1) *Assumptions and notation of Problem 1.1 in force. Consider the stochastic differential equation*

$$\begin{aligned} dY_t &= f_t(Y_t)dt + \sigma_t^f(Y_t)dW_t, \quad Y_0 = x_0, \\ (\sigma_t^f(y))^2 &= \frac{2}{\sum_{i=1}^m \lambda_i p(y, \theta_i(t))} \left[\int_b^y \left(\int_b^x \sum_{i=1}^m \lambda_i \frac{\partial p(u, \theta_i(t))}{\partial t} du \right) dx \right. \\ &\quad \left. + \int_b^y f_t(x) \sum_{i=1}^m \lambda_i p(x, \theta_i(t)) dx \right]. \end{aligned} \tag{2}$$

If $\sigma^f \in \Sigma(f, x_0)$*, then the SDE (2) solves Problem 1.1, in that*

$$p_{Y_t}(y) = \sum_{i=1}^m \lambda_i \; p(y, \theta_i(t)), \quad t \geq 0.$$

Proof. Write the Fokker–Planck equation for the candidate diffusion Y with the candidate solution $\sum_{i=1}^m \lambda_i p(\cdot, \theta_i(t))$ already inside, and then back-out the diffusion coefficient via two subsequent integrations starting from the lower point b. □

We will use a particular case of this proposition.

Corollary 1.3. *Assumptions as in the above proposition. If the basic densities* $p(\cdot, \theta_i(t))$ *evolving in* $\mathcal{M}(\mathcal{D}, \lambda)$ *are the marginal densities of a family of (instrumental) SDEs*

$$dX_t^i = f_t^i(X_t^i)dt + \sigma_t^i(X_t^i)dW_t, \quad x_0, \quad p(x, \theta_i(t)) := p_{X_t^i}(x),$$

all satisfying assumptions (A1), (A2), (A3) (with f^i*'s replacing* f*) and such that* $\sigma^i \in \Sigma(f^i, x_0)$*, then the solution of Problem (1.1) takes the form*

$$(\sigma_t^f(y))^2 = \sum_{i=1}^m \Lambda_t^i(y)(\sigma_t^i(y))^2 + \frac{2\sum_{i=1}^m \lambda_i \int_b^y (f_t(x) - f_t^i(x)) p(x, \theta_i(t)) dx}{\sum_{j=1}^m \lambda_j p(y, \theta_j(t))}, \tag{3}$$

$$\Lambda_t^i(y) := \frac{\lambda_i p(y, \theta_i(t))}{\sum_{j=1}^m \lambda_j p(y, \theta_j(t))} \quad \left(\text{with} \;\; \sum_{j=1}^m \Lambda_t^i(y) = 1\right).$$

An interesting particular case occurs when f satisfies

$$f_t(y) = \sum_{i=1}^{m} \Lambda_t^i(y) f_t^i(y). \tag{4}$$

In this case the second term in the right hand side of (3) vanishes, and we have that requiring the marginal of our final SDE to be a given λ-mixture of the marginals of the instrumental processes results in a drift and squared diffusion coefficients that are (state-dependent) Λ- "mixtures" of the drifts and squared diffusion coefficients of the instrumental processes. We move from the combinators λ (density) to the combinators Λ (coefficients).

Proof. It suffices to use the previous result by noticing that the Fokker–Planck equation for the instrumental processes reads:

$$\frac{\partial p(x,\theta_i(t))}{\partial t} = -\frac{\partial [f_t^i(x) p(x,\theta_i(t))]}{\partial x} + \tfrac{1}{2}\frac{\partial^2 [\sigma_t^i(x)^2 p(x,\theta_i(t))]}{\partial x^2}$$

and substituting in (2) the right hand sides of such Fokker–Planck equations. □

The problem solved in the above proposition could have been formulated to track a generic density-evolution $t \mapsto q_t$ which does not necessarily occur in a mixture family. In such a case, given assumptions analogous to (A3) and (A4), the appropriate diffusion coefficient would be

$$(\sigma_t^f(y))^2 = \frac{2}{q_t(y)} \left[\int_b^y \left(\int_b^x \frac{\partial q_t(u)}{\partial t} du \right) dx + \int_b^y f_t(x) q_t(x) dx \right]$$

which generalizes the solution in the above proposition. We preferred to present directly the parametric case. In the next sections, we shall consider first an interesting application of the above corollary to the fundamental cases of mixtures of normal and lognormal families, and second an application of the latter to the option pricing problem in mathematical finance, in particular as a possible means to model the so called "smile" phenomenon.

2 Diffusions whose densities follow mixtures of normal distributions

Mixtures of normals are ubiquitous in statistics, representing the standard in many applications. Also, mixture of normals are often used in econometrics to model time series that have tails fatter than the Gaussian. See for example Alexander (2001) for a discussion on normal mixtures applied to financial data, also in comparison with different distributions. In general, normal mixtures represent in a sense the least departure from the Gaussian family allowing for skewness and kurtosis different from the Gaussian ones. However, to the best of our knowledge, no explicit attempt has been made so far to design continuous time diffusion models displaying this kind of marginal distributions. The subject can be relevant at least in mathematical finance, as we will show when addressing the smile problem, but bears also an interest of its own in the study of the interaction between a diffusion process dynamics and particular families of distributions.

Let us then start from the normal family, that we parameterize via its first two moments. In this case

$$\mathcal{D} = \{p_{\mathcal{N}(m,v^2)}, \quad m, v \in \mathbf{R}\}, \quad b = -\infty.$$

We are given a curve in the λ-mixtures of normals, $t \mapsto \sum_{i=1}^m \lambda_i p_{\mathcal{N}(m_i(t),v_i(t)^2)}$, and we wish to find a diffusion process compatible with such a law. It is easy to see that if we consider the instrumental processes

$$dX_t^i = \mu_i(t)\ dt + \sigma^i(t) dW_t, \int_0^t \mu_i(s)ds = m_i(t), \int_0^t (\sigma^i(s))^2 ds = v_i(t)^2$$

with deterministic μ and σ's and null initial condition, by applying the above corollary in the particular case (4) we end up with the diffusion process whose drift and squared diffusion coefficient are given respectively by

$$f_t(y) = \sum_{i=1}^m \Lambda_t^i(y)\mu^i(t), \qquad \sigma_t^f(y)^2 = \sum_{i=1}^m \Lambda_t^i(y)\sigma^i(t)^2, \tag{5}$$
$$\Lambda_t^i(y) = \frac{\lambda_i p_{\mathcal{N}(m_i(t),v_i(t)^2)}(y)}{\sum_{j=1}^m \lambda_j p_{\mathcal{N}(m_j(t),v_j(t)^2)}(y)}.$$

Now assume the coefficients $t \mapsto \mu(t)$'s and $t \mapsto \sigma(t)$'s to be at least C^1 (and hence bounded on all finite time intervals), with the σ's bounded away from zero, $\sigma_i(t) \geq L > 0$ for all i and t. There are possible problems for a regular behaviour of the above f and σ^f when $(t, y) \to (0, 0)$. In order to avoid this, we may decide to modify our coefficients by imposing $\mu_i(t) = \bar{\mu}$, $\sigma_i(t) = \bar{\sigma}$ for all i and $t \in [0, \epsilon)$, with ϵ a given positive real number, typically small. We can then assume suitable transitory trajectories for the μ's and the σ's in say $[\epsilon, 2\epsilon)$ that recover the correct integrals m and v^2 for times larger than 2ϵ. The desired mixture is thus unmatched only in the (typically negligible) time interval $[0, 2\epsilon]$. Now $\Lambda_t^i(y) = \lambda_i$ for all $t < \epsilon$ and all y.

From the above assumptions, since by definition $0 \leq \Lambda_t^i(y) \leq 1$ for all t and y, we have also that f and σ^f are bounded in all regions with bounded time, implying non-explosion for the related SDE in all intervals of the kind $t \leq T$ for some $T > 0$. Furthermore, f and $(\sigma^f)^2$ are also seen to be C^1 in both t and y. This in turn implies that f is locally Lipschitz, whereas the fact that the $\sigma(t)$'s are bounded away from zero implies that so is σ^f in both t and y, thus ensuring that $(\sigma^f)^2 \in C^1$ implies $\sigma^f \in C^1$. Therefore σ^f is locally Lipschitz, and now we have all the elements to apply Theorem 12.1 in Section V.12 of Rogers and Williams (1996) to conclude that our SDE admits a unique strong solution. We have thus produced a diffusion process compatible with a given mixture of normal densities, and can state the following

Theorem 2.1. (SDEs whose marginal law follows a given normal mixture) *Consider an SDE*

$$dY_t = f_t(Y_t)dt + \sigma_t^f(Y_t)dW_t, \quad Y_0 = 0,$$

with drift and diffusion coefficient f and σ^f as in (5), where the m's and v's are C^2 time functions. Assume moreover that in an initial time interval $t \in [0, \epsilon)$ we have $m_i(t) = \bar{\mu}t$ and $v_i(t)^2 = \bar{\sigma}^2 t$. Then the considered SDE admits a unique strong solution and its solution has as marginal density the λ-normal mixture

$$p_{Y_t} = \sum_{i=1}^{m} \lambda_i p_{\mathcal{N}(m_i(t), v_i(t)^2)}$$

at time t.

An interesting feature of the obtained process concerns the covariance between the process itself and the diffusion coefficient in its dynamics. This quantity is of interest, for example, in mathematical finance. Let us denote by "corr_t" the correlation between two random variables, and by "cov_t" the covariance, both conditional on the information available at time t, the time being omitted if $t = 0$. We have, for the above SDE, the following *"instantaneous"* correlation between the *instantaneous change* in the process and the *instantaneous change* in the related diffusion coefficient at a given instant:

$$\text{corr}_t(dY_t, d\,\sigma_t^f(Y_t)) = \frac{d\langle Y, \sigma_\cdot^f(Y_\cdot)\rangle_t}{\sqrt{d\langle Y\rangle_t \;\; d\langle \sigma_\cdot^f(Y_\cdot)\rangle_t}} = \frac{dY_t \; d\,\sigma_t^f(Y_t)}{\sqrt{dY_t \; dY_t}\,\sqrt{d\,\sigma_t^f(Y_t)\; d\,\sigma_t^f(Y_t)}} = 1,$$

as is obvious from the fact that the diffusion coefficient is a deterministic function of the current value of Y.

However, things are rather different for the *terminal* correlation. A straightforward if lengthy computation is needed to show that

$$\begin{aligned}\text{cov}(Y_t, \sigma_t^f(Y_t)^2) &= E_0(Y_t\; \sigma_t^f(Y_t)^2) - E_0(Y_t)\; E_0(\sigma_t^f(Y_t)^2) \\ &= \sum_{i=1}^{m} \lambda_i m_i(t)\sigma_i^2(t) - \left(\sum_{i=1}^{m} \lambda_i m_i(t)\right)\left(\sum_{i=1}^{m} \lambda_i \sigma_i^2(t)\right).\end{aligned}$$

Consider now the case where all the means in the normal mixture densities are equal: $\mu_i(\cdot) = \mu(\cdot)$ for all i, and correspondingly $m_i(\cdot) = m(\cdot)$. In this case $f_t(y) = \mu(t)$ for all y and, perhaps surprisingly, especially if compared to the perfect instantaneous correlation, the above formula gives

$$\text{corr}(Y_t, \sigma_t^\mu(Y_t)^2) = \text{cov}(Y_t, \sigma_t^\mu(Y_t)^2) = 0.$$

This is a case where the *instantaneous* correlation is 1, whereas the terminal correlation after a time t, no matter how small, is 0. Thus we have a stochastic process whose instantaneous *changes* are perfectly correlated with the instantaneous *changes* of its squared diffusion coefficient, whereas at any time its value has 0 correlation with the squared diffusion coefficient value. It seems then that the squared diffusion coefficient has a special shape that immediately "decorrelates" itself from the process even after an arbitrarily small time. It is not difficult to prove an analogous statement for the *average* squared diffusion coefficient:

$$\mathrm{corr}\left(Y_T, \int_0^T \sigma_t^{\mu}(Y_t)^2 \, dt\right) = 0.$$

Such interesting results on terminal correlation versus instantaneous correlation will be further discussed in the financial applications.

Finally, going back to the above theorem, we notice that by reasoning along the same lines, diffusions displaying mixtures of lognormals as marginal densities can be easily obtained. Indeed, if Y is the diffusion process from the above theorem, we can set $S_t := \exp(Y_t)$ and derive easily the SDE for S via Ito's formula:

$$dS_t = S_t f_t(\ln(S_t))dt + \tfrac{1}{2} S_t \sigma_t^f(\ln(S_t))^2 dt + S_t \sigma_t^f(\ln(S_t))dW_t, \quad S_0 = 1.$$

It is a straightforward exercise to verify that the equality

$$p_{S_t}(y) = p_{Y_t}(\ln y)/y$$

implies that S is distributed as a mixture of lognormals if Y is distributed as a mixture of normals.

3 The smile phenomenon in option pricing

Now we briefly review a stylized version of the smile problem in financial modeling and explain the possible use of diffusions whose marginals follow λ-mixtures of lognormals.

3.1 The smile problem and market implied distributions

Let us consider a financial market with a "money market account" process B_t, with positive deterministic instantaneous interest rate $r(t) > 0$, so that $dB_t = r(t)B_t dt$. Let us also consider a process S_t modeling the evolution of some traded financial (risky) asset in our market, typically a stock.

The resulting financial market might admit arbitrage opportunities. A sufficient condition which ensures arbitrage-free dynamics is the existence of an equivalent martingale measure Q, sometimes termed risk-neutral measure. An equivalent martingale measure is a probability measure that is equivalent to the initial one and under which the process $\{S_t/B_t : t \geq 0\}$ is a martingale. Let us assume, in line with the basic Black and Scholes (1973) setup, that the risk-neutral dynamics of S is modeled by

$$dS_t = r(t)S_t dt + \nu(t)S_t \, dW_t, \quad S_0 = s_0, \quad t \in [0,T], \tag{6}$$

where s_0 is a positive deterministic initial condition, and ν is a well-behaving deterministic function of time (instantaneous volatility). The above process is a geometric Brownian motion and the probability density p_{S_t} of S_t, at any time t, is lognormal. Indeed,

$$\ln \frac{S_t}{S_0} \sim \mathcal{N}\left(R(0,t) - \tfrac{1}{2}V(t)^2,\ V(t)^2\right),\ R(a,t) := \int_a^t r(s)ds,\ V(t)^2 := \int_0^t \nu(s)^2 ds. \quad (7)$$

When $a = 0$, we write shortly $R(t)$ for $R(0,t)$. In the above equation (6) we modeled directly the risky asset dynamics under the unique equivalent martingale measure, so that W is assumed to be a Brownian motion under that measure, and it is immediate to check that S_t/B_t is indeed a martingale. It is this dynamics that matters when pricing options, as opposed to the real world one, which is related instead to historical estimation, statistical analysis and similar matters. Indeed, by applying the results by Harrison and Pliska (1981), the unique no-arbitrage price for a given $\mathcal{F}_T$-measurable contingent claim $H_T \in L^2(Q)$ is $V_t = B_t E^Q\left\{H_T/B_T|\,\mathcal{F}_t\right\} =: B_t E_t^Q\{H_T/B_T\}$ where $\{\mathcal{F}_t : t \geq 0\}$ denotes the filtration associated to the process S. The contingent claim is said to be a simple one when it is of the form $H_T = h(S_T)$ for a suitable function h.

One of the most common simple claims is a European call option written on the stock, with maturity T and strike K, which pays $H = (S_T - K)^+$ at time T. Its price is obtained by computing the expectation of the discounted payoff according to the lognormal distribution implied by (7), leading to the celebrated Black and Scholes (1973) call option formula, which we denote by "BSCall" and whose explicit expression we omit for brevity:

$$E_0^Q[(S_T - K)^+/B(T)] = \text{BSCall}(S_0, K, T, R(T), V(T)).$$

The quantity $V(T)/\sqrt{T}$ is the (average) volatility of the option, and according to this formulation, does not depend on the strike K of the option. Indeed, in this formulation, volatility is a characteristic of the stock S underlying the contract, and has nothing to do with the nature of the contract itself. In particular, it has nothing to do with the strike K of the option.

Now take two different strikes K_1 and K_2. Suppose that the market provides us with the prices of two related options on our stock with the same maturity T: $\text{MKTCall}(S_0, K_1, T)$ and $\text{MKTCall}(S_0, K_2, T)$.

Life would be simple if the market followed Black and Scholes' formula in a consistent way. But is this the case? Does there exist a *single* volatility parameter $V(T)$ such that both the following equations hold?

$$\text{MKTCall}(S_0, K_1, T) = \text{BSCall}(S_0, K_1, T, R(T), V(T)),$$

$$\text{MKTCall}(S_0, K_2, T) = \text{BSCall}(S_0, K_2, T, R(T), V(T)).$$

The answer is a resounding "no". In general, market option prices do not behave like this. What one sees when looking at the market is that two *different* volatilities

$V(T,K_1)$ and $V(T,K_2)$ are required to match the observed market prices if one is to use Black and Scholes' formula:

$$\text{MKTCall}(S_0,K_1,T) = \text{BSCall}(S_0,K_1,T,R(T),V(T,K_1)),$$

$$\text{MKTCall}(S_0,K_2,T) = \text{BSCall}(S_0,K_2,T,R(T),V(T,K_2)).$$

In other terms, each market option price requires its own Black and Scholes (implied) volatility $V^{\text{MKT}}(T,K)/\sqrt{T}$ depending on the option strike K.

The market therefore uses Black and Scholes' formula simply as a metric to express option prices as volatilities. The curve $K \mapsto V^{\text{MKT}}(T,K)/\sqrt{T}$ is the so called volatility smile of the T-maturity option. If Black and Scholes' model were consistent along different strikes, this curve would be flat, since volatility should not depend on the strike K. Instead, this curve is commonly seen to exhibit "smiley" or "skewed" shapes.

Clearly, only some strikes $K = K_i$ and maturities $T = T_j$ are quoted by the market, so that usually the remaining points have to be determined through interpolation or through an alternative model. Interpolation in K, for a fixed maturity T, can be easy but it does not give any insight as to the underlying stock dynamics compatible with such prices.

Indeed, suppose that we have a few market option prices for expiries $T = T_j$ and for a set of strikes $K = K_i$.

For each fixed $T = T_j$, by smooth interpolation we can obtain the price for every other possible K, i.e. we can build a function $K \mapsto \text{MKTCall}(S_0,K,T)$. Now, if this strike-$K$ price corresponds really to an expectation, we have

$$\text{MKTCall}(S_0,K,T) = e^{-\int_0^T r(s)ds} E_0^Q (S_T - K)^+ = e^{-\int_0^T r(s)ds} \int_K^\infty (x-K) p_T(x)\, dx, \quad (8)$$

where p_T is the true risk-neutral density of the underlying stock at time T. If Black and Scholes' formula were consistent, this density would be the lognormal density, coming for example from a dynamics such as (6), i.e. $p_T = p_{S_T}$. We have seen that this is not the case in the market. However, by differentiating the above integral twice with respect to K we see that, see also Breeden and Litzenberger (1978),

$$\frac{\partial^2 \text{MKTCall}(S_0,K,T)}{\partial K^2} = e^{-\int_0^T r(s)ds} p_T(K),$$

so that by differentiating the interpolated-prices curve we can find the density p_T of the underlying stock at time T that is compatible with the given interpolated prices. Nevertheless, the method of interpolation may interfere with the recovery of the density, since a second derivative of the interpolated curve is involved. Moreover, what kind of dynamics, alternative to (6), do the densities $p_{T_1}, p_{T_2}, \dots, p_{T_j}, \dots$ come from?

3.2 Local and stochastic volatility models

A partial answer to these issues can be given the other way around, by starting from an alternative dynamics. Indeed, assume that

$$dS_t = r(t)S_t dt + \sigma(t, S_t)\ S_t\ dW_t, \quad S_0 = s_0, \tag{9}$$

where σ can be either a deterministic or a stochastic function of S_t. In the latter case we would be using a so called "stochastic-volatility model", where for example $\sigma(t, S) = \xi(t)$, with ξ following a second stochastic differential equation, such as:

$$d(\xi(t)^2) = b(t, \xi(t)^2)dt + \chi(t, \xi(t)^2)dZ_t,$$

with the important specification

$$dZ_t dW_t = \rho\ dt.$$

Instead, in the so-called "local volatility models" the diffusion coefficient $\sigma(t, S_t)$ is a deterministic function of S_t.

One feature of stochastic volatility models that is usually deemed to render them superior with respect to local volatility models is "instantaneous decorrelation". Indeed, for stochastic volatility models we can have

$$\text{Corr}(dS_t,\ d\sigma^2(t, S_t)) = \rho < 1$$

whereas

$$\text{Corr}(dS_t,\ d\sigma^2(t, S_t)) = 1$$

for local volatility models, including our mixture dynamics models. This superiority no longer holds when considering terminal correlations, as we will remark later on. For the time being we concentrate on deterministic $\sigma(t, \cdot)$'s, leading to local-volatility models, such as for example $\sigma(t, S) = \eta\ S^\gamma$ (CEV model, see Cox (1975)), where γ ranges in a suitable interval and where η is a positive deterministic constant. Below we will propose a new $\sigma(t, \cdot)$ of our own, flexible enough for practical purposes.

We have seen above how the "true" risk-neutral densities $p_{T_1}, p_{T_2}, \ldots, p_{T_j}, \ldots$ of the underlying asset are linked to market option prices through second-order differentiation. The problem we will face is finding a dynamics alternative to (6) and as compatible as possible with the densities $p_{T_1}, p_{T_2}, \ldots, p_{T_j}, \ldots$ ideally associated with market prices. This will be done by fitting directly the prices implied by our alternative model to the market prices $\text{MKTCall}(S_0, K, T)$ for the considered set of strikes K_i and maturities T_j. To further clarify this point, it may be helpful to explain explicitly how an alternative dynamics such as (9) leads to a volatility smile to be fitted to the market smile. The way in which an alternative local-volatility model dynamics generates a smile is clarified by the following stylized operational scheme:

1. Set the pair (T, K) to a starting value;

2. Compute the model option price

$$\Pi(T,K) = e^{-\int_0^T r(s)ds} E_0^Q (S_T - K)^+$$

with S obtained through the no-arbitrage alternative dynamics (9).

3. Invert Black and Scholes' formula for this strike and maturity, i.e. solve

$$\Pi(T,K) = \mathrm{BSCall}(S_0, K, T, R(T), V(T,K))$$

in $V(T,K)$, thus obtaining the model implied volatility $V(T,K)$.

4. Change (T,K) and restart from point 2 until the last maturity/strike pair (T,K) is reached.

The fact that the alternative dynamics is not lognormal implies that we obtain curves in the strike $K \mapsto V(T,K)$ that are not flat. Clearly, one needs to choose $\sigma(t,\cdot)$ flexible enough for the surface $(T,K) \mapsto V(T,K)/\sqrt{T}$ to be able to resemble or even match the corresponding volatility surfaces coming from the market. Indeed, the model implied volatilities $V(T_j,K_i)/\sqrt{T_j}$ corresponding to the observed strikes and maturities have to be made as close as possible to the corresponding market implied volatilities $V^{\mathrm{MKT}}(T_j,K_i)/\sqrt{T_j}$, by acting on the coefficient $\sigma(\cdot,S)$ in the alternative dynamics.

At this point it should be clear why a λ-mixture dynamics can be of help. Existing local volatility models have either too little flexibility to calibrate a large number of points in a volatility surface, or are specified in a too general way requiring interpolation and other possibly dangerous artifices in order to be implemented. The CEV model for example has only one more parameter γ with respect to the basic Black and Scholes "flat" model in the time-homogeneous case, so that its fitting capabilities are rather poor. Dupire's (1997) approach is quite general, but if applied straightforwardly in its most general form, it requires a continuum of traded strikes and maturities, with the possible interpolation problems observed above, not to mention the lack of guarantees on existence of solutions for the resulting SDE.

3.3 Local volatility lognormal mixture diffusion dynamics

Consider instead our approach and write the λ-lognormal mixture diffusion. Set the instrumental processes to Black and Scholes processes,

$$dX_t^i = r(t)X_t^i dt + \nu_i(t) X_t^i dW_t, \quad s_0$$

and derive the diffusion coefficient corresponding to $f(t,y) = r(t)y$. Notice that here all instrumental processes have the same drift $f_i = f$ as the final process. In this particular case, taking into account the lognormal marginal distributions of the instrumental processes, by applying (3) under (4) we have

$$\sigma^2_{\mathrm{mix}}(t,y)y^2 := \sigma_t^f(y)^2 = y^2 \sum_{i=1}^m \Lambda_i(t,y)\nu_i(t)^2,$$

$$\Lambda_i(t,y) = \frac{\lambda_i \ p_{\mathcal{N}(\ln s_0 + R(t) - V_i(t)^2/2,\ V_i(t)^2)}(\ln y)}{\sum_{j=1}^m \lambda_j \ p_{\mathcal{N}(\ln s_0 + R(t) - V_j(t)^2/2,\ V_j(t)^2)}(\ln y)},$$

where R and V's are defined as in (7). In Brigo and Mercurio (2001b) we show that the SDE resulting from such coefficients, i.e.

$$dS_t = r(t) S_t \ dt + \sigma_{\rm mix}(t, S_t) \ S_t \ dW_t, \quad s_0 \tag{10}$$

admits a unique strong solution, provided one takes suitable regularity conditions on the time functions $t \mapsto \nu_i(t)$'s analogous to those on the $t \mapsto \sigma^i(t)$'s illustrated in Section 2. Indeed, it is straightforward to prove such existence and uniqueness result starting from the proof given in Section 2 for the normal-mixture case.

For our process S in (10), we confirm the curious result obtained in Section 2 from the comparison between instantaneous and terminal correlations. Notice that also in this case we have

$$\text{corr}_t(dS_t,\ d\sigma^2_{\rm mix}(t, S_t)) = 1,$$

considered to be a drawback of local volatility models. Yet, when considering terminal correlations, things change considerably.

Theorem 3.1. (Terminal correlation between underlying asset and average percentage variance in the lognormal mixture dynamics model for the smile)

Consider the random variable

$$v(T) := \int_0^T \sigma^2_{mix}(t, S_t) dt \ ,$$

$v(T)/T$ being the "average percentage variance" of the process S. Then

$$\boxed{corr(\sigma^2_{mix}(T, S_T), S_T) = 0, \quad and \quad corr(v(T), S_T) = 0 \quad for\ all \ \ T} \ . \tag{11}$$

Proof. First we show that

$$\text{corr}(\sigma^2_{\rm mix}(T, S_T), S_T) = 0,$$

i.e. that

$$E\{\sigma^2_{\rm mix}(T, S_T) \ S_T\} - E\{\sigma^2_{\rm mix}(T, S_T)\} E\{S_T\} = 0. \tag{12}$$

This is immediate by direct calculation:

$$E\{\sigma^2_{\rm mix}(T, S_T) \ S_T\} = \int \sum_{i=1}^m \Lambda_i(T, y) \nu_i(T)^2 y \ p_{S_T}(y) dy =$$

$$\int \sum_{i=1}^m \Lambda_i(T, y) \nu_i(T)^2 y \ \sum_{j=1}^m \lambda_j p_{X_T^j}(y) dy = \int \sum_{i=1}^m \lambda_i p_{X_T^i}(y) \nu_i(T)^2 y \ dy$$

$$= s_0 e^{\int_0^T r(s) ds} \sum_{i=1}^m \lambda_i \nu_i(T)^2$$

given the definition of the Λ's. Similarly, one computes

$$E\{\sigma^2_{\text{mix}}(T, S_T)\} = \sum_{i=1}^{m} \lambda_i \nu_i(T)^2,$$

from which (12) follows.

To show the other equality, notice that

$$dv(t) = \sigma^2_{\text{mix}}(t, S_t)dt,$$

and compute

$$d(v(t)S_t) = S_t \sigma^2_{\text{mix}}(t, S_t)dt + r(t)v(t)S_t dt + (\ldots)dW_t.$$

Taking expectations and Fubini's theorem

$$dE(v(t)S_t) = E(S_t \sigma^2_{\text{mix}}(t, S_t))dt + rE(v(t)S_t)dt.$$

Set $A_t = E(S_t \sigma^2_{\text{mix}}(t, S_t))$, which we computed above, and $C_t = E(v(t)S_t)$. The above equation reads

$$\dot{C}_t = r(t)C_t + A_t,$$

whose solution is

$$C_t = e^{\int_0^t r(s)ds} \int_0^t e^{-\int_0^u r(s)ds} A_u du.$$

By carrying out the computations one obtains $E(v(t)S_t)$. At this point it is easy to prove that

$$E(v(t)S_t) - E(v(t))E(S_t) = 0$$

by computing $E(v(t))$ trough Fubini's theorem. □

The above result is partly weakened by the fact that correlation is not a satisfactory measure of dependence outside the Gaussian world. However, a striking feature remains of two processes that are instantaneously perfectly correlated but such that for any infinitesimal time $T = \epsilon$ have zero terminal correlation.

Let us now set apart this correlation result and go back to understanding the reason why a λ-mixture dynamics is particularly appealing when pricing options. One of the main reasons lies in the price becoming a linear combination of prices with respect to underlying assets modeled according to the instrumental processes. Indeed, one has immediately, for a call option,

$$\Pi(T, K) = e^{-\int_0^T r(s)ds} E^Q\left\{(S_T - K)^+\right\} = e^{-\int_0^T r(s)ds} \int_0^{+\infty} (y - K)^+ \sum_{i=1}^{m} \lambda_i p_{X_T^i}(y)dy$$

$$= \sum_{i=1}^{m} \lambda_i e^{-\int_0^T r(s)ds} \int_0^{\infty} (y - K)^+ p_{X_T^i}(y)dy = \sum_{i=1}^{m} \lambda_i \ \text{BSCall}(S_0, K, T, R(T), V_i(T)) \ ,$$

the last equality following from the geometric Brownian motion structure of the underlying instrumental processes. This procedure is very general, and the price of a European-style simple claim is always the linear combination of the corresponding prices for the instrumental processes. In the lognormal-mixture case, when pricing a call option we obtain a linear combination of Black and Scholes prices. This is very appreciated by traders, who usually prefer to contain departures from the lognormal distribution and the corresponding Black and Scholes formula. In a sense, when in need of generalizing a lognormal distribution, a mixture of lognormals is the least departure from the original lognormal paradigm.

Important calibration benefits that should not go unnoticed are a consequence of the fact that a λ mixture-of-lognormals dynamics can price call options analytically. This is very helpful for calibrating the model to the market. In such a case one runs an optimization to find the values of the parameters V_i and λ_i that best reproduce a given set of market prices, and the target function of this optimization can be computed in closed form without resorting to numerical methods such as Monte Carlo simulation, trees, and finite difference schemes. Notice also that since we are free to select an arbitrary number m of instrumental processes, in principle our diffusion model features a limitless number of calibrating parameters. Once the model has been calibrated, one can use it to price more complicated (for example early exercise or path dependent) claims that have no quoted price in the market. Monte Carlo simulation through the Euler or Milstein discretization schemes (see for example Klöden and Platen (1995)) applied to our dynamics or recombining trees in the spirit of Nelson and Ramaswamy (1990) can be attempted, thanks to the explicit diffusion dynamics we provided.

In the present paper we have presented the mixture diffusion model in its most mathematical aspects. Other advantages and characteristics of these models and of their variants, based on shifted dynamics, and numerical investigation and calibrations to market data have been illustrated in Brigo and Mercurio (2000a, 2000b, 2001a, 2001b).

The introduction of general drift rates $\mu_i(t)$ not necessarily all equal to r in the instrumental processes and possible mixtures of densities coming from hyperbolic-sine processes are considered in Brigo, Mercurio and Sartorelli (2002).

A study of particular forms of time dependence of the ν's leading to desirable properties of the lognormal mixture dynamics and to a simple specification of the parameters in the model is carried out in Alexander and Brintalos (2003).

A generalization of the mixture dynamics apparatus to multivariate underlying assets and possible applications to the pricing of basket options in presence of volatility smile are considered in Rapisarda (2002) and Brigo, Mercurio and Rapisarda (2002).

3.4 Uncertain volatility geometric Brownian motion

We conclude the paper by pointing out an important relationship between the lognormal-mixture diffusion dynamics and an analogous uncertain volatility model given by a geometric Brownian motion with uncertain volatility.

In general it is known (Derman, Kani and Kamal (1997), Britten-Jones and Neuberger (2000), Gatheral (2001)) that every stochastic volatility model has a local volatility (i.e. scalar-diffusion) version that features the same marginal distributions in time (and thus the same initial prices for all plain vanilla options such as European calls). We may wonder whether our lognormal mixture diffusion dynamics (10) is the local volatility version of some stochastic volatility model. The answer is affirmative and we introduce the related model below.

Consider the following uncertain volatility model:

$$\begin{aligned} dS_t &= r(t)S_t dt + \xi(t)S_t dW_t, \quad S_0 = s_0, \\ (t \mapsto \xi(t)) &= \begin{cases} (t \mapsto \nu_1(t)) \text{ with probability } \lambda_1, \\ \quad\quad \ldots \\ (t \mapsto \nu_m(t)) \text{ with probability } \lambda_m, \end{cases} \end{aligned} \tag{13}$$

ξ independent of W and drawn at random at (an almost zero) time $\epsilon > 0$,

where all ν's are assumed to be regular enough and have a common value in $[0, \epsilon]$, $\nu_i(t) = \bar{\nu}$ for all i and $t \leq \epsilon$. We assume ξ to be independent of W and that the original probability space is large enough to allow for such a ξ (otherwise we may define ξ on a different space and then take the product space). Conditional on ξ, the process S is a geometric Brownian motion as in the Black-Scholes model. Thanks to independence, it is easy to show that for $t > \epsilon > u$,

$$Q\{S_t \in A | S_u = y\} = \sum_{i=1}^{m} \lambda_i Q\{S_t \in A | S_u = y, \xi = \nu_i\}$$

so that in particular

$$p_{S_t|S_u}(x; y) = \sum_{i=1}^{m} \lambda_i p_{S_t|S_u,\xi}(x; y, \nu_i),$$

i.e. the *transition density* of our uncertain volatility model is a mixture of lognormal transition densities, each corresponding to a volatility function ν_i. If we condition on an instant $u > \epsilon$, including the information on which value ν_i of ξ has realized itself at time $\epsilon < u$, an information that is contained in the path of S up to $u > \epsilon$, the transition density between u and t reduces merely to a lognormal density characterized by the relevant ν_i.

By considering the case $u = 0$, it is immediate to see that this model has the same marginals as the lognormal mixture diffusion seen earlier, although it leads to an incomplete market. Then at the initial time 0 it implies the same prices as the local volatility version seen earlier. Hedging is thus different and more complicated, and has to be based on additional hedging instruments. But what is interesting now is that transition densities are also known, not only marginals. This is not true for the local volatility version seen earlier.

There is, however, a close relationship between the lognormal mixture diffusion dynamics (10) and the uncertain volatility mixture dynamics (13).

Proposition 3.2. *The lognormal mixture diffusion dynamics (10) is the local volatility version of the uncertain volatility mixture dynamics (13). The two models are linked by the relationship*

$$\sigma^2_{mix}(t,x) = E\{\xi(t)^2 | S_t = x\}.$$

Proof. The proof is immediate by resorting to a variant of Bayes' formula:

$$E\{\xi(t)^2|S_t = x\} = E[\xi(t)^2 \sum_{k=1}^{m} 1\{\xi = \nu_k\}|S_t = x] = \sum_{k=1}^{m} E[\xi(t)^2\ 1\{\xi = \nu_k\}|S_t = x]$$

$$= \sum_{k=1}^{m} E[\xi(t)^2|S_t = x, \xi = \nu_k]Q\{\xi = \nu_k|S_t = x\} = \sum_{k=1}^{m} \nu_k^2(t) Q\{\xi = \nu_k|S_t = x\} = \sigma^2_{\text{mix}}(t,x)$$

since, by Bayes' formula,

$$Q\{\xi = \nu_k|S_t = x\} = Q\{S_t \in dx|\xi = \nu_k\}Q\{\xi = \nu_k\}/Q\{S_t \in dx\} = \Lambda_k(t,x).$$

□

Remark 3.3. *(Casting some light on the "zero terminal correlation" result of Theorem 3.1) The terminal correlation computed at time* 0 *between the asset* S *and its average variance* $\int_0^T \xi^2(t)dt$ *is easily seen to be zero, due to independence of* ξ *and* W*. The same property is shared by the local volatility version, as was pointed out in Theorem 3.1. Now we see that the local volatility version maintains the decorrelation pattern between volatility and underlying asset that is so natural for its uncertain volatility originator. The result of Theorem 3.1 looks less surprising in the light of this result for the uncertain volatility version.*

3.5 Evolution of the volatility smile in the two models

We now look at another important feature of the lognormal mixture diffusion that has not been investigated in our earlier papers. We are interested in looking at the smile evolution in time as implied by the model.

We investigate this matter numerically as follows. We consider a set of parameters coming from a possible calibration of model (10) with $m = 2$ to the foreign exchange market. The initial foreign exchange rate as from february 10, 2003 is $S_0 = 1.07$ US Dollars for 1 Euro. Calibration of the model to market data provides us with the parameters (we assume the ν's to be constant in time, except for a negligible initial time-interval $[0, 2\epsilon]$)

$$\lambda_1 = 0.9747,\ \lambda_2 = 0.0253,\ \nu_1(t) = 0.7572,\ \nu_2(t) = 0.0899.$$

The risk-neutral drift $r(t)$ is taken consistently with the differences of interest rates in the domestic and foreign curves: indeed, we know that under the risk neutral measure, the drift of an exchange rate is the difference of instantaneous interest rates between the domestic and the foreign markets, $r(t) := r_d(t) - r_f(t)$, where interest rates are assumed to be deterministic and r_d, r_f, R_d, R_f denote respectively the domestic and foreign instantaneous interest rates and their integrals. The initial smile for an option maturing in one year, $K \mapsto V(1y, K)$ is given in the $0y$-row of Table 2. We perform the following test. We consider options with one-year time to maturity $T - t = 1y$ set at future times $t > 0$, conditional on the average underlying being realized, i.e. conditional on $S_t = \bar{S}_t := E_0(S_t)$, the expectation taken under the risk neutral measure. We therefore price the option with our model (10), by resorting to an Euler scheme with time step of 1/1000y (Antignani (2003)) and inverting then the corresponding Black Scholes formula. Indeed, let us define $V(t, T, K)$ as the solution of the equation

$$e^{-R_d(t,T)} E_t[(S_T - K)^+ | S_t = \bar{S}_t] = \text{BSCall}(\bar{S}_t, K, T - t, R_d(t, T), R_f(t, T), V(t, T, K)). \tag{14}$$

By solving the above equation we infer what we call the conditional future smile at t for maturity T, i.e. $K \mapsto V(t, T, K)/\sqrt{T - t}$. We will take $t = 1y, 2y, 3y, 6y, 7y$ and $T = t + 1y$. We thus focus on options maturing in one year and on their smile as implied by model (10), which has been employed to compute the expectation on the left hand side of (14). Interest rates are taken from the market and assumed to be deterministic. Table 1 reports the market values we used. We recall that in a deterministic interest-rates world $R(t, T) = R(0, T) - R(0, t)$.

T	$e^{-R_d(0,T)}$	$e^{-R_f(0,T)}$
1y	0.974454	0.985738
2y	0.946724	0.960891
3y	0.914757	0.925555
4y	0.879548	0.885228
5y	0.841922	0.84227
6y	0.803363	0.799019
7y	0.764915	0.756466

Table 1: Domestic and foreign discount factors (Euro and USD)

We obtain the results reported in Table 2 as annualized percentage implied volatilities. We notice that the one-year smile flattens considerably in time (and similar considerations apply to longer maturity smiles). The initial smile (first row) shows an excursion from 16.17% to 10.72% and then up to 13.61%, whereas the last smile (7y/8y smile, last row) moves from 10.26% down to 9.55% and up again to 9.77%. As we see by looking at the rows of the table, the smile flattens considerably in time.

Again, the one-dimensional diffusion model (10) mimics the uncertain volatility model (13) of which it is the local volatility version. Indeed, since after time ϵ we know the realization of ξ, we know which volatility ν_i realized itself and, conditional on this, our process (13) is a geometric Brownian motion and implies a flat smile. Therefore,

		$K/\bar{S}_t$								
t	$\bar{S}_t$	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
0y	1.07	16.17	13.73	11.98	11.02	10.72	10.84	11.30	12.26	13.61
1y	1.08257	11.23	10.44	10.10	9.98	9.96	10.00	10.10	10.30	10.62
2y	1.08628	10.10	9.95	9.86	9.83	9.83	9.88	9.97	10.07	10.24
3y	1.08294	10.60	10.09	9.92	9.83	9.81	9.82	9.85	9.91	10.02
6y	1.06401	11.14	10.20	9.84	9.71	9.68	9.66	9.67	9.72	9.85
7y	1.05773	10.26	9.83	9.67	9.60	9.57	9.55	9.57	9.64	9.77

Table 2: Conditional future 1y smile $K \mapsto V(t, t+1y, K)$ for $t = 0, 1, 2, 3, 6, 7$ years

after time ϵ the smile flattens completely around the realized ξ in the model (13), and the one-dimensional diffusion version mimics this behaviour by progressively flattening the implied smile in time.

4 Conclusions and further research

In the present paper, we have found a candidate diffusion process whose marginal law follows a given evolving mixture of probability densities. We derived a stochastic differential equation (SDE) admitting a unique strong solution whose density evolves as a mixture of Gaussian densities. We introduced a seemingly paradoxical result on the comparison between the instantaneous and the terminal correlation between the obtained process and its time-averaged squared diffusion coefficient. As an application to option pricing, we considered diffusion processes whose marginal densities are mixtures of lognormal densities, showing how such processes can be used to model the market smile phenomenon. Furthermore, we have pointed out how the lognormal mixture dynamics is the one-dimensional diffusion version of a geometric Brownian motion with uncertain volatility, leading to an uncertain volatility Black-Scholes model, which allowed us to suitably reinterpret the earlier correlation result. Finally, we checked numerically the evolution of the smile in time and found that the diffusion model mimics again the uncertain volatility model by a substantial flattening of the smile implied curve in time.

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