Inflation-Indexed Credit Default Swaps

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Abstract

The aim of this work is to develop a pricing model for a kind of contract that we term "inflation indexed credit default swaps (IICDS)". IICDS' payoffs are linked to inflation, in that one of the legs of the swap is tied to the inflation rate. In particular, the structure exchanges consumer price index (CPI) growth rate plus a fixed spread minus the relevant libor rate for a protection payment in case of early default of the reference credit. This is inspired by a real market payoff we manged in our work. The method we introduce will be applied to our case but is in fact much more general and may be envisaged in situations involving inflation / credit / interest rate hybrids. The term IICDS itself can be associated to quite different structures. Many variables enter our IICDS valuation. We have the CPI, the nominal and real interest rates, and the default modeling variables. For our pricing purposes we need to choose a way of modeling such variables in a convenient and practical fashion. Our choice fell on the familiar short rate model setting, although frameworks based on recent market models for credit and inflation could be attempted in principle, for example by combining ideas on Credit Default Swap Market Models (Schönbucher 2004, Brigo 2005) with ideas on Inflation Market Models (Belgrade, Benhamou and Koehler 2004, Mercurio 2005). We discuss numerical methods such as Euler discretization and Monte Carlo simulation for our pricing procedure based on gaussian and CIR short rate models for rates and default intensity. We analyze the numerical results in details and discuss the impact of correlation between the different rates on the valuation.

Contents

Chapter 1

Introduction

The aim of this work is to develop a pricing model for a recent type of credit derivative contracts. Such contracts have their payoffs linked to some current rate of inflation. This is to say they have a particular floating-to-floating interest rate structure in which one of the floating legs is tied to the inflation rate. Meanwhile, being these credit derivatives, their whole contingent claim depends on the creditworthiness of a third party. We will generally name contracts of this kind "Inflation-Indexed Credit Default Swaps" or IICDS.

It will be clear that these contracts find their economic sense when two (or more) parties agree on the following assumptions in interpreting the market: the protection seller believes that the spread between inflation and nominal interest rates will rise but thinks this will not affect the overall economic situation and that the credit quality of the contractual underlying economic agent will not change substantially whilst the protection buyer thinks inflation will rise and that this may deteriorate the creditworthiness of economic agents and of the underlying in particular. Usually the buyer of such contracts has opened positions with one or more of such economic agents and fears the possibility of a credit event affecting his wealth or current income.

Many variables enter the inflation indexed credit default swap payoffs. We have the Consumer Price Index (CPI), the nominal and real interest rates, and the default modeling variables. For our pricing purposes we needed to choose a way of modeling such variables in a convenient and practical fashion. The choice of the best suited "technology" fell on the familiar short rate model setting, although frameworks based on recent market models for credit and inflation could be attempted in principle, for example by combining ideas on Credit Default Swap Market Models (Schonbucher 2004, Brigo 2005) with ideas on Inflation Market Models (Benhamou 2004, Mercurio 2005).

We rely on the Jarrow-Yildirim and on the Hull-White models for the nominal and the short rates, on the log-normal setting for the Consumer

Price Index dynamics and on the Cox-Ingersoll-Ross model to determine the stochastic intensity of default within the contract.

Our pricing procedure tries to define the fixed δ spread that has to be added to one of the legs to render the contract fair at inception time $t = 0$. This will be done under different assumptions on correlation patterns between the previously outlined variables.

As usual the price of our derivative is given by the expected value of its discounted payoff. Due to the complexity of the IICDS payoff there is no analytical closed form solution to compute such expectation and so numerical methods such as Euler discretization and Monte Carlo simulation will be required in the pricing procedure.

In the following sections we will briefly introduce the general framework. We describe the methods used for modeling inflation and for modeling default. In the last sections we will use the defined methods to assemble our pricing model for IICDS's and to test it empirically. At last we present the final conclusions.

Chapter 2

An Overview of Inflation Modeling

2.1 Inflation

Over the last decades a generalized and constant rise in the prices of goods, partly driven by the growing worldwide demand and partly due to the rise of the cost of energetic resources, claimed the introduction of new types of financial instruments.

These instruments have been defined so as to preserve individuals invested wealth at a minimum guaranteed purchasing power throughout the years.

This minimum guaranteed purchasing power is achieved by linking the payoffs offered by these instruments to the growth rate of prices, i.e. to inflation.

Inflation-indexed financial instruments are not new entries in the financial industry: these products have been regularly issued for thirty years by all kinds of institutions; what is actually new is the recent application this inflation-index feature has had on the world of derivative instruments and in particular on hybrid products using Credit Default Swaps features.

The periodic inflation rate is defined as the ratio between the values of a given price index computed at two different times. The most used price indexes are the CPIs or Consumer Price Indexes. These are prices of baskets of reference goods and the change in their value is the best proxy for the rate of inflation.

Since the evolution of CPI is random, to price inflation-indexed derivatives we must retrieve such evolution via stochastic models. In this work we will refer to the Brigo-Mercurio (2006) short- rate reformulation of the Jarrow-Yildirim (2003) model. The model is built on the Heath-Jarrow-

Morton (1992) framework and is based on the foreign-currency analogy.

Here the evolution of instantaneous real and nominal rates is modeled together with the evolution of the Consumer Price Index. Nominal rates are interpreted as the domestic rates while real rates are interpreted as foreign economy rates. The CPI is treated as the "exchange rate" between the nominal and real economies.

2.2 Inflation Modeling: General Framework

According to the foreign-currency analogy of Jarrow-Yildirim (2003), the evolution of inflation is modeled via the following approach: nominal and real rates are treated as domestic and foreign rates respectively and the CPI is treated as if it were the "exchange rate" between associated markets.

If we denote by $I(t)$ the value of the CPI index at time t, we have that, to buy the reference basket of goods in $t = 0$, one has to spend $I(0)$ monetary units.

Computing the ratio between the value of the index at time t and at time 0 we obtain the total return of the CPI in $[0, t]$:

$$\frac{I(t)}{I(0)} = 1 + \text{inflation rate between time 0 and } t.$$

The ratio $I(t)/I(0)$ can be interpreted as an exchange rate: multiplying a given amount of "real currency" at time t by this ratio we obtain the corresponding amount of "nominal currency" at time t. By taking the reciprocal of such ratio one is asking which is the real amount with respect to time 0. He is thus "exchanging" a nominal value into a real value at time t. Actually this is very similar to the approach one uses when converting two different currencies. The only difference is that here we compare the nominal cost of a basket of goods at some given time t with its initial cost making it into a real value. In the FX market we instead exchange two different currencies at the same time making one currency in its value with respect to the other currency.

It comes straightforward that setting $I(0) = 1$, it is possible to convert nominal values to real values at a generic time t just dividing by the CPI level at time t. This means that if an asset at time t has a nominal value of X its value in real terms is $X/I(t)$.

2.3 Zero Coupon and Year-on-Year Inflation-Indexed Swaps

Before resorting to the Jarrow and Yildirim (2003) model for pricing inflation-indexed model dependent derivatives, we shall take a short overview of the most important inflation-indexed swaps, such as zero coupon (ZC) inflation-indexed swaps and year-on-year (YY) inflation-indexed swaps. This is necessary because these contracts can be considered as calibration inputs of our particular pricing model. ZCIIS are model independent derivatives and can be priced by general no-arbitrage arguments while YYIIS involve some interest rate modeling and will be our reason for introducing the short rate framework.

A zero coupon inflation-indexed swap is a contract that has a single cash flow occurrence at a final time in T years. In such contract, in fact, party A pays to party B, at maturity, a floating payoff equal to

$$N\left[\frac{I(T)}{I_0} - 1\right], \tag{2.1}$$

while party A receives, at maturity, from party B the fixed amount

$$N[(1+K)^T - 1], \tag{2.2}$$

where K is a fixed interest rate and N is the contract notional.

A year-on-year inflation-indexed swap is slightly different from ZC swaps. In this case we have a set of fixed payments from party B to party A at each time T_i:

$$N\phi_i K, \tag{2.3}$$

where ϕ_i is the year fraction of the fixed leg over the interval $[T_{i-1}, T_i]$. Party A, conversely, at each time T_i, pays the floating amount

$$N\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]. \tag{2.4}$$

In this case let ψ_i denote the year fraction of the floating payment over the time interval $[T_{i-1}, T_i]$. Actually, the CPI fixings typically do not coincide with the payment scheduled T's but can be displaced of a few weeks. Here to simplify exposure we assume the two sets of dates to coincide. As for the ZCIIS, K can be set to the fixed rate that renders the YY swap contract fair at time 0.

The valuation of such contracts can be exploited via the usual no-arbitrage pricing theory.

Starting with the ZCIIS we can see that the fixed leg valuation is straightforward and needs no in-depth analysis: one simply discounts the known final fixed payment with the nominal zero-coupon bond. The floating leg valuation, on the contrary, needs some attention.

From simple no-arbitrage considerations, the value of a ZCIIS at any given time t is the risk-neutral expected value, with respect to filtration $\mathcal{F}_t$, of its nominal discounted payoff at maturity:

$$ZCIIS(t,T,I_0,N) = N\mathbb{E}_n\left\{e^{-\int_t^T n(u)du}\left[\frac{I(T)}{I_0}-1\right]|\mathcal{F}_t\right\}. \tag{2.5}$$

One should recall that the usual risk neutral pricing measure we are using features the nominal bank account as numeraire. The numeraire is thus $e^{\int_0^T n(u)du}$. To simplify the computation of such expectation we use a foreign-currency analogy result. Supposing we are at time t, such result states that the expected present value of a foreign payoff $S(T)$ (at maturity), discounted back at t with the foreign interest rate and taken with respect to the foreign risk-neutral probability f , multiplied by the exchange rate X at time t, is equal to the expectation of the domestically discounted foreign payoff multiplied by the exchange rate at time T, taken with respect to the domestic risk-neutral probability d.

In mathematical terms this looks like:

$$X(t)\mathbb{E}_f\left[D_f(t,T)S(T)|\mathcal{F}_t\right] = \mathbb{E}_d\left[D_d(t,T)S(T)X(T)|\mathcal{F}_t\right], \tag{2.6}$$

where D_f and D_d are stochastic discount factors. This analogy, applied to our context, and considering a unit payoff X, yields the following result:

$$I(t)\mathbb{E}_r\left[D_r(t,T)|\mathcal{F}_t\right] = \mathbb{E}_n\left[D_n(t,T)I(T)|\mathcal{F}_t\right], \tag{2.7}$$

which, given the usual definition of zero-coupon bond price, can be rewritten as:

$$I(t)P_r(t,T) = \mathbb{E}_n\left[D_n(t,T)I(T)|\mathcal{F}_t\right]. \tag{2.8}$$

By plugging this result in (2.5), we obtain the following model independent ZCIIS price:

$$ZCIIS(t,T,I_0,I_t,N) = N\left[\frac{I(t)}{I_0}P_r(t,T) - P_n(t,T)\right]. \tag{2.9}$$

Needless to say, at time 0 this equation simplifies to

$$ZCIIS(0,T,N) = N\left[P_r(0,T) - P_n(0,T)\right]. \tag{2.10}$$

This is a fine result since, as already said, the pricing formula is model independent. This is like saying that no assumption on the future evolution of interest rates is required to perform valuation. This conclusion is empirically very helpful because it gives us an easy way for stripping real discount factors from the term structure of nominal discount factors and zero-coupon swaps. This stripping procedure is performed, at the present time, by equating the value of the ZCIIS to zero. Remembering the fixed leg definition given in (2.2) we obtain the following result by equating (2.10) to the nominal present value of (2.2):

$$P_r(0,T) = P_n(0,T)(1+K(T))^T. \tag{2.11}$$

The market quotes, with daily frequency, fixed values of $K = K(T)$ for different maturities that render the various ZCIIS fair in the considered date $t = 0$. By substituting these values of K and the respective nominal discount factors, in the above formula, it is straightforward to obtain the term structure of real discount factors .

Albeit not so different in their essence from the ZCIIS, year-on-year inflation-indexed swaps are not so easy to price and require, in performing their valuation, another kind of approach.

According to (2.4), the value at time t of a single floating payment of a YYIIS, due in $T_i > t$, is:

$$YYIIS(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i \mathbb{E}_n \left\{ D_n(t,T_i) \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] |\mathcal{F}_t \right\}, \tag{2.12}$$

where ψ_i is the usual year fraction of the floating leg. Using the tower property of conditional expectations and remembering the definition of $D_n(t,T)$ we can write such payoff as follows:

$$YYIIS(\cdot) = N\psi_i \mathbb{E}_n \left\{ e^{-\int_t^{T_{i-1}} n(u)du} \mathbb{E}_n \left[e^{-\int_{T_{i-1}}^{T_i} n(u)du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) |\mathcal{F}_{T_{i-1}} \right] |\mathcal{F}_t \right\}. \tag{2.13}$$

Paying some attention to (2.13), one can see that the inner expectation is nothing but a ZCIIS$(T_{i-1}, T_i, I(T_{i-1}))$ with unit nominal. By substituting the homologous of formula (2.10), for time T_{i-1}, in (2.13) one obtains the following structure for the YYIIS:

$$\begin{aligned} &N\psi_i \mathbb{E}_n \{ e^{-\int_t^{T_{i-1}} n(u)du} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] |\mathcal{F}_{T_{i-1}} \} \\ &= N\psi_i \mathbb{E}_n \{ D_n(t, T_{i-1}) P_r(T_{i-1}, T_i) |\mathcal{F}_t \} - N\psi_i P_n(t, T_i). \end{aligned} \tag{2.14}$$

Using the change of measure technique (i.e. applying Girsanov's theorem) we can go from the nominal risk-neutral probability measure to the n

T_{i-1}-forward probability measure. Such procedure justifies taking $D_n(t, T_{i-1})$ outside the expectation operator. This finally yields:

$$N\psi_i P_n(t, T_{i-1})\mathbb{E}_n^{T_{i-1}}\{P_r(T_{i-1}, T_i)|\mathcal{F}_t\} - N\psi_i P_n(t, T_i). \tag{2.15}$$

Problems in the valuation of a YYIIS arise when in need to compute the above expectation. This expectation is model-dependent and to evaluate it we resort to the Jarrow and Yildirim (2003) model that is defined as a special case of the Hull-White (1990) model.

2.4 A short guide to the Hull-White interest rate model

In order to fully understand the Hull-White (1990) approach to the inflation dynamics derived by Jarrow and Yildirim (2003) it is useful to briefly define the Hull and White model for the instantaneous short rate.

Such model improves the time homogeneous model for short rates defined by Vasicek (1977). Such core model has in fact problems in retrieving the exact fit to the current term structure of interest rates when this is taken as an input, i.e. its calibration to market data returns parameters that make such model fit to a limited number of term structure shapes. This is due to the time invariance of the parameters in the Vasicek model.

A bad term structure interpolation, or more precisely, having the term structure as an output rather than an input, affects the concrete applicability of the model in pricing interest rate derivatives. To overcome such limitations Hull and White (1990) proposed a modification with time varying coefficients (hence abandoning the time homogeneous approach). The original formulation contemplated by Hull and White was with three time varying coefficients but here we resort to a simpler case in which only the mean reverting coefficient is free to vary across time. This adds infinite degrees of freedom in term structure fitting. The model we use is the following:

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t). \tag{2.16}$$

Here a and σ are positive constant coefficients and the boundary condition on the s.d.e is $r(0) = r_0$. W is a Brownian motion under the risk neutral probability. Notice that here, in exposing the Hull and White model, r is the usual short rate, i.e. the nominal rate we denote by n when dealing with inflation.

The term $\theta(t)$ leads to an exact fit of the initial term structure of interest rates if it is chosen as follows:

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{2.17}$$

where

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T}, \tag{2.18}$$

is the instantaneous forward rate derived from current market data.

With the Hull and White Vasicek extension for instantaneous short rates we are thus able to model the fundamental rate r while gaining a perfect fit to market data and keeping the analytical tractability guaranteed by the gaussianity assumption on r. This comes with a cost, i.e. the non zero probability of negative interest rates, as opposed to other models like the Cox-Ingersoll-Ross (1985). It can be proved that such possibility is almost negligible in practical applications.

Now we try and illustrate a simple application of the Hull and White model in the valuation of a zero-coupon bond. As seen earlier, the price at time t of a ZC with unit notional and maturity T, in a stochastic interest rate setting, is nothing but the value of the following expectation:

$$\begin{aligned} P(t,T) &= \mathbb{E}\Big[D(t,T)|\mathcal{F}_t\Big] = \\ &= \mathbb{E}\Big[\exp\Big(-\int_t^T r(s)ds\Big)|\mathcal{F}_t\Big]. \end{aligned}$$

Such expectation is easily computed once we know the distribution of the random variable r and its complete specification. In a gaussian framework like ours, all we need at this point is the mean and variance of such random variable. This is achieved thanks to our knowledge of the process r.

Starting from $dr(t)$ and defining a derivative process $Y = r(t)e^{at}$, we apply Ito's lemma to find dY. Once we get this differential, we take the integral from time s to t and we switch variables again, obtaining the following expression for $r(t)$ (see Brigo-Mercurio (2001)):

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-s)}\theta(u)du + \sigma \int_s^t e^{-a(t-s)}dW(u) \tag{2.19}$$

$$= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-s)}dW(u), \tag{2.20}$$

with

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2. \tag{2.21}$$

After some tedious algebra, we can finally compute $\int_t^T r(u)du$ and hence come up with its distribution:

$$\int_t^T r(u)du \sim N\left(B(t,T)[r(t)-\alpha(t)]+\ln\frac{P^M(0,t)}{P^M(0,T)}+\frac{1}{2}[V(0,T)-V(0,t)], V(t,T)\right), \tag{2.22}$$

where

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{2.23}$$

and

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]. \tag{2.24}$$

Given that the integral of r is a normal random variable and given that we know it can be found at the exponent in our payoff formulation, we can easily find the expectation of the zero coupon bond value using the moment generating function, i.e.:

$$\mathbb{E}[e^{\lambda X}] = e^{a\lambda + \frac{1}{2}b\lambda^2}, \quad \text{with } X \sim N(a,b). \tag{2.25}$$

Applying such function and rearranging terms we get:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{2.26}$$

with

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2a})B(t,T)^2\right\}. \tag{2.27}$$

2.5 The Jarrow-Yildirim model

Before illustrating the model we introduce some notation.

As in Brigo-Mercurio (2006), nominal and real economy quantities are respectively denoted with the subscripts n and r.

The instantaneous nominal and real forward rates, whose dynamics Jarrow and Yildirim (2003) start from, are defined as follows:

$$f_x(t,T) = -\frac{\partial \ln P_x(t,T)}{\partial T} \quad , \quad x \in \{n, r\}. \tag{2.28}$$

Integrating both sides with respect to maturity, we have that the nominal discount factor is:

$$\begin{aligned} P_n(t,T) &= \mathbb{E}\Big[D_n(t,T)|\mathcal{F}_t\Big] = \\ &= \exp\Big(-\int_t^T f_n(t,s)ds\Big), \end{aligned}$$

and the real discount factor is

$$\begin{aligned} P_r(t,T) &= \mathbb{E}\Big[D_r(t,T)|\mathcal{F}_t\Big] = \\ &= \exp\Big(-\int_t^T f_r(t,s)ds\Big). \end{aligned}$$

We then obtain the term structure of nominal discount factors as $T \to P_n(t,T)$ and that of real discount factors as $T \to P_r(t,T)$ for $T \geq 0$.

Jarrow and Yildirim (2003), in describing the evolution of nominal and real instantaneous forward rates and to describe the evolution of the CPI, define the following dynamics under the real world probability space $(\Omega, \mathcal{F}, P)$, associated with filtration $\mathcal{F}$:

$$df_n(t,T) = \alpha_n(t,T)dt + \beta_n(t,T)dW_n^{P(t)} \tag{2.29}$$

$$df_r(t,T) = \alpha_r(t,T)dt + \beta_r(t,T)dW_r^{P(t)} \tag{2.30}$$

$$dI(t) = I(t)\mu(t)dt + I(t)\sigma_I(t)dW_I^P(t). \tag{2.31}$$

with $I(0) = I_0 \geq 0$, and

$$f_x(0,T) = f_x^M(0,T), \quad x \in \{n, r\}.$$

We define

- (W_n^P, W_r^P, W_I^P) as a vectorial Brownian motion with correlations $\rho_{n,r}$, $\rho_{r,I}$ and $\rho_{n,I}$;
- β_n and β_r are deterministic functions;
- α_n, α_r and μ are processes adapted to $\mathcal{F}$;
- $\sigma_I(t) = \sigma_I$ is a constant with $\sigma_I \geq 0$;
- $f_n^M(0,T)$ and $f_r^M(0,T)$ are the nominal and real instantaneous forward rates observed in the market in $t = 0$ with maturity $t = T$.

Here we move from the Jarrow-Yildirim (2003) model, built on the Heath-Jarrow-Morton (1992) instantaneous forward rates framework, to the equivalent Hull-White short rate model formulation. Furthermore, we model our interest rate process directly under the risk neutral measure.

We implement a gaussian Hull-White stochastic model for the short rates, as suggested in Brigo-Mercurio (2006), instead of a model for the instantaneous forward rates, to render calculations easier. To do so, we choose to model instantaneous volatilities as the following deterministic functions of time:

- $\beta_n(t,T) = \sigma_n e^{-a_n(T-t)}$,
- $\beta_r(t,T) = \sigma_r e^{-a_r(T-t)}$,

with σ's and a's positive constants.

The equivalent instantaneous short rate formulation under the n-dynamics of the nominal and real instantaneous rates and of the CPI is as follows:

$$dn(t) = [\theta_n(t) - a_n n(t)]dt + \sigma_n dW_n(t) \tag{2.32}$$

$$dr(t) = [\theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]dt + \sigma_r dW_r(t) \tag{2.33}$$

$$dI(t) = I(t)[n(t) - r(t)]dt + \sigma_I I(t)dW_I(t), \tag{2.34}$$

with (W_n, W_r, W_I) a vectorial Brownian motion, under the risk neutral measure, with correlations $\rho_{n,r}$, $\rho_{r,I}$ and $\rho_{n,I}$.

The terms $\theta_n(t)$ and $\theta_r(t)$ are the usual deterministic functions from the Hull-White model used to exactly fit the input current term structure of nominal and real rates. As from Brigo-Mercurio (2006) (see also (2.17) above) these are defined as follows:

$$\theta_x(t) = \frac{\partial f_x(0,t)}{\partial T} + a_x f_x(0,t) + \frac{\sigma_x^2}{2a_x}(1 - e^{-2a_x t}), x \in \{n, r\} \tag{2.35}$$

For a detailed proof see Brigo-Mercurio (2006).

We can now solve for $I(t)$. By setting $Y = \ln I(t)$ and remembering Ito's lemma we have that:

$$dY(t) = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial I(t)}dI(t) + \frac{1}{2}\frac{\partial^2 Y}{\partial I(t)^2} < dI(t) >^2 . \tag{2.36}$$

By calculating the partial derivatives and substituting for the respective elements we get:

$$dY(t) = [n(t) - r(t)]dt - \frac{1}{2}\sigma_I^2 dt + \sigma_I dW_I(t) \tag{2.37}$$

Here we can finally take the integral from t to T and with a second change of variables we get the explicit process for $I(T)$:

$$I(T) = I(t)e^{\int_t^T [n(t)-r(t)]du - \frac{1}{2}\sigma_I^2(T-t)+\sigma_I(W_I(T)-W_I(t))}. \tag{2.38}$$

$I(T)$ is log-normally distributed according to the gaussianity of the nominal and real short rates and the Brownian motions in the exponential.

2.6 Pricing with the Jarrow-Yildirim/Hull-White model

2.6.1 Pricing of Year-on-year Inflation Indexed Swaps

The first application of the Jarrow-Yildirim (2003) model is within the pricing of the YYIIS we left unfinished in Section 2.3. Such contract had the following structure:

$$N\psi_i P_n(t,T_{i-1})\mathbb{E}_n^{T_{i-1}}\{P_r(T_{i-1},T_i)|\mathcal{F}_t\} - N\psi_i P_n(t,T_i),$$

and our problem was the difficulty in computing $\mathbb{E}_n^{T_{i-1}}\{P_r(T_{i-1},T_i)|\mathcal{F}_t\}$. To compute such expectation we need to model the real rate under the nominal T_{i-1}-forward probability measure so we must use the change of numeraire technique once again. The Jarrow-Yildirim definition of the real instantaneous rate under the nominal probability as from Section 2.5 is:

$$dr(t) = [\theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]dt + \sigma_r dW_r(t)$$

Applying the change of measure to such dynamics we get:

$$dr(t) = [\theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t) - \rho_{n,r}\sigma_r\sigma_n B_n(t,T_{i-1})]dt + \sigma_r dW_r^{T_{i-1}}(t), \tag{2.39}$$

with $W^{T_{i-1}}$ a Brownian motion under the nominal T_{i-1}-forward measure.Under such measure, the expected value of $P_r(T_{i-1},T_i)$ is obtained after quite cumbersome calculations and (as in Brigo-Mercurio (2006)) is given by:

$$P_r(T_{i-1},T_i) = \frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)}, \tag{2.40}$$

with

$$\begin{aligned} C(t,T_{i-1},T_i) = \sigma_r B_r(T_{i-1},T_i)\Bigg[B_r(t,T_{i-1})\Bigg(\rho_{r,I}\sigma_I - \frac{1}{2}\sigma_r B_r(t,T_{i-1}) \\ + \frac{\rho_{n,r}\sigma_n}{a_n+a_r}(1+a_r B_n(t,T_{i-1}))\Bigg) - \frac{\rho_{n,r}\sigma_n}{a_n+a_r}B_n(t,T_{i-1})\Bigg]. \end{aligned} \tag{2.41}$$

and P_r, A_r and B_r as in Section 2.4:

$$\begin{aligned}
P_r(t,T) &= A_r(t,T)e^{-B_r(t,T)r(t)},\\
B_r(t,T) &= \frac{1}{a}\left[1-e^{-a_r(T-t)}\right],\\
A_r(t,T) &= \frac{P_r^M(0,T)}{P_r^M(0,t)}\exp\left\{B_r(t,T)f_r^M(0,t)-\frac{\sigma_r^2}{4a_r}(1-e^{-2a_r})B_r(t,T)^2\right\}.
\end{aligned}$$

We now have all the elements we need for pricing a one period year-on-year inflation indexed swap; this is:

$$Y(t,T_{i-1},T_i,\psi_i,N) = N\psi_i P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)} - N\psi_i P_n(t,T_i). \tag{2.42}$$

The value of the whole contract is the summation of all the payments and in $t=0$ this is:

$$\begin{aligned}
Y(0,T_M,\Psi,N) &= N\psi_1[P_r(0,T_1)-P_n(0,T_1)] \qquad (2.43)\\
&+N\sum_{i=2}^{M}\psi_i\left[P_n(0,T_{i-1})\frac{P_r(0,T_i)}{P_r(0,T_{i-1})}e^{C(0,T_{i-1},T_i)}-P_n(0,T_i)\right]\\
&= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{1+\tau_i F_n(0,T_{i-1},T_i)}{1+\tau_i F_r(0,T_{i-1},T_i)}e^{C(0,T_{i-1},T_i)}-1\right],
\end{aligned}$$

where

$$F_x(t,T_{i-1},T_i) = \frac{1}{\tau_i}\left[\frac{P_x(t,T_{i-1})}{P_x(t,T_i)}-1\right], \quad x\in\{n,r\}. \tag{2.44}$$

2.6.2 Pricing of Inflation Indexed Caps and Floors

Another contract related to our final pricing model is the Inflation Indexed Cap. Here we take a brief overview of such contract and highlight its pricing procedure via the Jarrow-Yildirim model. To simplify our treatment we start from the core components of an Inflation Indexed Cap i.e. Inflation Indexed Caplet or IIc. An IIc is a call option contract that pays inflation, which, as seen previously can be considered as the "rate of return" of the CPI index, minus a predefined strike. Obviously the contingent claim pays out only if such difference is positive. The payoff structure of the IIc is as follows:

$$N\psi_i\left[\left(\frac{I(T_i)}{I(T_{i-1})}-K\right)^+\right], \tag{2.45}$$

where notation retains our usual meaning and $K = 1+k$ where k is the strike level.

The price at time t of such payoff derives from no-arbitrage theory and is equal to the present value of its expectation under the appropriate probability measure. This is:

$$\begin{aligned} IIc(t, T_{i-1}, T_i, \psi_i, K, N) &= N\psi_i \mathbb{E}_n\left[e^{-\int_t^{T_i} n(u)du}\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)^+ |\mathcal{F}_t\right] \\ &= N\psi_i P_n(t, T_i)\mathbb{E}_n^{T_i}\left[\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)^+ |\mathcal{F}_t\right]. \end{aligned} \tag{2.46}$$

When pricing with the Jarrow-Jildirim model we know we have Gaussian nominal and real interest rates that lead to a log-normally distributed CPI and as a consequence, to log-normally distributed CPI ratios. This log-normality of the Consumer Price Index holds when moving from the nominal risk neutral probability measure Q_n to the nominal forward measure $Q_n^{T_i}$. This is an important point since we need to price such contingent claim under its appropriate probability measure $Q_n^{T_i}$. Thanks to the previous property we now can apply a useful result from probability to complete the contract valuation. Setting $I(T_i)/I(T_{i-1}) = X$ and, knowing that X is log-normally distributed, we can compute its expectation and the variance of its logarithm. This permits us to value the contingent claim. In fact, with $\mathbb{E}(X) = f$ and Std[X] $= s$ we have:

$$\mathbb{E}[(X-K)^+] = f\Phi\left(\frac{\ln\frac{f}{K} + \frac{s^2}{2}}{s}\right) - K\Phi\left(\frac{\ln\frac{f}{K} - \frac{s^2}{2}}{s}\right), \tag{2.47}$$

where Φ is the CDF of a Standard Normal Gaussian RV.

Now, remembering the price of the YYIIS we can easily obtain the expected value of X, i.e.:

$$\mathbb{E}_n^{T_i}\left[X = \frac{I(T_i)}{I(T_{i-1})}|\mathcal{F}_t\right] = \frac{P_n(t, T_{i-1})}{P_n(t, T_i)}\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}e^{C(t, T_{i-1}, T_i)} = f.$$

For what concerns the variance of the logarithm of X we have:

$$\mathrm{Var}_n^{T_i} = \left[\ln\frac{I(T_i)}{I(T_{i-1})}|\mathcal{F}_t\right] = V^2(t, T_{i-1}, T_i) = s^2.$$

After some cumbersome calculations we have that such variance is (see Brigo-

Mercurio (2006)):

$$
\begin{aligned}
V^2(t,T_{i-1},T_i) = {} & \frac{\sigma_n^2}{2a_n^3}(1-e^{a_n(T_i-T_{i-1})})^2(1-e^{-2a_n(T_{i-1}-t)}) + \sigma_I^2(T_i-T_{i-1}) \\
& + \frac{\sigma_r^2}{2a_r^3}(1-e^{a_r(T_i-T_{i-1})})^2(1-e^{-2a_r(T_{i-1}-t)}) - 2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_n a_r(a_n+a_r)} \\
& \cdot(1-e^{a_n(T_i-T_{i-1})})(1-e^{-a_r(T_i-T_{i-1})})(1-e^{-(a_n+a_r)(T_{i-1}-t)}) \\
& + \frac{\sigma_n^2}{a_n^2}\left[T_i-T_{i-1}+\frac{2}{a_n}e^{-a_n(T_i-T_{i-1})}-\frac{1}{2a_n}e^{-2a_n(T_i-T_{i-1})}-\frac{3}{2a_n}\right] \\
& + \frac{\sigma_r}{a_r^2}\left[T_i-T_{i-1}+\frac{2}{a_r}e^{-a_r(T_i-T_{i-1})}-\frac{1}{2a_r}e^{-2a_r(T_i-T_{i-1})}-\frac{3}{2a_r}\right] \\
& - 2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_n a_r}\left[T_i-T_{i-1}-\frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}-\frac{1-e^{-a_r(T_i-T_{i-1})}}{a_r}\right. \\
& \left.+\frac{1-e^{-(a_n+a_r)(T_i-T_{i-1})}}{a_n+a_r}\right] + 2\rho_{n,I}\frac{\sigma_n\sigma_I}{a_n}\left[T_i-T_{i-1}-\frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}\right] \\
& - 2\rho_{r,I}\frac{\sigma_r\sigma_I}{a_r}\left[T_i-T_{i-1}-\frac{1-e^{-a_r(T_i-T_{i-1})}}{a_r}\right].
\end{aligned}
\tag{2.48}
$$

Here all notation maintains the previously given meaning.

Now by substituting the expectation and the variance of the logarithm of X in formula (2.47), discounting with the appropriate discount factor and remembering notional and year fractions, we obtain the price of the Inflation Indexed Caplet. An Inflation Indexed Cap (IIC) is nothing but a stream of Inflation Indexed Caplets.

Chapter 3

Introduction to credit risk

In recent times, policy regulators and institutions worldwide are stressing the importance of introducing an efficient counter-party risk management in financial operations.

Attention to modeling the probability of default or the creditworthiness of economic subjects has sharply increased also due to the latest bankruptcies that shock financial communities at an international level with important names such as Enron, Parmalat or Worldcom involved, to name just a few. This led to a higher level of analysis and foreseeing in the investment procedures and a willingness to hedge the possibility of mishaps deriving from counter-party risk.

In the meanwhile banks, financial institutions and any other corporation have been struggling to align themselves to the International Accounting Standards that obliges firms to mark-to-market their financial investments almost on a daily basis.

This particular environment has been a fertilizer to the development and diffusion of credit derivatives. Such instruments have recently become extremely popular in the financial industry.

3.1 A brief overview on credit default swaps

Counter-party risk is an intrinsic component of almost any type of economic transaction: it is the risk that a subject, who owes us money on the basis of a contractual agreement, does not respect the obligation to pay it back. This can be due to its deteriorating economic situation or even, which is the worst case, to its default. One could argue that, due to its central importance within any kind of contractual agreement, credit risk should have its own price and such price should enter in the contract valuation. Credit derivatives try to determine such price and offer significant insight on the dynamics underlying the credit risk framework. They are also instruments that allow to hedge the default risk.

A CDS is a contract between two or more subjects. One subject (protection buyer) pays a certain periodical rate, generally fixed at inception, and the other subject (protection seller) pays protection in case of a pre-determined credit event. The periodical payments stop in case of default. The credit event that triggers the protection payoff could be represented by payment delay or default of a third entity (the "reference" credit) on a contractual obligation.

A simplified stylized credit default swap, with unit nominal, protecting from time T_α to time T_β, has the following payoff structure composed by the premium leg (first summation) and the protection leg (second summation):

$$\begin{aligned}\Pi_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} D_n(t,T_i)\psi_i \mathbf{1}_{\{\tau \geq T_i\}} R \\ &- \sum_{i=\alpha+1}^{\beta} D_n(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} \mathrm{LGD}, \end{aligned} \tag{3.1}$$

where

- $[T_\alpha, T_\beta]$ is the interval when protection against default is considered
- $T_{\alpha+1}, T_{\alpha+2}, ..., T_\beta$ are the times when the premium payments R are considered.
- $D_n(t,T)$ is the nominal stochastic discount factor.
- $\mathbf{1}_{\{.\}}$ is the indicator function which is 1 if the statement between curly brackets is true and 0 otherwise.
- τ is the default time of the reference credit.
- R is the fixed rate that renders the contract fair at time 0.
- LGD is the Loss Given Default or percentage value of the nominal payment due by the protection seller. This usually is computed as LGD=1-RR. RR is the recovery rate, i.e. the percentage of the initial nominal investment that one recovers after default.

We can summarize the previous contract, a postponed CDS, as a contract paying protection, to the protection buyer, if a third entity defaults, versus a fixed premium rate R paid to the protection seller. In this formulation the LGD is not paid at the moment of default but is postponed to the first T_i following default. This feature simplifies the contract since there is no accrual-interest on the premium leg, due to the fact that all payments are made at the same instant.

At this point we can introduce the value of a CDS at time t with starting date T_α and maturity T_β. This is simply the expected value of its payoff Π:

$$CDS_{\alpha,\beta}(t, R, LGD) = \mathbb{E}[\Pi_{\alpha,\beta}(t)|\mathcal{G}_t]. \tag{3.2}$$

Here the $\mathcal{G}_t$ σ-algebra is different from the usual default-free $\mathcal{F}_t$ σ-algebra. The filtration $\mathcal{G}_t$ in fact is the complete set of information including default status up to time t, i.e. wether the firm has defaulted or not up to time t, and if so when exactly. In mathematical terms:

$$\mathcal{G}_t \;\; = \;\; \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t). \tag{3.3}$$

Most of times conditioning on $\mathcal{F}_t$ is still preferable and it is made possible by using the filtration switching formula (more on this later).

Valuation of a credit default swap, in a stochastic interest-rate framework, depends on the assumptions one makes on the rate dynamics and on the time of default τ.

3.2 Modeling the default time

We defined default as the event in which an obligor cannot face its contractual liabilities. In mathematical terms, default is described via the default time τ. This random instant of time is modeled in several ways but the most used are structural models and reduced form models (also known as intensity models in certain contexts).

Structural models, also called firm value models, are based on Merton's seminal work (1974). This approach is based on an elementary equation describing a firms value.

It is known that the value of an economic entity is the sum of its assets and of its debts. If its liabilities are larger than its assets the firm faces hard times in paying the debt back and so entails in a default event. The original formulation by Merton imposes a log-normal dynamic for the value of the firms assets and if this erratic value falls below the fixed debt level at a final time T then the credit event is triggered.

First passage time models (Black and Cox, 1976) are a bit more sophisticated since default time can arrive at an earlier stage rather than at a final time T. These models use the same framework of barrier option pricing models; this is because bankruptcy is triggered whenever the random value of assets hits from above the debt threshold. Such threshold can be itself stochastic or, in simpler approaches, fixed (even if it still can be time varying). These model are with no doubt more realistic than the usual structural models because they face situations that can happen in the real world: an example could be a contractual covenant that forces the firm to declare bankruptcy as soon as an interest or capital payment is not honored.

A second interesting feature of structural models derives from the assumption of log-normality in value dynamics. Log-normal dynamics in fact are the same as equity dynamics in the Black-Scholes (1973) model, for which several option formulas are available. Also, if the firms value is assumed to be observable, default comes less at a surprise with respect to intensity models.

In our context we prefer the reduced form (i.e. intensity) model framework. These models lack the economic interpretability of structural models, but such cost is greatly offset buy their main feature: we can describe default time τ using the same technology used in the interest rate modeling environment.

Intensity models start from the idea of describing the default time by means of the Poisson statistic law. The default time τ is nothing but the first jump of a Poisson process. This amounts to saying that default is an exogenous component independent of market data. The simplest Poisson process is the "time homogeneous Poisson process": such process $\{K_t, t \geq 0\}$ is a right continuous, unit-jump increasing process with initial value $K_0 = 0$ and stationary independent increments. If one lets τ be the first jump time of process K_t, then the probability that such process is different from 0 at any time t is defined as follows:

$$\mathbb{Q}(K_t \neq 0) = \mathbb{Q}(\tau \leq t) = 1 - e^{-\gamma t}, \tag{3.4}$$

with $\gamma > 0$. Other properties of Poisson processes are:

- $\lim_{t \to 0} \mathbb{Q}(K_t \geq 2)/t = 0$, meaning that as the interval of time shrinks there is no probability of having more than one jump.
- $\lim_{t \to 0} \mathbb{Q}(K_t = 1)/t = \gamma$ so γ is the instantaneous jump probability.
- γt is the mean and the variance of the process K_t.

In a " time homogeneous" Poisson process, times between jumps ($\tau_1 - 0, \tau_2 - \tau_1$, ...) are identical and independently (i.i.d.) distributed as exponential random variables with, in our case, parameter γ and mean $(1/\gamma)$.

Since we are interested in the first jump we only need its distribution, that is exponentialRV(γ). According to the previous statements:

$$\mathbb{Q}(\tau \in [t, t+dt] | \tau \geq t) = \frac{\mathbb{Q}(\tau \in [t, t+dt])}{\mathbb{Q}(\tau > t)} \simeq \gamma dt, \text{ for small } dt, \tag{3.5}$$

i.e. γdt is the probability of having the firm defaulting in time interval $[t, t+dt)$, given it has not defaulted up to time t.

Simple reduced form models rely on this framework to describe the probability of having a credit event in a given period. In such default models, γ is the intensity of default. As said before, with $\gamma\tau \sim$ exponentialRV(1), we have that:

- The probability of surviving up to time t is $\mathbb{Q}(\tau > t) = e^{-\gamma t}$.
- The probability of defaulting between time t and time $t + dt$ (i.e. in a small time interval) is:

$$\mathbb{Q}(\tau \in [t, t+dt] | \tau \geq t) = \frac{\mathbb{Q}(\tau > t) - \mathbb{Q}(\tau > t+dt)}{\mathbb{Q}(\tau > t)} = 1 - e^{-\gamma dt} \simeq \gamma dt, \tag{3.6}$$

  for small values of dt.
- The probability of defaulting in a generic interval $(u, t]$ is:

$$\mathbb{Q}(u < \tau \leq t) = e^{-\gamma u} - e^{-\gamma t}. \tag{3.7}$$

We can see from the above equations that the mathematical structure of such probabilities of survival/default resembles, quite closely, the mathematical structure of deterministic discount factors:

$$\mathbb{Q}(\tau > t) = e^{-\gamma t} \rightarrow P(0, t) = e^{-rt}.$$

As said previously, this is an important analogy because we can model said probabilities using the same framework we used to model discount factors and, in general, interest rates.

An important consideration is due on the meaning of the time-invariant constant γ; this can be seen as a credit spread that has to be applied to the risk free rate to obtain the price of a defaultable bond.

Another Poisson process used in intensity models is the "time inhomogeneous" process. In such models the intensity of default γ is no longer constant but varies across time. We then must define the cumulated intensity function also called Hazard function:

$$\Gamma(t) = \int_0^t \gamma(u) du. \tag{3.8}$$

In this case, the time between jumps $(\tau_1 - 0, \tau_2 - \tau_1, ...)$ is no longer i.i.d. due to time varying γ. We still have that the probability of surviving default from time 0 to time t is:

$$\mathbb{Q}(\tau < t) = e^{-\int_0^t \gamma(u)du} = e^{-\Gamma(t)}, \tag{3.9}$$

due to the fact that now $\Gamma(\tau) = \xi \sim$ exponentialRV(1), with ξ independent from all default free market information. For such reason we get default time τ by inverting this last expression and obtaining:

$$\Gamma^{-1}(\xi) = \tau. \tag{3.10}$$

The fact that the default time is exponential suggests an easy way to simulate it.

The mathematical formulas of survival probabilities up to any given date t, or in any time interval $[u, t)$ or again in any short interval $[t, t + dt]$ are as stated before. The only difference is at the exponent where γt is substituted with $\int_0^t \gamma(u)du$.

Even in "time inhomogeneous" intensity models we find analogies with the interest rate framework, in particular with the deterministic short rate $r(t)$ approach:

$$\mathbb{Q}(\tau > t) = e^{-\int_0^t \gamma(u)du} \rightarrow P(0, t) = e^{-\int_0^t r(u)du}.$$

Time varying γ allows to reproduce an entire term structure of credit spreads.

The basic Poisson-based reduced form model is the Cox process. Such model does not only introduce time-varying intensity but allows for it to be stochastic. This is the default intensity process we will use to implement the inflation indexed CDS pricing model.

Keeping up with the Brigo and Mercurio (2006) notation, let us introduce the random intensity λ_t>0 that follows the diffusion law:

$$d\lambda(t) = a(t, \lambda_t)dt + \sigma(t, \lambda_t)dW_\lambda(t). \tag{3.11}$$

This generic process for λ_t is continuous and $\mathbb{F}_t$-adapted (i.e. λ_t is known given the default-free market information up to time t). This means than all the randomness in the stochastic intensity is introduced by the default-free market. This does not have to be confused with the randomness of τ due to ξ, that is independent of λ.

For such double stochasticity in λ and ξ, the Cox process is also known as "doubly-stochastic" Poisson process.

As with the "time inhomogeneous" process, also the Cox process follows a Poisson law and here the cumulated intensity of default is given by:

$$\Lambda(\tau) = \xi \sim \text{exponentialRV}(1). \tag{3.12}$$

If $\lambda > 0$, which we assume from now on, the default time τ is the inverse of such hazard function, we have in fact:

$$\Lambda^{-1}(\xi) = \tau. \tag{3.13}$$

As for the previous processes, knowing the distribution of ξ yields an easy approach in simulating such exponential variable and, as a straightforward consequence, in simulating τ.

- First simulate ξ in scenario i as $\xi_i = -\ln(1-u_i)$ where $u \sim Uniform[0,1]$.
- Get the general default time τ_i in scenario i by inverting Λ, i.e. computing $\Lambda^{-1}(\xi_i) = \tau_i$.

Having the default time described by a Cox process provides us with the following survival probability:

$$\mathbb{Q}(\tau > t) = \mathbb{Q}\left(\xi > \int_0^t \lambda(u)du\right) = \mathbb{E}\left[\mathbb{Q}(\xi > \int_0^t \lambda(u)du|\mathcal{F}_t)\right] = \mathbb{E}\left[e^{-\int_0^t \lambda(u)du}\right]. \tag{3.14}$$

Once again we clearly see the analogy with discount factors in the stochastic short rate setting:

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[e^{-\int_0^t \lambda(u)du}\right] \rightarrow P(0,t) = \mathbb{E}\left[e^{-\int_0^t r(u)du}\right].$$

Here the stochasticity of λ allows not only for a term structure of credit spreads but also for a volatility of such term structure.

In our pricing model we consider a CIR process for λ_t. In the following section we give a sketchy introduction to such process.

3.3 A brief introduction to the Cox-Ingersoll-Ross model for the intensity of default

In a setting where the stochastic intensity is a random variable we have the following formula for a defaultable zero-coupon bond:

$$P^D(t,T) = \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}]. \tag{3.15}$$

For sake of explanation, we consider the short rate $n(t)$ as independent of the stochastic intensity λ so we have:

$$P^D(t,T) = \mathbb{E}[D(t,T)]\mathbb{E}[\mathbf{1}_{\{\tau>T\}}]. \tag{3.16}$$

The expectation of the indicator function is nothing but the risk neutral probability, for the firm that issued the bond, of surviving up to payment date T. This is $\mathbb{Q}(\tau > t)$, and in a stochastic intensity framework, has the mathematical formulation $\mathbb{E}\left[e^{-\int_0^t \lambda(u)du}\right]$. Keeping such things in mind, together with the stochastic discount factor formula, we have the following equivalent expression for the defaultable bond:

$$\begin{aligned} P^D(t,T) &= \mathbb{E}\left[e^{-(\int_0^t n(u)du+\int_0^t \lambda(u)du)}\right] \\ &= \mathbb{E}\left[e^{-\int_0^t n(u)du}\right]\mathbb{E}\left[e^{-\int_0^t \lambda(u)du}\right] \\ &= \mathbb{E}\left[e^{-\int_0^t n(u)du}\right]\mathbb{Q}(\tau > t). \end{aligned} \tag{3.17}$$

To compute the second expectation we need to know the distribution of the random variable λ. The distribution of λ depends on the process we assign to such random variable: this is where we resort to the Cox-Ingersoll-Ross (1985) process.

As previously said, λ is the stochastic intensity of default. The cumulated intensity $\Lambda(t) = \int_0^t \lambda(s)ds$ has to be inverted in order to retrieve τ, according to (3.13). Λ needs to be invertible. For this to happen, we must have values of λ belonging to the positive part of the $\mathbb{R}$ axis, hence we need a process that retains positivity. The CIR or square root process is the most tractable short-rate process through which we can guarantee the positivity of λ.

The CIR process has the following formulation under the risk-neutral measure:

$$d\lambda(t) = k[\mu - \lambda(t)]dt + \nu\sqrt{\lambda(t)}dW(t), \lambda(0) = \lambda_0. \tag{3.18}$$

We need k, μ, ν, λ_0 to be positive constants and we need to impose the following condition to ensure that the process remains strictly positive:

$$\mu > \frac{\nu^2}{2k}.$$

The process λ, in the CIR framework, features a non-central χ^2 distribution and has the following mean and variance:

$$\mathbb{E}(\lambda(t)|\mathcal{F}_s) = \lambda(s)e^{-k(t-s)} + \mu\left(1 - e^{-k(t-s)}\right), \tag{3.19}$$

$$\mathrm{Var}(\lambda(t)|\mathcal{F}_s) = \lambda(s)\frac{\nu^2}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \mu\frac{\nu^2}{2k}\left(1 - e^{-k(t-s)}\right)^2. \tag{3.20}$$

As seen before, the survival probability $\mathbb{Q}(\tau > t)$ is the expected value of minus the exponential integrated λ function:

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[e^{-\int_0^t \lambda(u)du}\right]. \tag{3.21}$$

Given that we now λ's distribution, we can compute such expectation and, after, we get:

$$\mathbb{Q}(\tau > t) = A(t,T)e^{-B(t,T)\lambda(t)}, \tag{3.22}$$

as in Vasicek but with:

$$A(t,T) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h+(h+k)(\exp\{(T-t)h\}-1)}\right]^{2k\mu/\nu^2}, \tag{3.23}$$

$$B(t,T) = \frac{2(\exp\{(T-t)h\}-1)}{2h+(h+k)(\exp\{(T-t)h\}-1)}, \tag{3.24}$$

$$h = \sqrt{k^2+2\nu^2}. \tag{3.25}$$

Chapter 4

Inflation-Indexed CDS pricing model

In the previous sections our effort was focused on illustrating the main theoretical insights needed to build and understand our inflation-indexed credit derivative pricing model.

In this chapter we finally describe the payoff of our contract and introduce the pricing model. We will try to practically compute the value of the CDS under several assumptions. The main task is, in fact, to obtain the fixed spread δ that renders the contract fair, at an hypothetical time $t = 0$, under different correlation structures between rates and the stochastic intensity of default. We will not compute the price of the Inflation-Indexed CDS at a time different from $t = 0$ due to the recent introduction of such contracts and hence to the lack of liquid data that can provide us with quoted values of δ. This is an experimental field in the derivatives industry and so we are committed in defining a correct theoretical framework more than in efficient pricing procedures.

4.1 Defining the contractual payoff

The first step in building a suitable and efficient pricing model is to correctly define the derivative contract payoff.

The contract we analyze is an Inflation-Indexed credit default swap. We assume it to be a postponed CDS in the sense that all the protection and premium payments are due at the generic time T_i, i.e. the end point of a payment period. More precisely, if default occurs at an instant τ between T_{i-1} and T_i the protection payment LGD (Loss Given Default) occurs in T_i. This simplifies the payoff since no accrual term has to be included.

The one-period Inflation-Indexed CDS payoff at a generic time t is as follows:

- It has a premium leg that pays quarterly the annual inflation rate, defined as the ratio between the CPI at the beginning and at the end of two subsequent times minus one, plus the sought δ spread that makes the contract fair, as long as the third party does not default; discounted at time t this is:

$$\Pi^{\text{Premium leg}}_{T_{i-1},T_i}(t) = N\psi_i D_n(t,T_i)\left(\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]^+ + \delta\right)\mathbf{1}_{\{\tau > T_i\}}, \quad (4.1)$$

  where N is the contractual notional amount, $I(T)$ is, as seen previously, the CPI level at time T, $\mathbf{1}_{\{\bullet\}}$ is the indicator function, ψ_i the premium leg year fraction and D_n the nominal stochastic discount factor. As can be seen from the above leg, the inflation rate to be payed is floored at zero.

- It has a protection leg in which the Loss Given Default (LGD) on the notional amount is payed if the firm encounters a credit event during the contractual period. This contract features a protection leg that has also a three month LIBOR payment towards the protection buyer, i.e.:

$$\begin{aligned}\Pi^{\text{Protection leg}}_{T_{i-1},T_i}(t) = {} & N\phi_i D_n(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau > T_i\}} \\ & + N\text{LGD}D_n(t,T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} \end{aligned} \quad (4.2)$$

  Here ϕ_i is the protection leg year fraction and the whole leg is discounted at time t by the the usual D_n.

Fixing the initial date T_α and the final date T_β we get the total discounted payoff by summing the whole set of payments. Such total discounted payoff as seen by the protection seller (premium legs minus protection legs) is:

$$\begin{aligned}\Pi_{T_\alpha,T_\beta}(t) = {} & N\sum_{i=\alpha+1}^{\beta}\psi_i D_n(t,T_i)\left(\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]^+ + \delta\right)\mathbf{1}_{\{\tau > T_i\}} \\ & -N\left(\sum_{i=\alpha+1}^{\beta}\phi_i D_n(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau > T_i\}} + \text{LGD}\sum_{i=\alpha+1}^{\beta}D_n(t,T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}}\right)\end{aligned} \quad (4.3)$$

As for any contingent claim, the value of the IICDS at time t is the risk neutral expectation of its discounted payoff at maturity. We must condition such expectation, as previously said, not to the default-free filtration $\mathcal{F}_t$ but to the complete $\mathcal{G}_t$ filtration which includes default monitoring, since this is the actual information we observe in the market at valuation time. This gives us the following value of our contingent claim:

$$IICDS_{T_\alpha,T_\beta}(t) = \mathbb{E}\left[\Pi_{T_\alpha,T_\beta}(t)|\mathcal{G}_t\right] \tag{4.4}$$

It is still possible to resort to the usual default-free information set by using the filtration switching formula:

$$\mathbb{E}(\Pi_{T_\alpha,T_\beta}(t)|\mathcal{G}_t) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)}\mathbb{E}(\Pi_{T_\alpha,T_\beta}(t)|\mathcal{F}_t). \tag{4.5}$$

For a detailed proof see Brigo and Mercurio (2006). Such change of filtration although, at $t = 0$, is not needed explicitly.

4.2 Pricing

Due to the recent introduction of such contracts and to their over the counter nature, calibration of the model parameters to market data happens to be excessively cumbersome. We therefore resort to Jarrow-Yildirim (2003) and to Brigo-Mercurio (2006) for providing some possible dynamics parameters. This is in line with our initial purpose of defining a theoretical approach and pointing out specific features in the valuation procedure such as correlation patterns rather than having precise price estimates.

In all our different fair δ spread estimates we will consider the following start and maturity dates: $T_\alpha = 0$ and $T_\beta = 5$ years. The premium leg and the protection leg have both a three month stylized payment frequency so we have the following year fractions: $\phi_i = \psi_i = 0.25$.

4.2.1 Pricing IICDS when the intensity of default is independent from interest rates and the CPI

In this section our aim is defining the correct δ spread that renders the IICDS fair at time zero when the intensity of default λ is independent of nominal and real short interest rates and is also independent of the CPI level. This means that $\rho_{I,\lambda} = \rho_{r,\lambda} = \rho_{n,\lambda} = 0$. We have the following price for the IICDS at time $t = 0$, which we set to zero in order to find the fair δ:

$$IICDS_{0,5}(0) = \mathbb{E}[\Pi_{0,5}(0)|\mathcal{F}_0] = 0.$$

Remembering that the total discounted payoff is the sum of the premium leg and of the protection leg, we can equate their expectations and obtain:

$$N\mathbb{E}\Bigg[\sum_{i=1}^{20}\psi_i D_n(0,T_i)\Bigg(\Bigg[\frac{I(T_i)}{I(T_{i-1})}-1\Bigg]^+ +\delta\Bigg)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_0\Bigg]$$

$$= N\mathbb{E}\Bigg[\sum_{i=1}^{20}\phi_i D_n(0,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau>T_i\}}+\text{LGD}\sum_{i=1}^{20}D_n(0,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_0\Bigg].$$

As usual we have that:

- $D_n(0,T_i)$ is the stochastic nominal discount factor.
- $I(T_i)$ is the CPI level at time T_i.
- δ is the sought fixed spread that renders the contract fair in $t=0$.
- $\mathbf{1}_\bullet$ is the indicator function tied to the default event.
- $L(T_{i-1},T_i)$ is the LIBOR rate resetting in T_{i-1} for T_i.
- LGD is the Loss Given Default percentage for a unit notional.
- N is the contract notional while ψ_i and ϕ_i are respectively the protection and premium leg year fractions.

The summations go from 1 to 20 because 20 is the number of quarterly periods in the contract (5 years). We also have $T_0=0$.

Remembering that there is no dependence structure between the rates and the intensities of default we can solve, via simple algebra, the above equation for the spread δ. We can use the tower property of conditional expectations to compute the expectations of the indicator functions and this gives us the probabilities of survival. We then factor the expectations in the payoff obtaining the following expression:

$$\begin{aligned}\delta(0,5y) = \Bigg[\Bigg(&\sum_{i=1}^{20}[P_n(0,T_{i-1})-P_n(0,T_i)]\\ &-\sum_{i=1}^{20}\phi_i P_n(0,T_i)\mathbb{E}\Bigg[\Bigg(\frac{I(T_i)}{I(T_{i-1})}-1\Bigg)^+|\mathcal{F}_0\Bigg]\Bigg)\mathbb{Q}(\tau>T_i)\\ &+\text{LGD}\sum_{i=1}^{20}P_n(0,T_i)\mathbb{Q}(T_{i-1}<\tau\le T_i)\Bigg]\Bigg\backslash\Bigg(\sum_{i=1}^{20}\phi_i P_n(0,T_i)\mathbb{Q}(\tau>T_i)\Bigg).\end{aligned} \tag{4.6}$$

The terms at the numerator on the right hand side are, respectively:

- A defaultable annuity paying the three month Libor rate .
- An inflation indexed defaultable cap.
- A term including the LGD payment if the credit event occurs

The term at the denominator on the right hand side is a defaultable annuity paying the year fraction term. Our aim is now to quantify such δ in case of no correlation between the rates, the CPI and the default intensities. We have to value the single components of the above equation using the interest rate models seen in the previous chapters. The first term, as said previously, is a simple annuity on the three month Libor rate that is paid if no default occurs during the contract life. The valuation of the annuity is fairly simple considering that at time $t = 0$ we know the entire term structure of the nominal discount factors $P_n(0, T_i)$. This is true also for the term including the LGD payment and for the annuity at the denominator of formula (4.6) whose valuations are straightforward. With respect to the definition of all the probabilities of survival that appear in the equation, we must remember we are in a stochastic default intensity framework. We must then resort to formula (3.22) which is based on the CIR model. Concerning the valuation of the inflation indexed cap, we must implement formula (2.47) seen in Chapter 2. In our work we practically tested the pricing model by developing a program written in the MatLab environment. Such program implements all the previously seen models and returns the value of δ that sets to zero the NPV of the contract at time $t = 0$. This is obviously done implementing the analytical equation seen in formula (4.6). The last passage before empirical testing is to fix the parameter values. For this aim we use data from the Jarrow and Yildirim paper and from Brigo-Mercurio. For the parameters of the interest rate models seen in Section (2.5) we have:

n_0	r_0	a_n	a_r	σ_n
0.045	0.024	0.03398	0.04339	0.00566

σ_r	σ_I	$\rho_{n,r}$	$\rho_{n,I}$	$\rho_{r,I}$
0.00299	0.00874	0.01482	0.06084	-0.32127

The Loss Given Default is set at 0.6 (sixty percent) of the unit notional. For the parameters of the intensity of default model seen in Section (3.3) we have:

λ_0	k	μ	ν
0.035	0.35	0.045	0.15

As previously said the parameters of the diffusion process for the intensity of default, i.e. the λ parameters are taken from Brigo-Mercurio (2006). Such parameters should be calibrated for each firm from its bonds or from CDS's that have the firm as underlying defaultable entity. All the parameters for the intensity of default model have been checked to guarantee positivity of the model. The intensity levels corresponding to these parameter values are quite high and ideally correspond to names with low rating. Clearly, these are the cases where the default risk contribution is more relevant.

Along with the analytical model we built a Monte Carlo pricing model, programmed in MatLab code as well. Such model will be used later on, when pricing will be done with correlation patterns different from zero between interest rates and default intensities. Now we will use the Monte Carlo procedure just once to verify the soundness of our analytical model. We will compare the values of δ for both the analytical formula and the Monte Carlo method, taking into account the 95 percent confidence window for the Monte Carlo result. Such confidence interval is built around the value of the NPV rather than being expressed in δ terms. The numerical Monte Carlo fair value of δ has been obtained through a "trial and error" approach. That is, one inserts a value for δ and the program yields the NPV of the contract with the related confidence window.

	Analytical	Monte Carlo
δ	0.0631	0.0629
NPV	0	$3.4376\text{E}-5$
95% C. I. Radius	N/A	$5.7786\text{E}-4$

Here δ is an annual rate so the equivalent quarterly rate is 0.0154. The number of Monte Carlo paths is 30000, leading to high precision of the pricing results and to a narrow confidence interval.

From the table above it can be seen that there is a slight misalignment between the δ obtained with the analytical pricing procedure and the δ yielded by the numerical program. This is possibly due to the discretization error of the Euler scheme implemented by our simulator. Nevertheless, we reduced the discretization step to as little as one day to gain as much precision as possible. We can say that in a large number of runs the delta of the analytical model is inside the confidence interval for the delta of the simulation model.

In the following paragraphs we will take a look at what happens to the price when correlation patterns between rates and default intensity change to more realistic non-null values.

4.2.2 Pricing IICDS when the intensity of default is correlated with interest rates and the CPI

In this section our aim is to price the inflation-indexed credit default swap when there is a non-null correlation between rates and the CPI with the intensity of default, i.e. with any of $\rho_{I,\lambda}, \rho_{r,\lambda}, \rho_{n,\lambda}$ different from zero. We will set correlations to different fixed values and analyze how the price of the contract reacts to such variations of the dependence structure.

We still consider the value of the contract at time $t = 0$ so we have the following discounted premium and protection legs payoff expectation:

$$\mathbb{E}\Big[\sum_{i=1}^{20} \psi_i D_n(0, T_i)\Big(\Big[\frac{I(T_i)}{I(T_{i-1})} - 1\Big]^+ + \delta\Big)\mathbf{1}_{\{\tau > T_i\}}|\mathcal{F}_0\Big]$$

$$-\mathbb{E}\Big[\sum_{i=1}^{20} \phi_i D_n(0, T_i) L(T_{i-1}, T_i)\mathbf{1}_{\{\tau > T_i\}} + \mathrm{LGD}\sum_{i=1}^{20} D_n(0, T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}}|\mathcal{F}_0\Big] = 0.$$

In case of non zero correlation we cannot price the IICDS with analytical closed form equations since we cannot factor the expectations. This is due to the fact that the variables inside the payoff expectation are no longer independent. Therefore there is no way of separating their expected values and of computing them one by one under their specific probability measure. For such reason we must resort to numerical methods. We must simulate the trajectories of r, n, I, and λ to obtain the paths of the probability of default, of inflation, of the discount factors and of the LIBOR rate. Once we have a large number of simulated payoffs as functions of these simulated variables we can take the sample mean over the number of scenarios and obtain a Monte Carlo approximation of the expected payoff value, hence of the price of the IICDS.

We use the mathematical framework provided above to define finite approximations of the variables we must simulate. This is because we have no analytical solution for their joint transition densities, i.e. no way of solving the stochastic differential equations when correlation is taken into account.

We use a simple Euler scheme for the stochastic differential equations approximation and run a plain vanilla Monte Carlo simulation with a one-day time step to reduce the discretization error. We calculate the Monte Carlo standard error and provide the 95% confidence radius of each simulation for the construction of the confidence intervals. To allow for the S.D.E.'s to be correlated we use the Cholesky factorization of the variance-covariance matrix of instantaneous shocks. We have the following discretized differential equations under the n-bank account numeraire (nominal risk neutral measure):

$$
\begin{aligned}
n(t_i) &= n(t_{i-1}) + [\theta_n(t) - a_n n(t_{i-1})](t_i - t_{i-1}) + \sigma_n \sqrt{(t_i - t_{i-1})} W_n(i), \\
r(t_i) &= r(t_{i-1}) + [\theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t_{i-1})](t_i - t_{i-1}) + \sigma_r \sqrt{(t_i - t_{i-1})} W_r(i), \\
I(t_i) &= I(t_{i-1}) + I(t_{i-1})[n(t_{i-1}) - r(t_{i-1})](t_i - t_{i-1}) + \sigma_I I(t_{i-1}) \sqrt{(t_i - t_{i-1})} W_I(i), \\
\lambda(t_i) &= \lambda(t_{i-1}) + k[\mu - \lambda(t_{i-1})]dt + \nu \sqrt{\lambda(t_{i-1})} \sqrt{(t_i - t_{i-1})} W_\lambda(i),
\end{aligned}
$$

where each W is a normal standard random variable and $\Delta t = (t_i - t_{i-1}) =$ One Day. The time step is very short to reduce discretization errors. We clearly recognize the structure of the S.D.E.'s seen in the previous chapters.

The W shocks are correlated. As said earlier, we allow for correlation between the four variables using the Cholesky decomposition of the variance-covariance matrix so, when implementing the Monte Carlo simulator with MatLab, the equations used to perform the joint simulation and based on independent shocks are more complex than the ones seen above. This is due to the fact they have the Cholesky terms inside and more than one random disturbance per equation. Every variable after the first, in fact, has at least one random disturbance in common with the other variables. The four correlated shocks are thus triangularly obtained from the four independent shocks, in the classical Cholesky setup.

The numerical pricing procedure computes the values of the single random variables, from which the contract depends, by implementing the mathematical models previously seen. In practice it computes the expectation of each payoff subcomponent and returns the NPV of the derivative contract. As anticipated, the δ is computed with a trial and error procedure. One inserts a value for δ in the program and the program computes the net present value; if this is zero the inserted δ is effectively the fair spread, otherwise one changes δ in the right direction and tries again. Along with the NPV estimate the program returns a Monte Carlo standard error for every simulation. This can be used to build confidence intervals for the NPV estimate. We now use the Monte Carlo simulator to determine the soundness of our mathematical model and to compute the price of the IICDS for several correlation patterns. Remaining model parameter values are the same as for the analytical model in Section 4.2.1.

We find 30000 paths are a good tradeoff between computational efficiency, estimate precision and therefore confidence interval width.

4.2.3 Pricing the IICDS when $\rho_{\lambda,n} = \rho_{\lambda,r} = \rho_{\lambda,I} \neq 0$

In this paragraph we start pricing the fair spread for an Inflation Indexed Credit Default Swap when correlations between interest rates, CPI and intensity of default have values other than zero. The first step will be testing the price when correlations are equal to each other. This is done with the highest possible correlation level and with the lowest possible correlation level keeping the matrix positive semi-definite. The maximum and the minimum equivalent levels attained for the three correlations are respectively 0.51 and of -0.51. If correlations were shifted at higher or lower levels the eigenvalues of the correlation matrix would become negative. The following table shows the values obtained for δ along with the NPV and the standard errors for the construction of the confidence intervals.

Correlation Level	δ	NPV	St.err.
$\rho_{\lambda,n} = \rho_{\lambda,r} = \rho_{\lambda,I} = 0.51$	0.0625	$1.0032\text{E} - 4$	$3.4166\text{E} - 4$
$\rho_{\lambda,n} = \rho_{\lambda,r} = \rho_{\lambda,I} = -0.51$	0.0629	$-1.6660\text{E} - 4$	$2.4553\text{E} - 4$

As usual, δ is an annual rate.

As we can see, the contract is not very sensitive to homogeneous changes in the correlation patters: the percentage difference between the two deltas is in fact only 0.64. If correlations move with the same intensities in the same direction, the value of the contract is not deeply affected. Notice also that the fair spread is the same as in case of zero correlations seen previously. This suggests that the fair spread is not that sensitive to correlations when the remaining parameters of the dynamical model (default-free volatilities and correlations) are chosen as we did. Now we try and investigate whether relaxing the assumption

$$\rho_{\lambda,n} = \rho_{\lambda,r} = \rho_{\lambda,I}$$

suggests a stronger influence of correlations.

4.2.4 Pricing the IICDS with several correlation patterns

In this section we will price the IICDS when correlations assume different values between each other. This will give us a hint on how the price of the derivative is affected by changes in the dependence patterns between the relevant variables. The aim is to see how the price reacts when extreme values are given to the correlations. These values are always the highest possible absolute values that maintain positivity of the eigenvalues of the correlation matrix. As usual the number of paths cycled by our numerical model are 30000 and the δ's are expressed as annualized rates.

Correlation Level	δ	NPV	St.err.
$\rho_{\lambda,n} = 0.9$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0$	0.0623	$-2.0190\text{E}-4$	$3.7620\text{E}-4$
$\rho_{\lambda,n} = -0.9$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0$	0.0632	$-5.9796\text{E}-5$	$1.9613\text{E}-4$
$\rho_{\lambda,r} = 0.9$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0$	0.0628	$-3.5164\text{E}-5$	$3.1305\text{E}-4$
$\rho_{\lambda,r} = -0.9$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0$	0.0628	$-4.7853\text{E}-5$	$2.8735\text{E}-4$
$\rho_{\lambda,I} = 0.9$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0$	0.0628	$3.0429\text{E}-5$	$2.8319\text{E}-4$
$\rho_{\lambda,I} = -0.9$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0$	0.0628	$-8.8742\text{E}-6$	$3.1391\text{E}-4$
$\rho_{\lambda,n} = 0$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0.58$	0.0628	$1.8421\text{E}-4$	$3.0012\text{E}-4$
$\rho_{\lambda,n} = 0$ $\rho_{\lambda,r} = \rho_{\lambda,I} = -0.58$	0.0628	$-2.5609\text{E}-4$	$3.0298\text{E}-4$
$\rho_{\lambda,r} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0.58$	0.0625	$1.7589\text{E}-4$	$3.4228\text{E}-4$
$\rho_{\lambda,r} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,I} = -0.58$	0.063	$-8.4842\text{E}-5$	$2.4945\text{E}-4$
$\rho_{\lambda,I} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0.58$	0.06235	$-2.0554\text{E}-4$	$3.5574\text{E}-4$
$\rho_{\lambda,I} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,r} = -0.58$	0.0631	$-7.3537\text{E}-5$	$2.2896\text{E}-4$
$\rho_{\lambda,n} = 0.45$ $\rho_{\lambda,r} = \rho_{\lambda,I} = -0.45$	0.06265	$-5.5987\text{E}-5$	$3.4348\text{E}-4$
$\rho_{\lambda,n} = -0.45$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0.45$	0.0629	$-2.4441\text{E}-4$	$2.5361\text{E}-4$
$\rho_{\lambda,r} = 0.45$ $\rho_{\lambda,n} = \rho_{\lambda,I} = -0.45$	0.063	$-1.8242\text{E}-4$	$2.6846\text{E}-4$
$\rho_{\lambda,r} = -0.45$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0.45$	0.0626	$-2.1037\text{E}-4$	$3.2767\text{E}-4$
$\rho_{\lambda,I} = 0.45$ $\rho_{\lambda,n} = \rho_{\lambda,r} = -0.45$	0.0631	$2.1032\text{E}-4$	$2.3746\text{E}-4$
$\rho_{\lambda,I} = -0.45$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0.45$	0.0626	$-1.2263\text{E}-4$	$3.5451\text{E}-4$

As we can see from the tables above, the price of our derivative is quite insensitive to changes in correlation patterns. The only sensitive impact on prices is seen when the correlation between the nominal interest rate and the intensity of default varies (first two rows of the table). This is mainly due to the fact that the nominal interest rate is our chosen numeraire and we take expectations under its probability measure. Both the Libor rate and the stochastic discount factor D_n depend on n so as soon as we move its correlation we obtain an impact on the level of the Inflation Indexed Credit Default Swap price. In the following paragraph we will see if it is possible to enhance the effects of correlation patterns on the contract price. We will do this by stressing the volatility levels of nominal and real interest rates and the volatility level of the Consumer Price Index, to further highlight the impact of diversified correlation.

4.2.5 Pricing the IICDS with higher volatility levels

As seen in the previous section, the price of the IICDS is not heavily affected by changes in the correlation between variables. This could be due to the fact that the volatility levels of nominal and real rates and of the CPI are relatively low, so that their randomness is limited anyway and correlations have little importance in general. Our aim in this paragraph is to see if a change in volatility levels can sensibly affect the impact of correlation patterns on the price of the IICDS, given that higher volatilities imply more randomness. We will set $\sigma_n^{new} = 3 * \sigma_n = 0.01698$ that corresponds to a percentage initial volatility of 37.73%. For the real interest rate volatility we set $\sigma_r^{new} = 3 * \sigma_r = 0.00897$ that is a percentage volatility of 37.37%. The CPI volatility is set to $\sigma_I^{new} = 0.1$. In the following table we synthesize our results.

Correlation Level	δ	NPV	St.err.
$\rho_{\lambda,n} = 0.9$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0$	0.0435	$-4.5675\text{E}-4$	$6.4410\text{E}-4$
$\rho_{\lambda,n} = -0.9$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0$	0.0463	$1.2458\text{E}-4$	$2.5938\text{E}-4$
$\rho_{\lambda,r} = 0.9$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0$	0.045	$1.2323\text{E}-4$	$5.1373\text{E}-4$
$\rho_{\lambda,r} = -0.9$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0$	0.0451	$2.7970\text{E}-4$	$4.8593\text{E}-4$
$\rho_{\lambda,I} = 0.9$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0$	0.045	$-6.7976\text{E}-5$	$4.2555\text{E}-4$
$\rho_{\lambda,I} = -0.9$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0$	0.045	$3.9552\text{E}-4$	$5.5795\text{E}-4$
$\rho_{\lambda,n} = 0$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0.58$	0.0449	$-1.9420\text{E}-5$	$4.6427\text{E}-4$
$\rho_{\lambda,n} = 0$ $\rho_{\lambda,r} = \rho_{\lambda,I} = -0.58$	0.0449	$-3.7014\text{E}-4$	$5.3223\text{E}-4$
$\rho_{\lambda,r} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0.58$	0.0441	$4.1515\text{E}-4$	$5.5975\text{E}-4$
$\rho_{\lambda,r} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,I} = -0.58$	0.0458	$1.6281\text{E}-4$	$4.1666\text{E}-4$
$\rho_{\lambda,I} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0.58$	0.0441	$-4.2116\text{E}-4$	$6.0374\text{E}-4$
$\rho_{\lambda,I} = 0$ $\rho_{\lambda,n} = \rho_{\lambda,r} = -0.58$	0.0459	$1.5397\text{E}-4$	$3.5409\text{E}-4$
$\rho_{\lambda,n} = 0.45$ $\rho_{\lambda,r} = \rho_{\lambda,I} = -0.45$	0.0443	$3.4874\text{E}-4$	$5.9528\text{E}-4$
$\rho_{\lambda,n} = -0.45$ $\rho_{\lambda,r} = \rho_{\lambda,I} = 0.45$	0.0456	$-2.9193\text{E}-4$	$3.6917\text{E}-4$
$\rho_{\lambda,r} = 0.45$ $\rho_{\lambda,n} = \rho_{\lambda,I} = -0.45$	0.0455	$3.6805\text{E}-4$	$4.4454\text{E}-4$
$\rho_{\lambda,r} = -0.45$ $\rho_{\lambda,n} = \rho_{\lambda,I} = 0.45$	0.0443	$1.1667\text{E}-4$	$5.3894\text{E}-4$
$\rho_{\lambda,I} = 0.45$ $\rho_{\lambda,n} = \rho_{\lambda,r} = -0.45$	0.0456	$-2.7172\text{E}-4$	$3.4538\text{E}-4$
$\rho_{\lambda,I} = -0.45$ $\rho_{\lambda,n} = \rho_{\lambda,r} = 0.45$	0.0443	$-3.5922\text{E}-4$	$6.0728\text{E}-4$

As can be seen by the table above, an increase in volatilities has a sensible impact on the fair spread of the derivative that goes from a 6 to a 4 percent level. By converse a rise in the volatilities does not affect in an evident way the impact of the correlation patterns on the IICDS price. We can see that the most evident impact of a change in correlation with normal volatilities in Section 4.2.4 is about 1.12%. The impact of correlation when volatilities of rates are multiplied by 3 and that of the CPI by more than ten times brings to a maximum impact of correlation around 6.44% (first two rows of the table). Moreover, the most sensitive changes are, as before, those given by a variation in the correlation between the nominal interest rate and the intensity of default.

It is clear at this point that variations in the correlation between interest rates, the CPI and the intensity of default have a moderate impact on the price of the IICDS. A heavier impact can be seen when changes in the volatility levels are taken into account, in that the contract value seems to be more sensitive to volatilities than to correlations, as is to be expected. However we must point out the fact that market volatilities nowadays are more similar to the enhanced volatilities than to the Jarrow-Yildirim (2003) volatilities used previously. This means that a maximum variation of 6.44% in the fair spread, given by a change in the correlation pattern, becomes more significant and that surely such level of variation must be taken into account by a market maker or a trader of such contracts.

Chapter 5

Conclusions

In this work we defined and priced the payoff structure of a recent type of hybrid inflation-credit derivative, i.e. the Inflation Indexed Credit Default Swap. For such aim we used a combination of theoretical models to define the core stochastic variables underlying the contract payoff. To model inflation we used the Jarrow-Yildirim framework. We explained how this model is based on the foreign-currency analogy: the evolution of instantaneous real and nominal rates is modeled along with the evolution of the Consumer Price Index in a FX/interest rates framework. Nominal rates are interpreted as the domestic rates while real rates are interpreted as foreign economy rates. Here the CPI index is the "exchange rate" between the nominal and real economies. The interest rates have been modeled in a Hull-White (1990) stochastic setting while the CPI was given a log-normal diffusion process.

To model the credit component we used a doubly stochastic reduced form model with intensity of default λ following a CIR diffusion process. By taking the expectation of the indicator functions under the n risk neutral measure we obtained the probabilities of survival that were necessary to price our derivative.

Once the theoretical framework was defined we empirically tested our pricing model with a Monte Carlo numerical procedure and with an analytical model for the simplified case of null correlation between rates, CPI and default intensity. This gave us the possibility to test how various correlation patterns affected the price of the derivative. This was done with different levels of volatilities for the nominal and real interest rates and for the CPI.

We have found that using the volatilities from the Jarrow-Yildirim (2003) paper, the impact of variations of the correlation patterns on the price of the IICDS are quite low. The maximum price variation is in the order of 1.12%. When we considerably increased the volatilities (300 percent for the real and nominal rates volatilities and about 1000 percent for the CPI volatility) the impact of correlation variations in the price of the derivative are more relevant but still moderate, amounting to 6.44 percent at most.

It is clear from the results of our work that the price of these new hybrid derivatives, mixing inflation modeling with credit, is relatively insensitive to variations in the correlation structures between default and interest rates, although the impact can become relevant for higher and probably more realistic volatilities. Ours is a quite rough approach and much of the work was done to define the payoff structures and implement a significant pricing model rather than giving a precise valuation under a fine-tuned calibrated model. Subsequent useful studies could be done to provide a more complete insight on the intensities of variations in the variable parameters and how these can be used to eventually hedge a portfolio of such credit derivatives. Adding jumps in the intensities process could also add realism in the default dynamics.

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