Randomness, Dynamics and Risk

From Quantum Theory and Chaos
to Signal Processing and Finance

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Welcome All! Inaugural lecture overview I

1. Randomness?
   - Probability and Stochastics
2. Stochastic and Ordinary Differential Equations
   - Local mean and local standard deviation
   - No randomness: Clockwork universe?
3. Chaos
   - Extreme sensitivity to initial conditions: practical unpredictability
   - Chaos, Randomness and Free Will
4. Is there anything truly random in Nature?
   - Ignorance? Coins and Dice, Statistical Mechanics, Economics
   - Quantum Mechanics: double slit experiment
   - QM: Interference? Of what? And with what?
   - QM: is this what true randomness should always look like?
   - Quantum Mechanics: A kinder version of Schroedinger's Cat
   - Probability: Quantum or Classical?

Welcome All! Inaugural lecture overview II

5. Randomness, Dynamics and Risk in Finance
   - Derivatives markets: context and beginnings
   - Derivatives: Option Pricing and Probability Measures
   - Problems with Derivatives Methodology Assumptions
   - Is trading, hedging and investing time really continuous?
   - Richer dynamics for random assets: Volatility smile modeling
   - Infinite dimensional objects: Interest rate curves random dynamics
   - Modeling default risk random dynamics for many names
   - Emerging/neglected risks, Valuation Adjustments and CCPs
   - Nonlinearities, contagion and the end of Platonic pricing

6. Signal Processing
   - Stochastic Nonlinear Filtering
   - The projection filter
   - Rocket science

Welcome All! Inaugural lecture overview III

7. Notes, references and further reading
   - Theories of Probability and Classical vs Quantum Probability
   - Stochastic Differential Equations
   - Chaos
   - Nonlinear filtering
   - Derivatives and No Arbitrage: Forerunners
   - Derivatives and No Arbitrage

Randomness?

Ignorance/hidden variables randomness or true randomness?

- Randomness in everyday life. Can't predict much of what happens
- Randomness can be our ignorance/incapability to model or calculate phenomena, or real random processes in nature.
- Pinpointing a true source of randomness in nature that is known to *current* science can be quite difficult

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- Pinpointing a true source of randomness in nature that is known to *current* science can be quite difficult
- **How about the maths of randomness, whatever its source?**

Probability: Interpretations, Calculus and Dynamics

What is probability? Interpretations and Calculus

- Despite a variety of interpretations where probability is
  - A degree of entailment (Logical, Keynes...)
  - A degree of belief by an individual (subjective, De Finetti/Ramsey...)
  - A frequency (frequentist, von Mises...)
  - A propensity (Popper...)
  - ...
- ... axioms probability calculations should follow have been fixed by Kolmogorov (1933) and are almost universally agreed upon

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Dynamics of randomness and probability: Stochastics

- The evolution of randomness and probabilities *in time* is the work of Stochastic Analysis. An example of Randomness in motion:

Stochastic vs deterministic differential equations

Randomness in motion: Examples

The future evolution of a financial asset, the future trajectory of a rocket undergoing perturbations, the future evolution of a population, the future position of a submarine probe, tides future levels...

Stochastic vs deterministic differential equations

Randomness in motion: Examples

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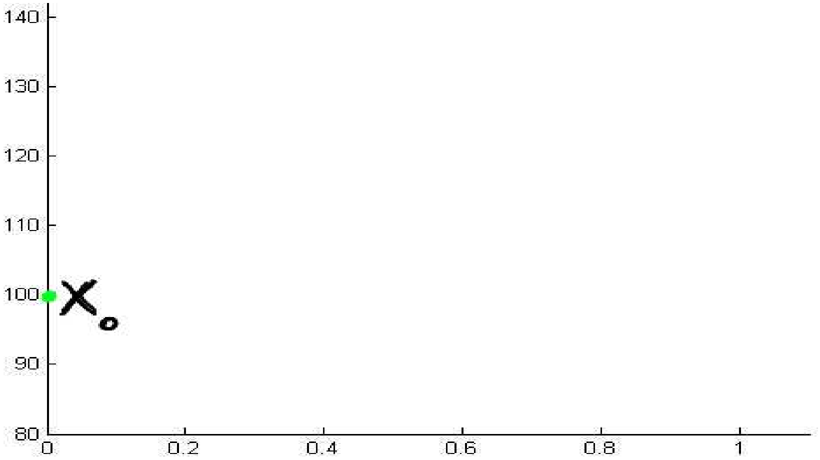
A Stochastic Differential Eq. (SDE) looks like this ($r = 5\%$ growth rate):

$$\underbrace{dX(t)}_{\substack{\text{Change in } X \\ \text{between } t \text{ and } t+dt}} = \underbrace{rX(t)}_{\substack{\text{function of } X \\ \text{in } t, \\ \text{"MEAN} \\ \text{CHANGE"}}} dt + \underbrace{\sigma\, X(t)}_{\substack{\text{Amplitude} \\ \text{coefficient} \\ \text{of shock} \\ (\textit{Volatility})}} \underbrace{dW_t}_{\substack{\text{New} \\ \text{random} \\ \text{shock}}}$$

Let us suppose this is the future price of an asset with return 5% and see how this varies with σ (or a future popolation toy model)

SDE: $dX_t = 0.05X_t dt + 0.1X_t dW_t$, $X_0 = 100$, ODE $dX_t = 0.05X_t dt$



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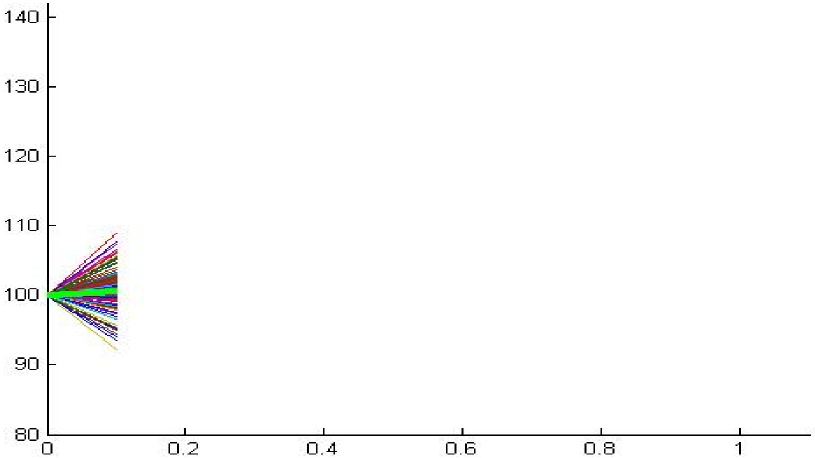

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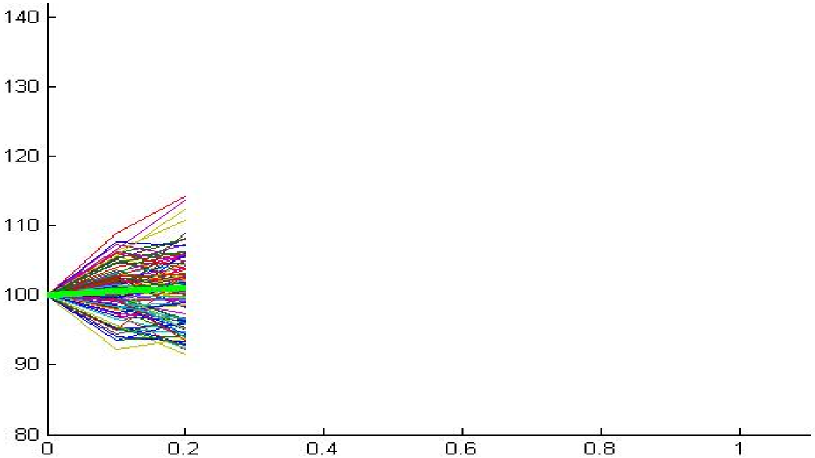

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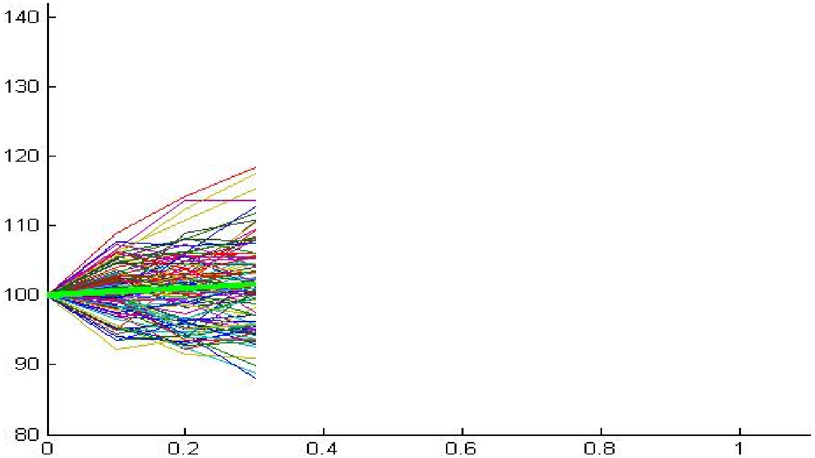

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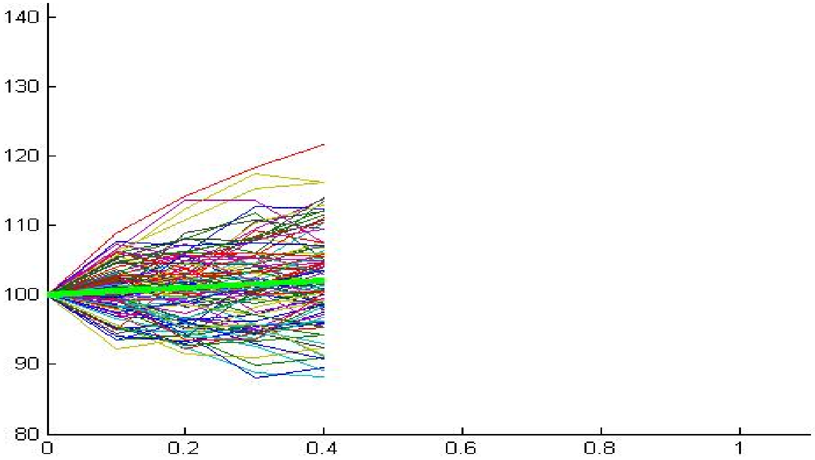

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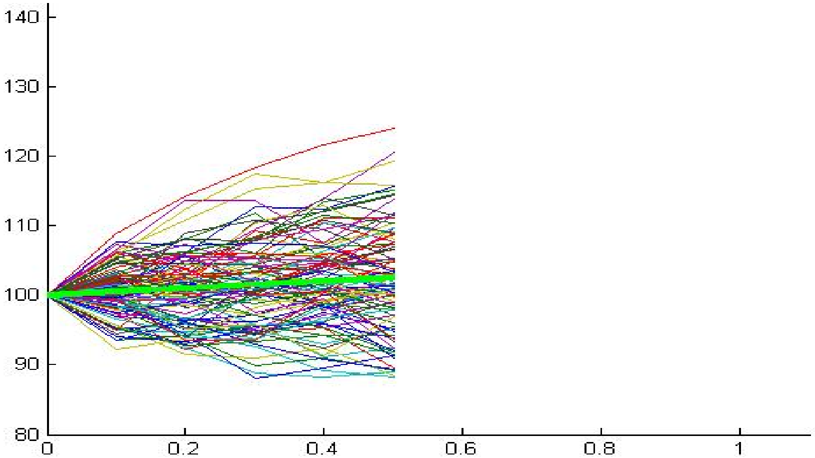

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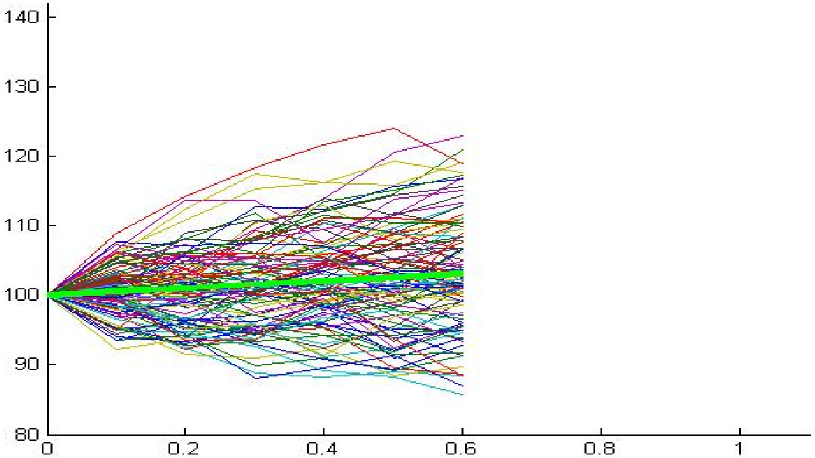

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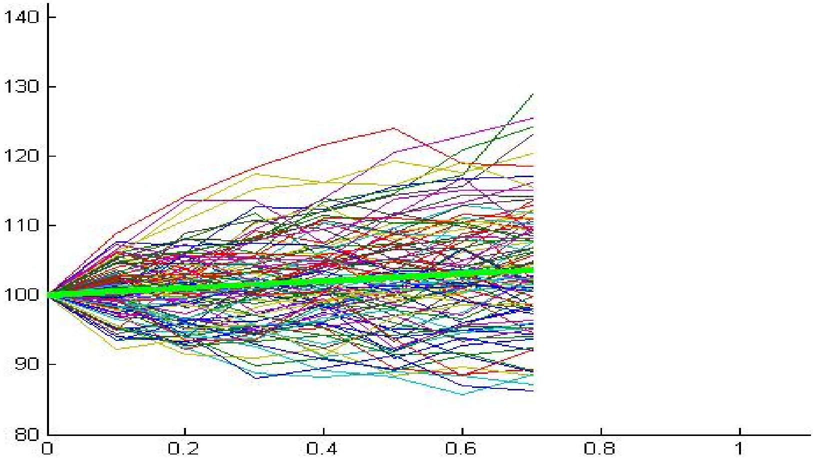

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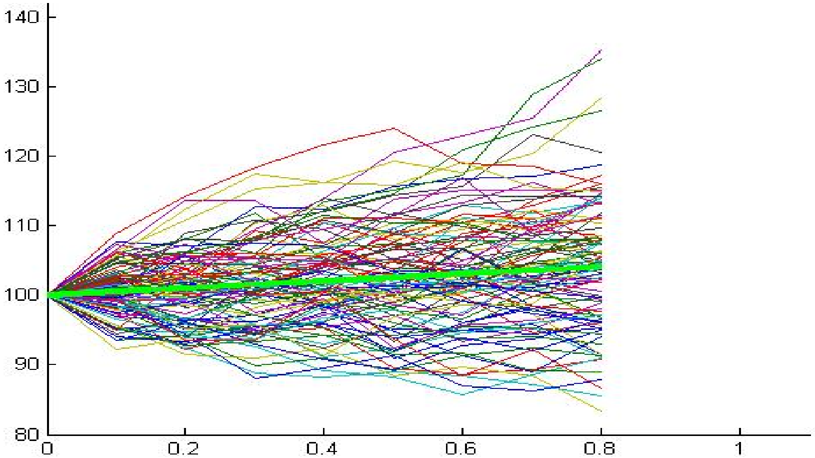

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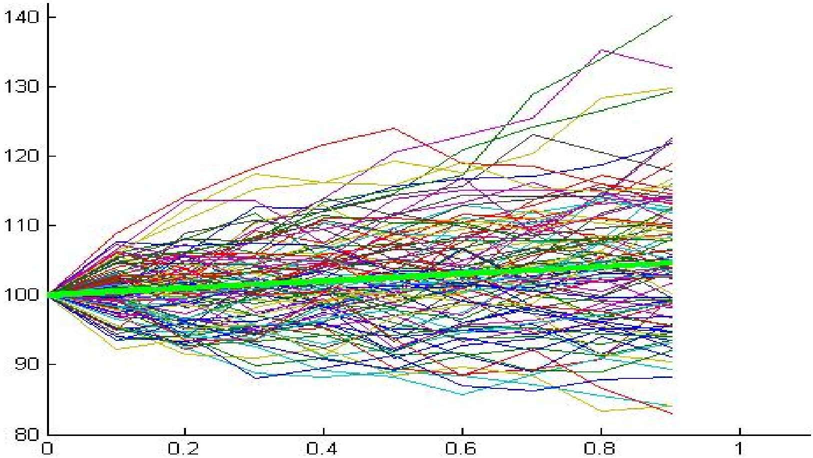

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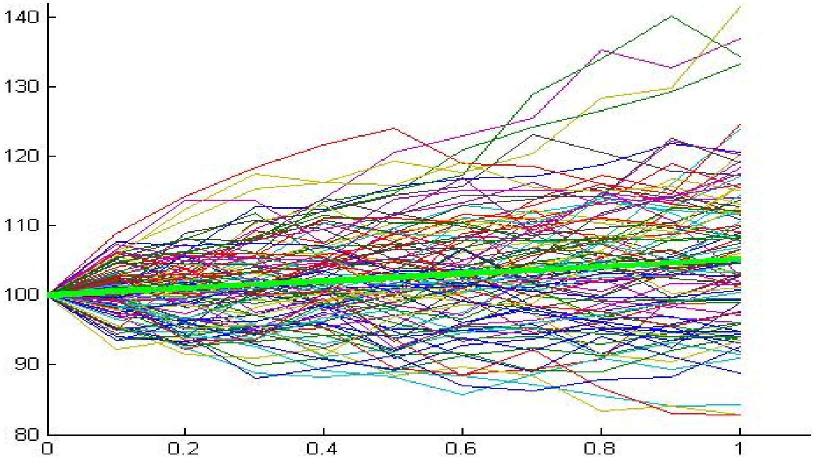

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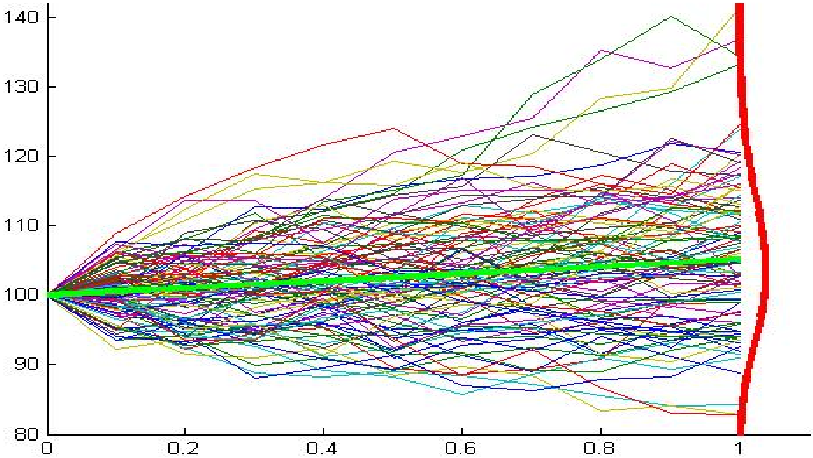
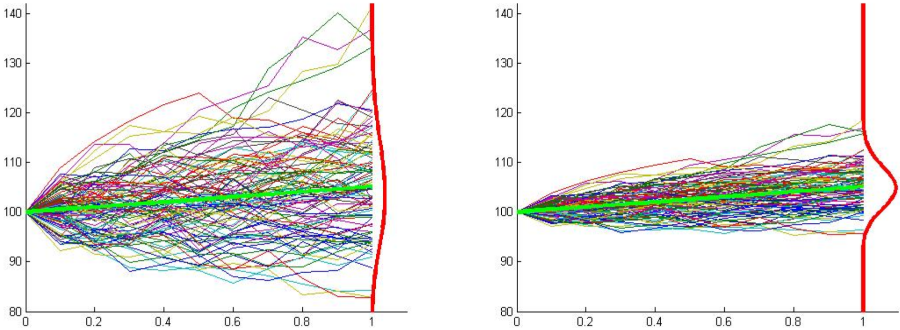


$dX_t = 0.05 X_t dt + \sigma X_t dW_t$, $X_0 = 100$:

$\sigma = 0.1$ vs $\sigma = 0.04$



What if there is only the green line? Determinism?

The clockwork universe differential equations

Newton's 2nd law, Force = mass x acceleration: here X_t =position(t).

$$\frac{d}{dt}\left(\frac{d}{dt}\,position(t)\right) = \frac{\textbf{Force}(position(t))}{\text{mass}}, \quad position(0), \quad \frac{d}{dt}\,position(0)$$

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Laplace (interestingly author of "Théorie analytique des probabilités"):

"... an intelligence which could comprehend all the forces by which nature is animated and the respective positions ... , this intelligence ... would embrace in the same formula both the movements of the largest bodies in the universe and those of the lightest atom; to it nothing would be uncertain [...] *Is everything preordained?* Free Will?

Deterministic but unpredictable? Chaos theory

Consider the following difference equation:

$$X_{t+1} - X_t = 4X_t(1 - X_t) - X_t, \quad X_0.$$

NO randomness: We don't have random terms dW_t like in previous equations. Given X_t, the next X_{t+1} is fully determined.

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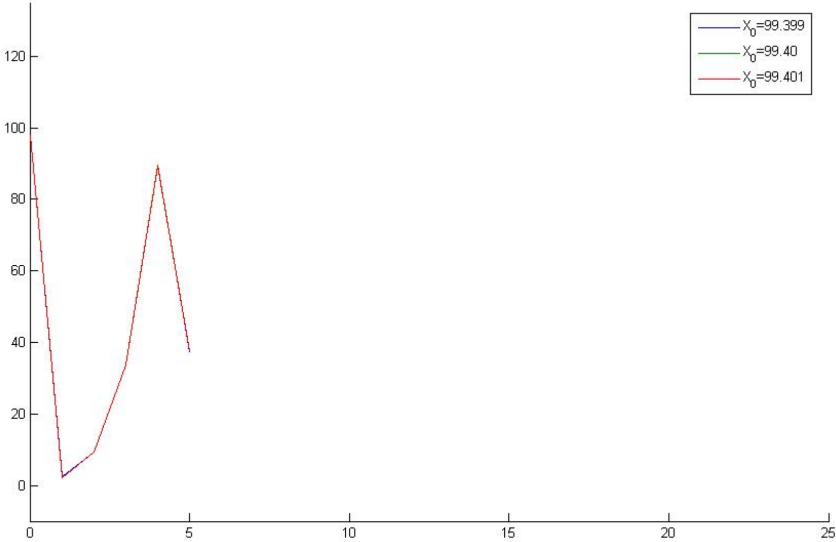
However, we may take *slightly different* initial conditions X_0

$$X_0 \in \{99.399, 99.4, 99.401\}$$

What happens for the three very close choices?

$$X_{t+1} = 4X_t(1 - X_t), \quad X_0 = 99.399,\ 99.4,\ 99.401$$



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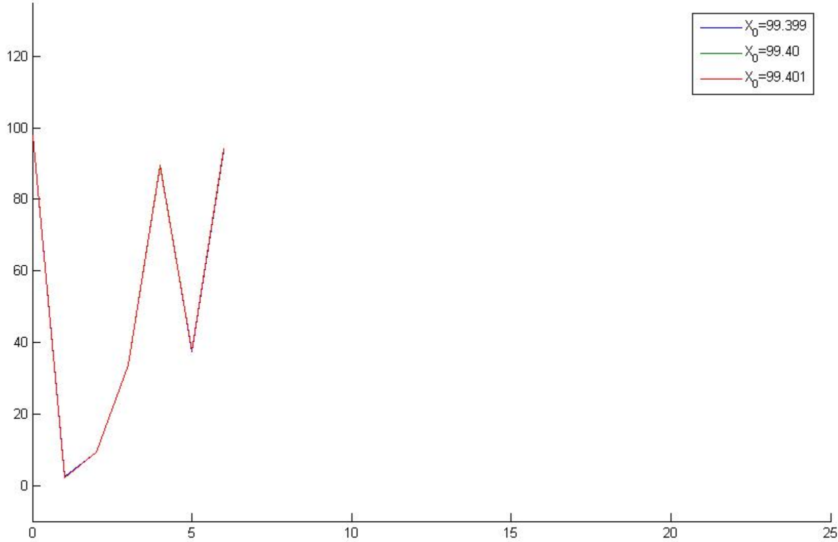

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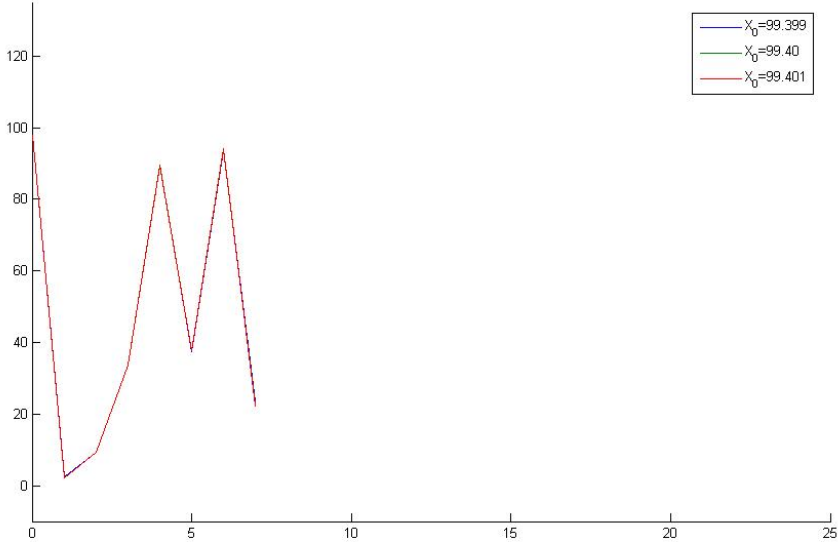

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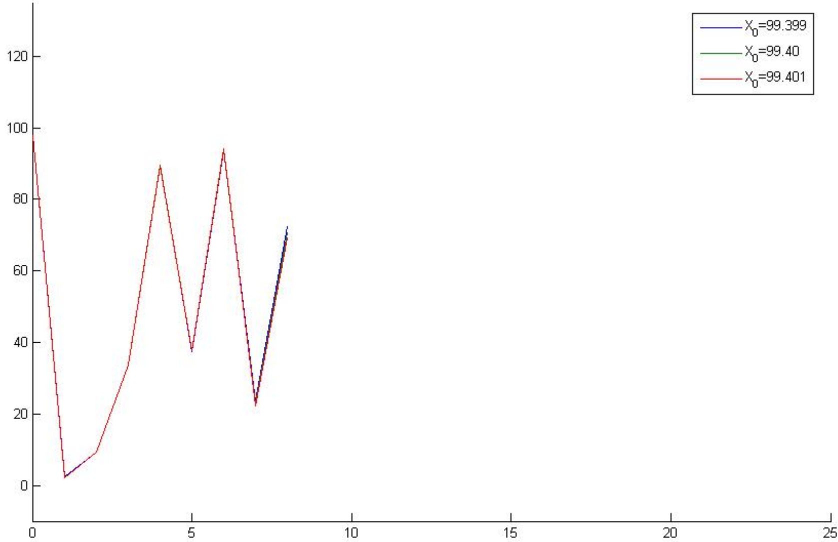

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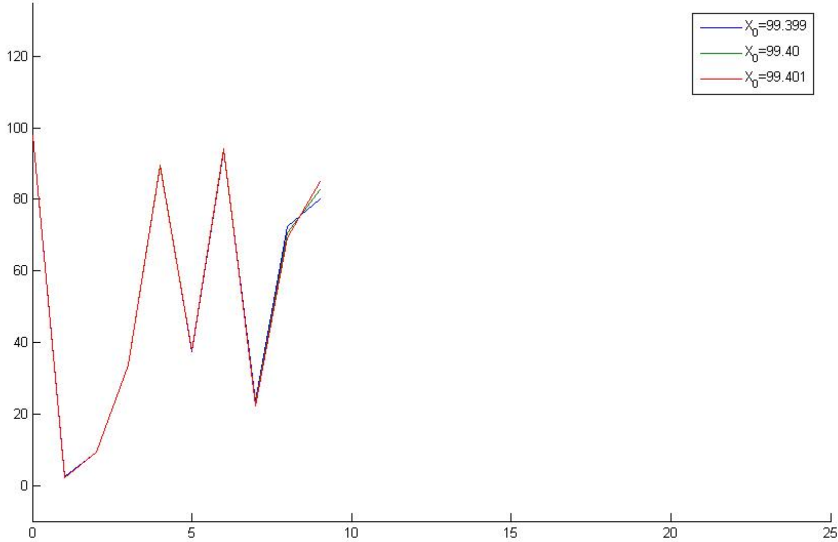

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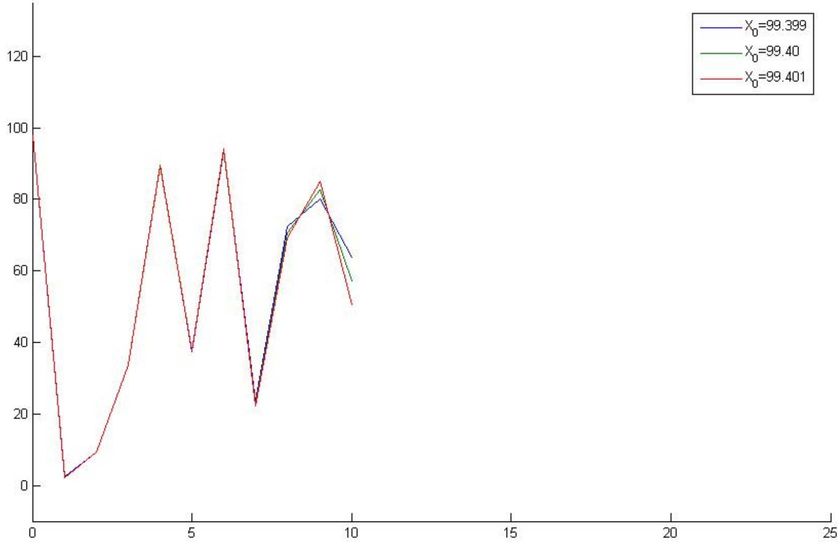

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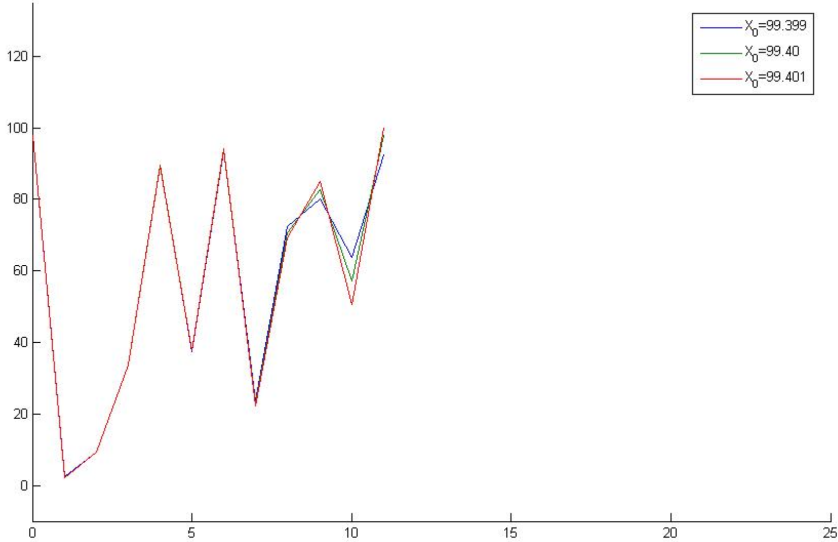

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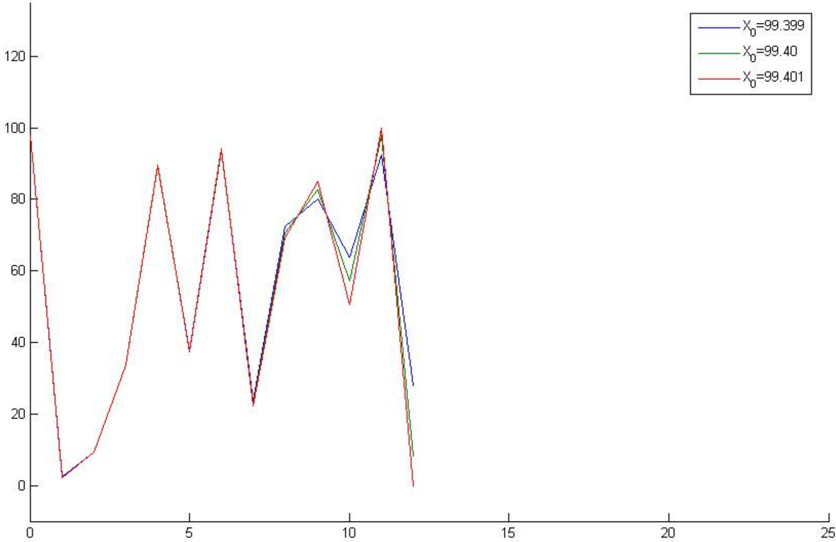

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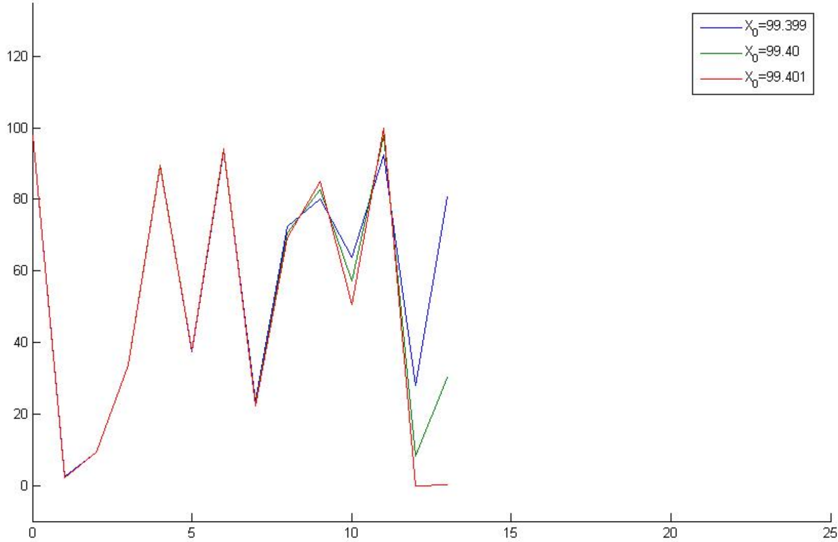

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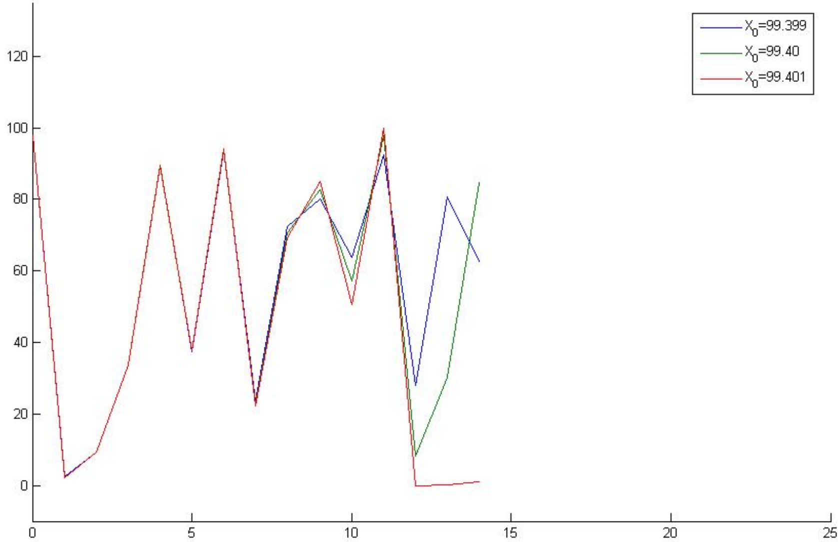

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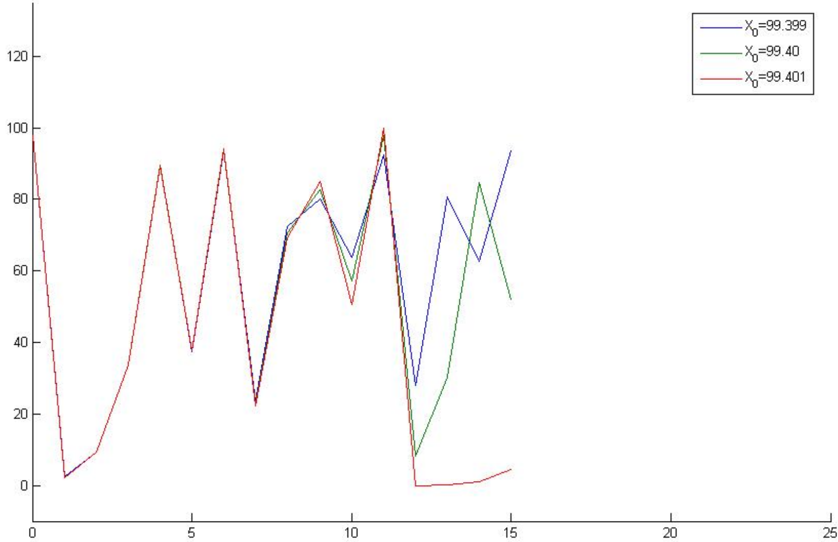

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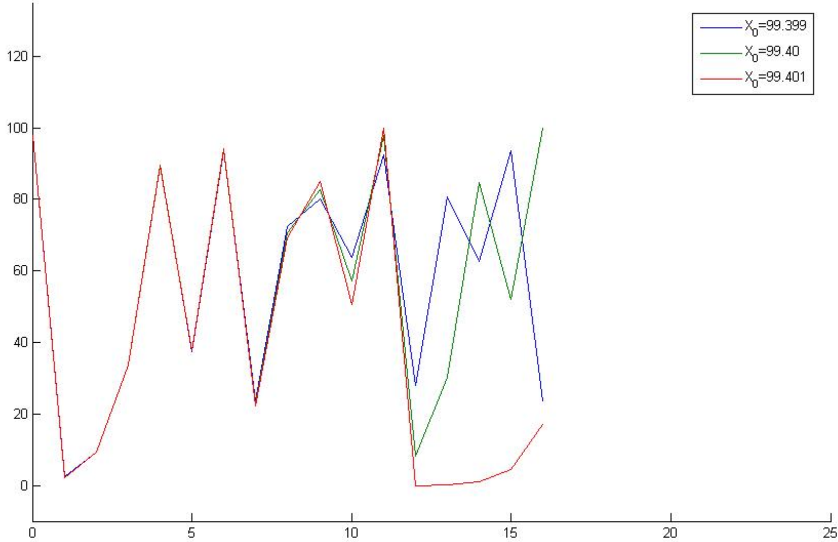

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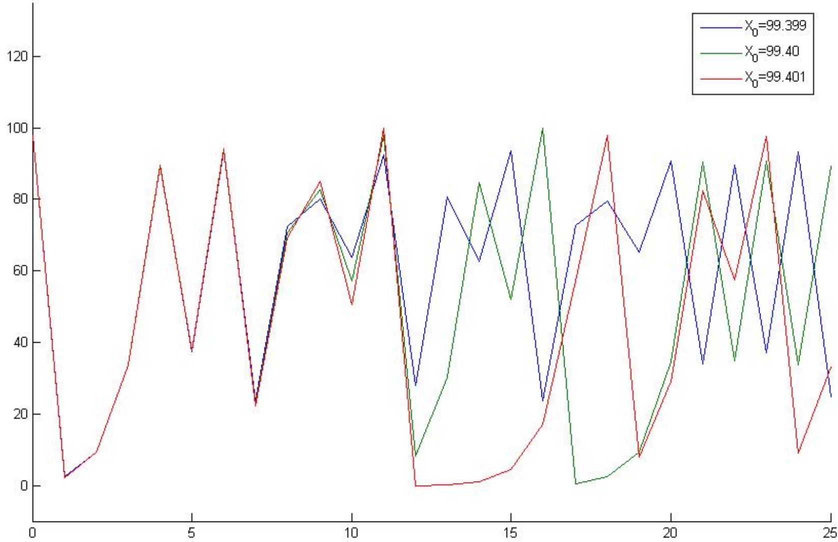


Saving free will: Randomness?

For almost identical starting points, the system can behave very differently (example: simplified models of atmospheric weather).

Chaos: Practical unpredictability but still determinism

We are not capable of predicting the future of chaotic systems because of the impossibility of perfectly measuring the initial condition of a system [see also uncertainty principle in Quantum Mechanics (QM)]. Still, in a deterministic universe the future is preordained even though unknowable by computation. There would be no free will.[a]

[a] Assuming you think the notion of free will makes sense and that you are an incompatibilist, ie you consider determinism and free will incompatible

Can Randomness help here?

If Nature were somehow truly random, then the future would not be preordained but open to more outcomes.

Is randomness just our ignorance? I

In most mathematical models (and in our previous examples with SDEs) randomness is a tool to model ignorance or incapability to account for complex effects, too many particles, hidden variables...

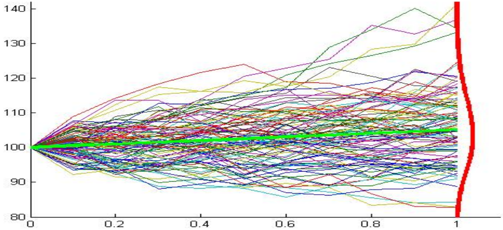
Example: Dice throw

We don't bother measuring the initial position of dice, the initial impulse we apply, air resistance, etc, so it is unpredictable to us, but randomness here and in related examples is due to our ignorance, or to chaos, and not to any intrinsically random feature in nature.

Is randomness just our ignorance? II

Example: FTSE 100

The financial index in the future will depend on market participants choices, macroeconomics, policies, market comovements... As all this cannot be modeled properly, many effects are grouped into a random term (like W_t) that is supposed to account for our ignorance through different scenarios. Stylized example finance: $dX_t = mX_t dt + \sigma X_t \, dW_t$



$m = 5\%, \ \sigma = 10\%, X_0 = 100.$

Is randomness just our ignorance? III

Statistical Mechanics (SM) (non-Quantum)

- Micro level: Particles move according to deterministic and time–symmetric laws (eg Newton's, see earlier)
- Ignorance: impossibility to know precisely initial positions and speeds of the huge system of particles and to carry out the exact calculations. So we resort to a probabilistic description.
- Temperature, pressure, heat capacity, the entropy arrow of time... all emerge through our statistical description.

Is randomness just ignorance? Not always (maybe)

Quantum Mechanics (QM)

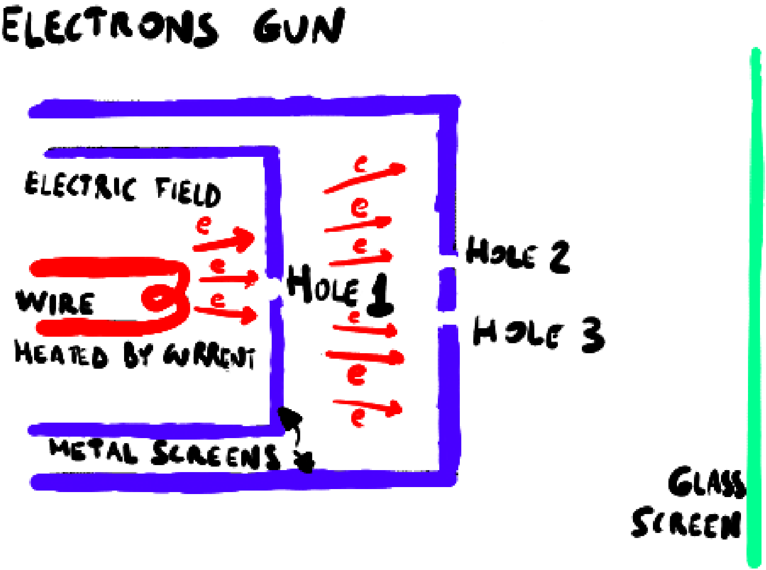
Randomness intrinsic property of nature at the quantum scale and not our ignorance on hidden "classical" variables.

But... QM's fundamental Randomness not accepted by all

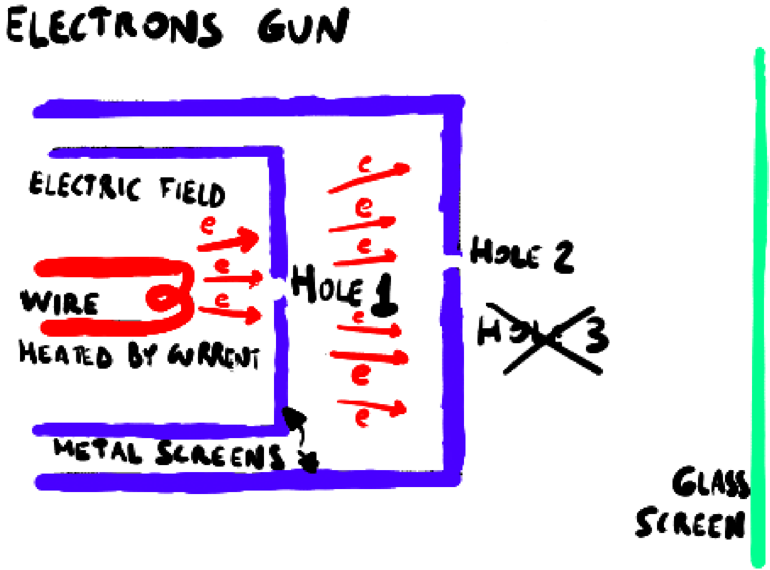
If QM randomness is not fundamental (alternatives include Pilot wave / global hidden variables or Many Worlds), we lose the only currently known true source of randomness in nature.

Now let's look at fundamental randomness in QM.

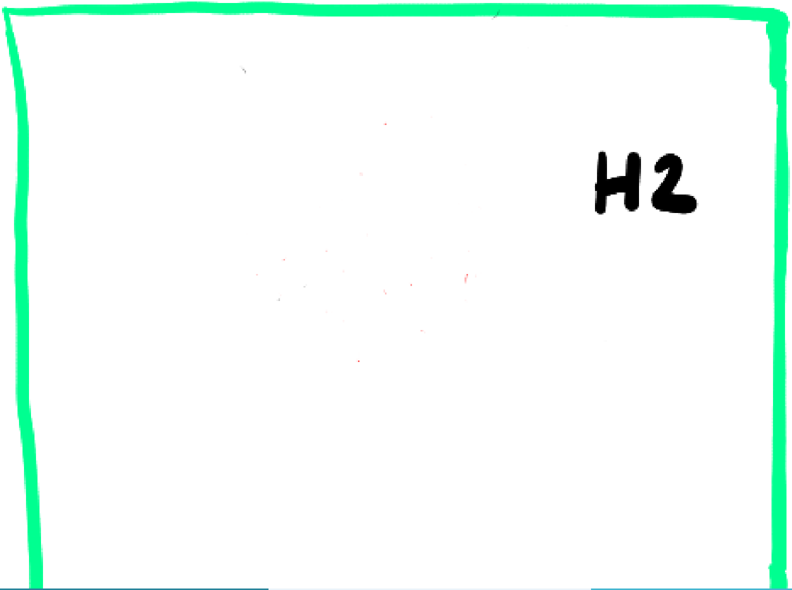
Double slit experiment (John Bell's summary)



Double slit experiment: Closing one slit



Glass screen, shoot 1 electron at the time. Particles?



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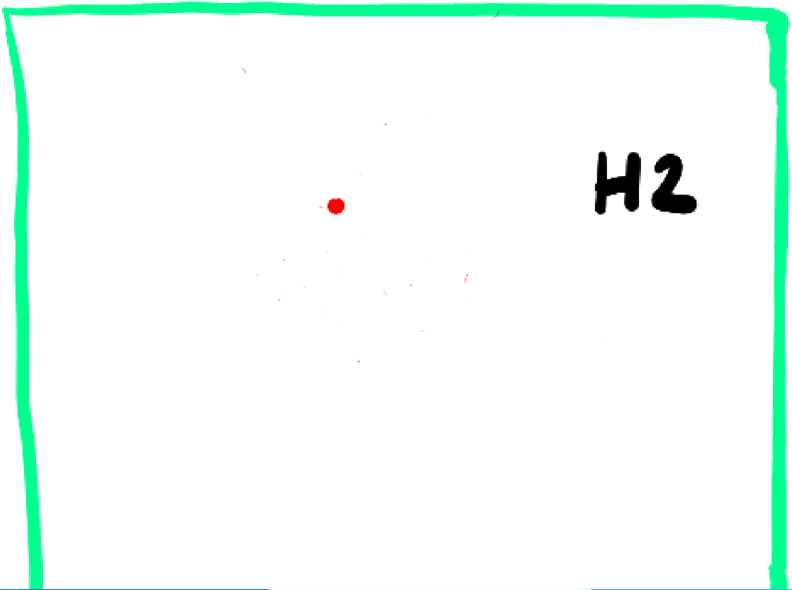

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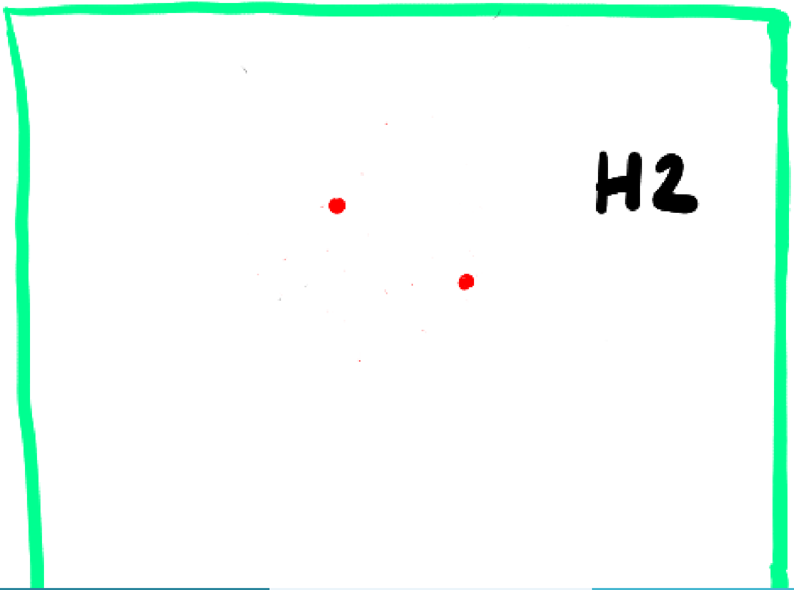

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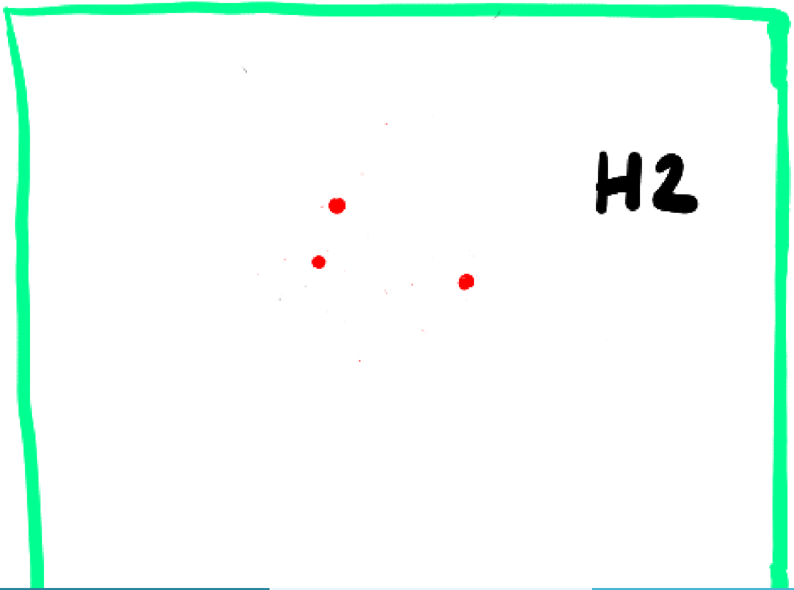

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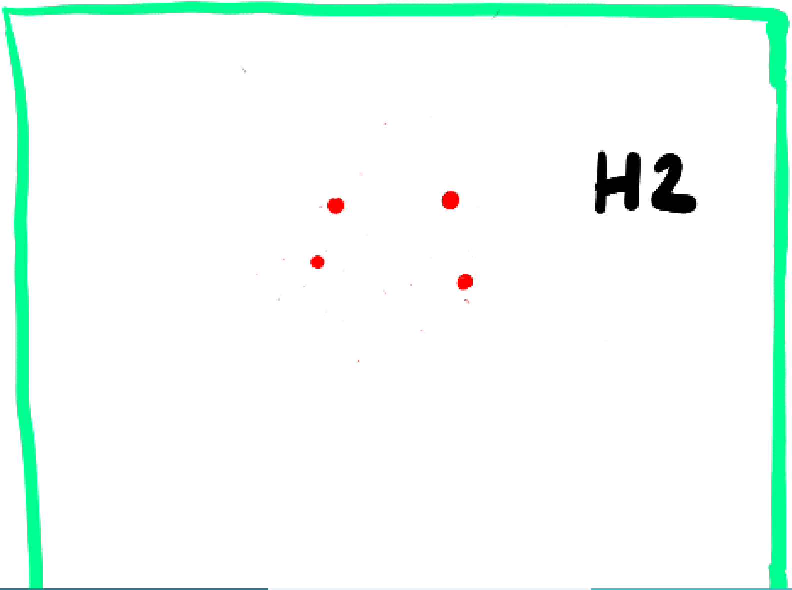

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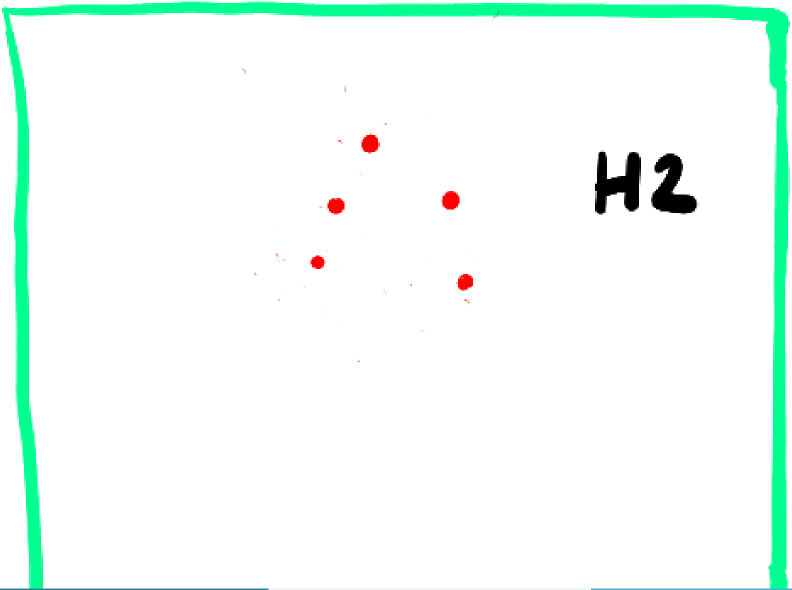

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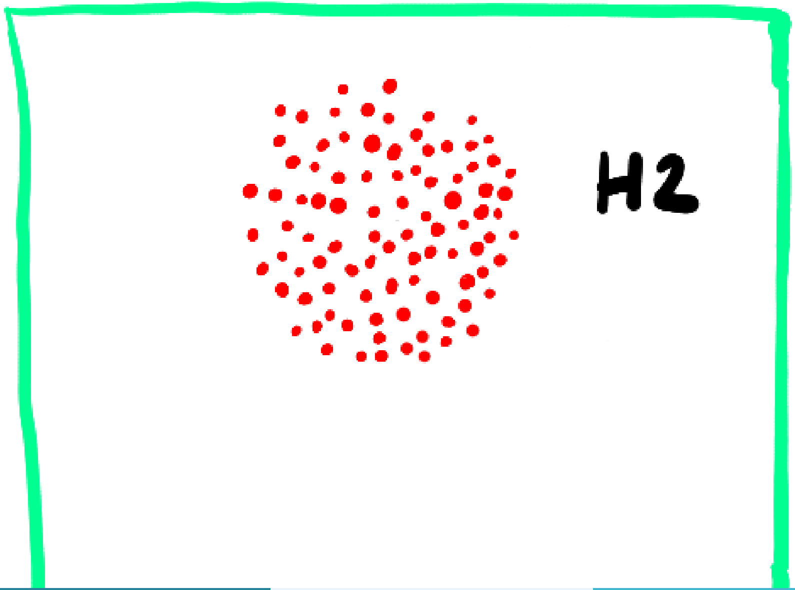
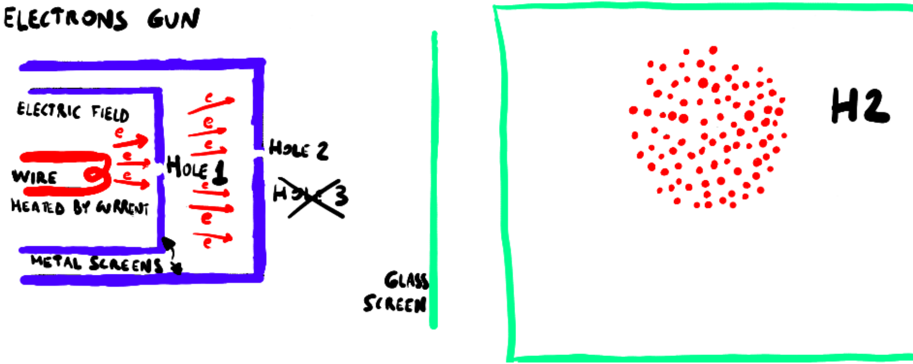


Double slit experiment: Closing one slit



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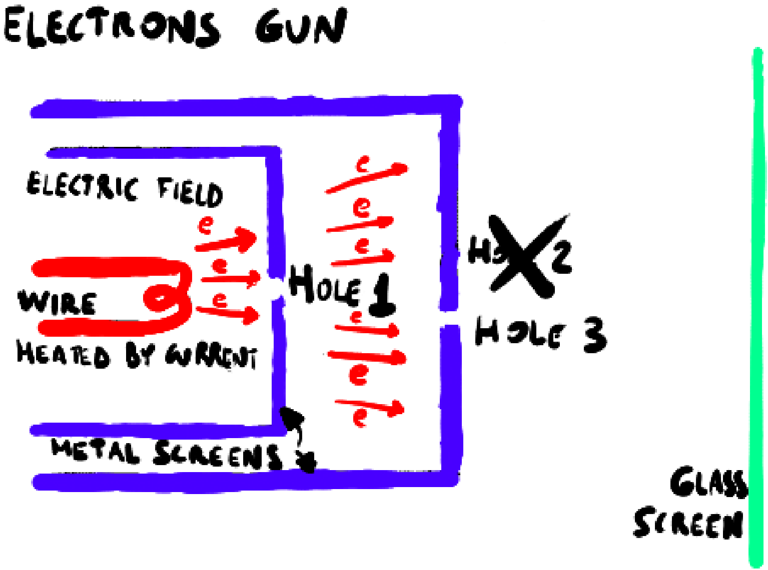

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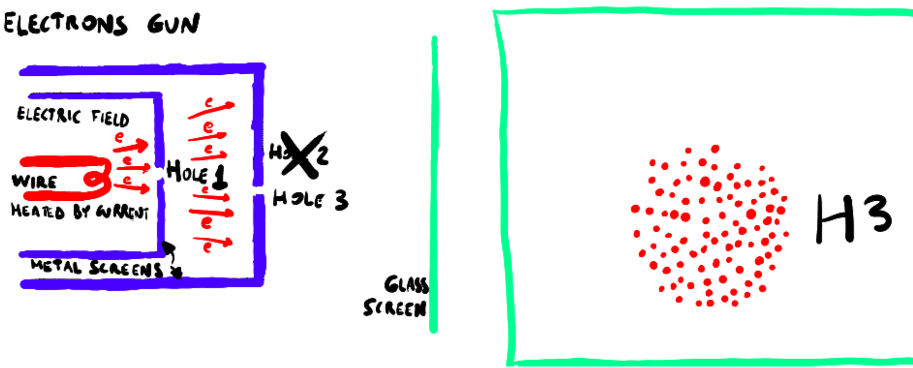
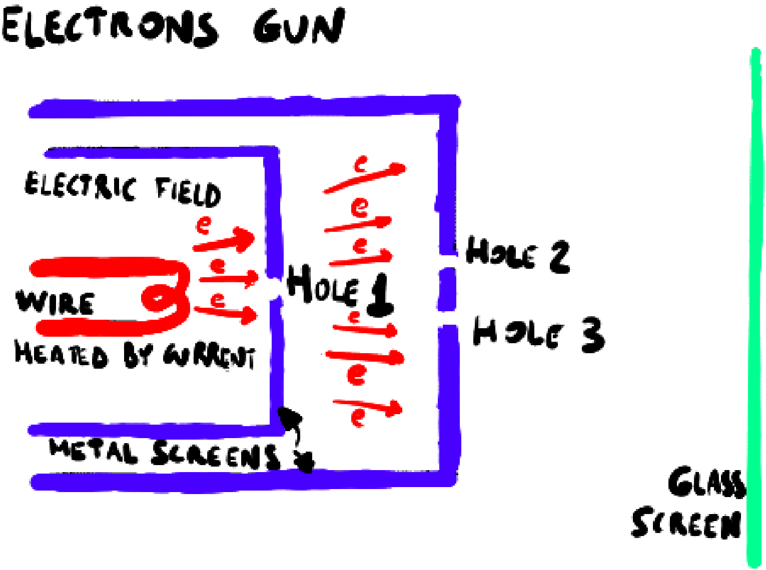


Both slits open: If particles, we expect this



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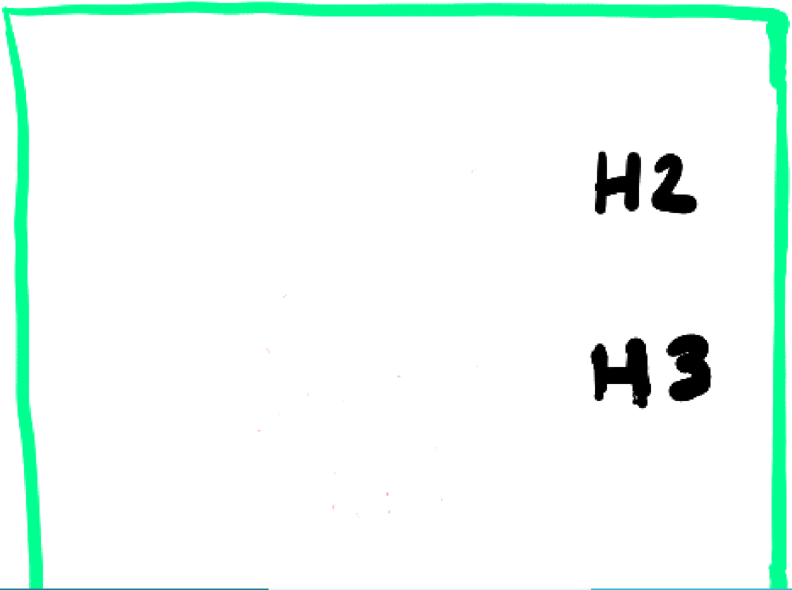

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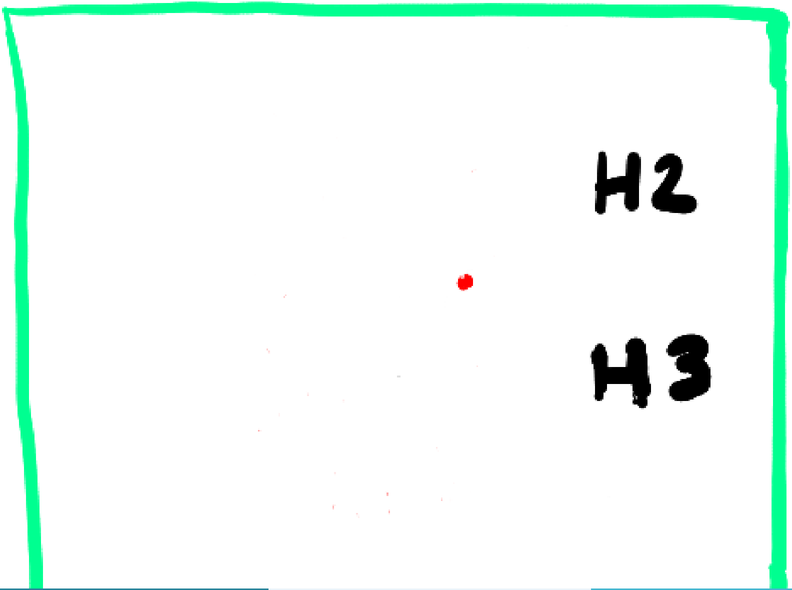

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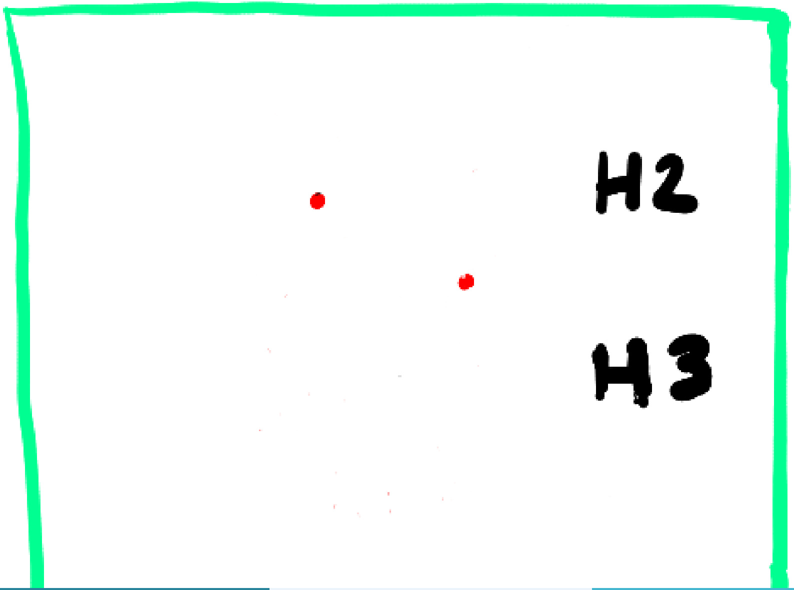

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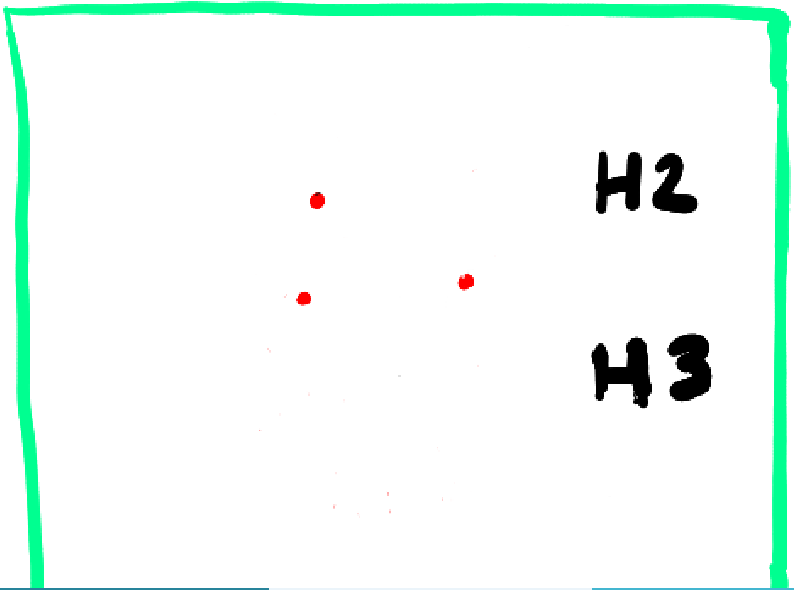

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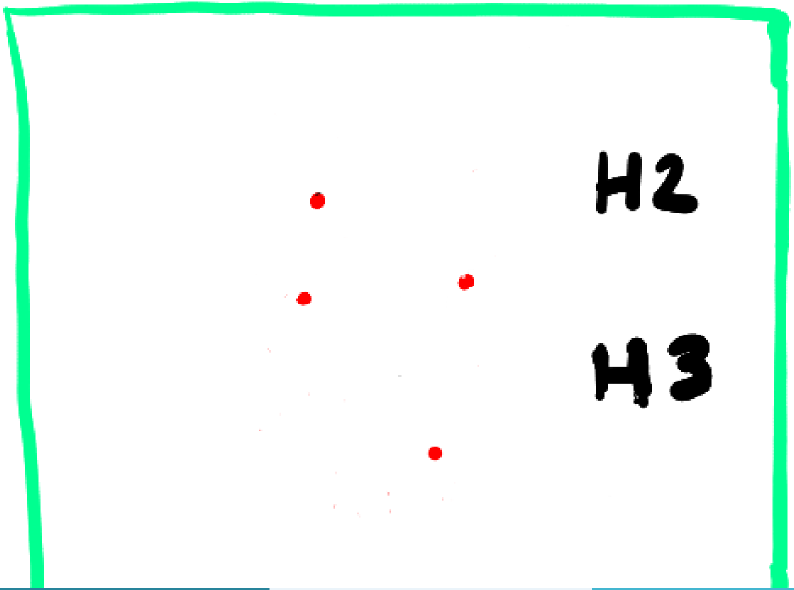

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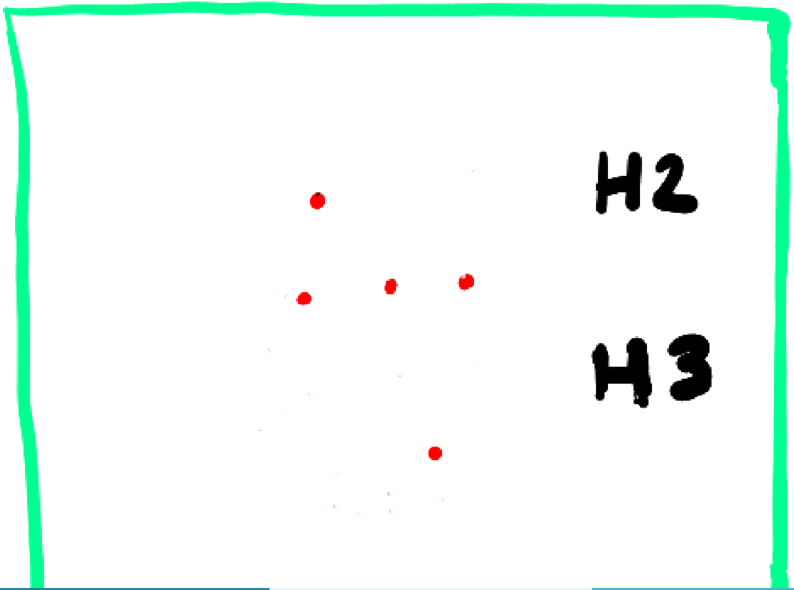

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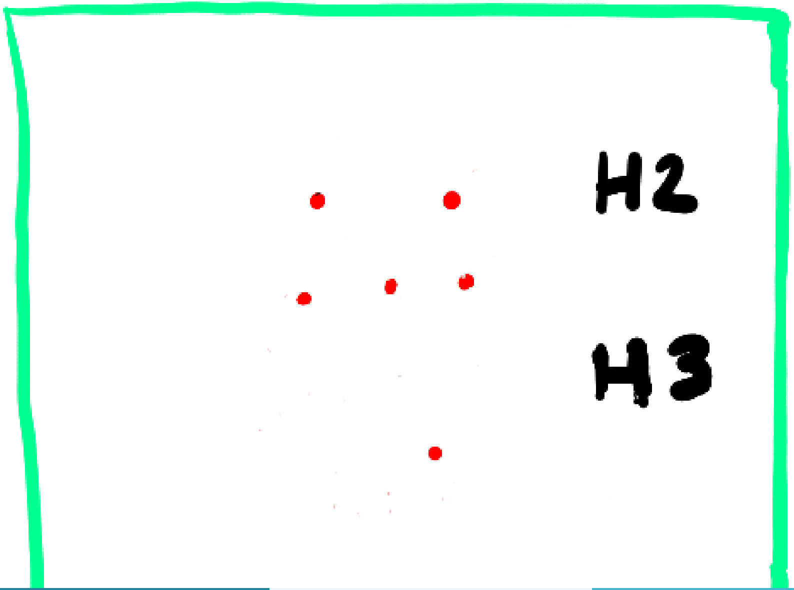

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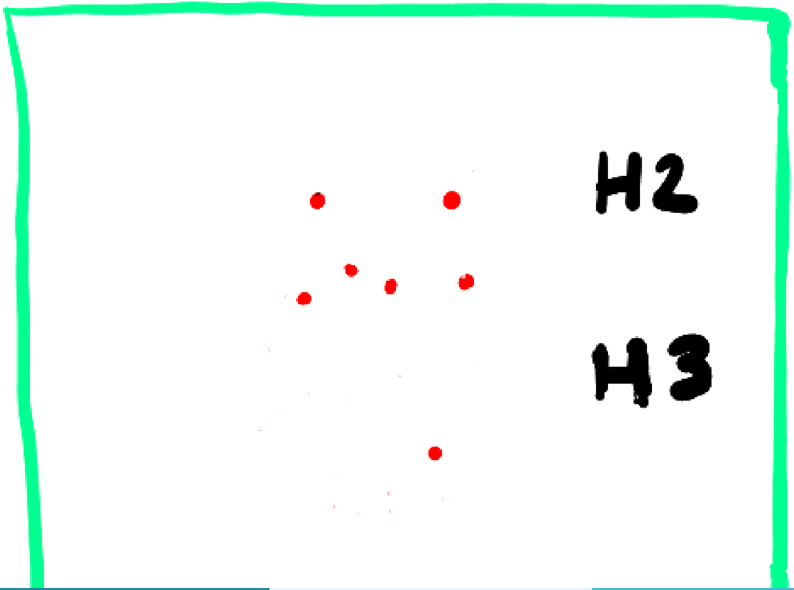

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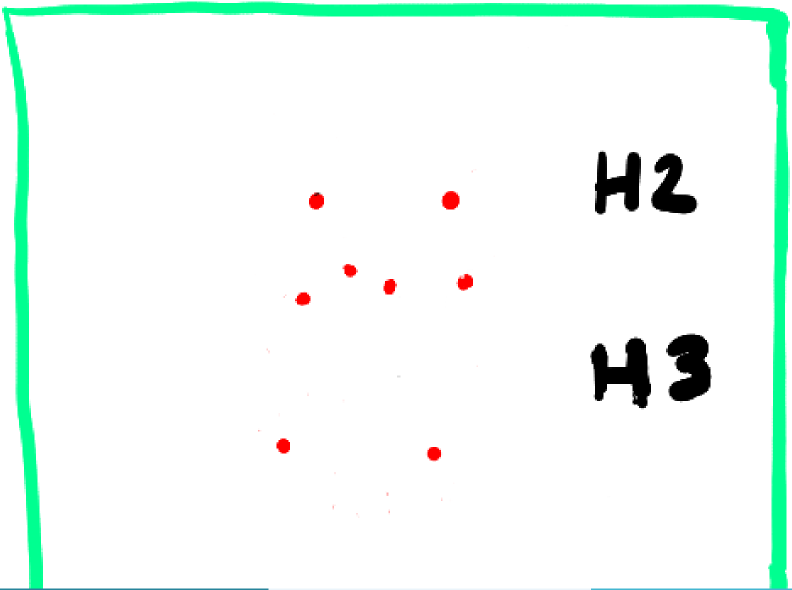

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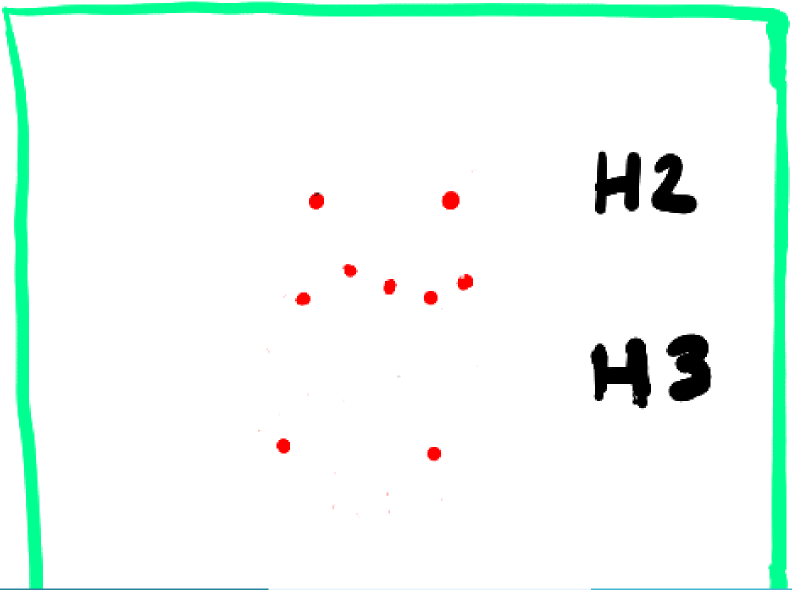

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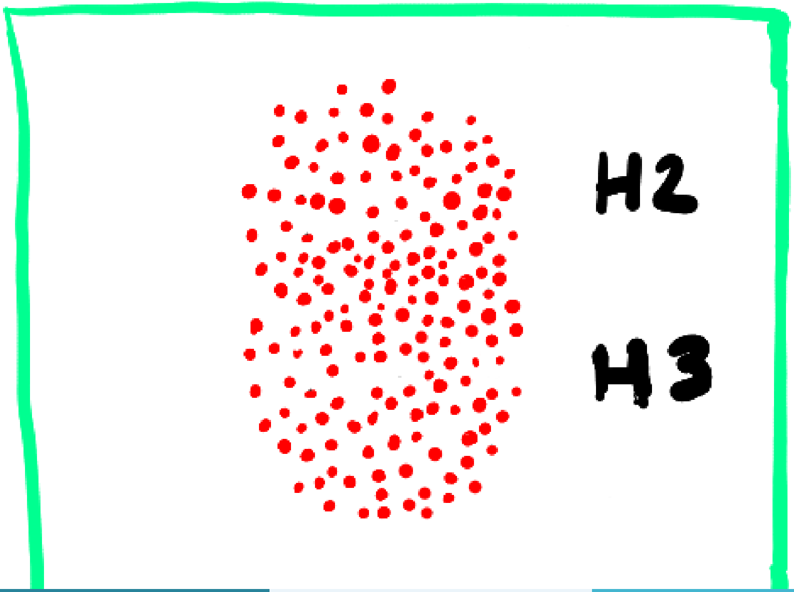

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What we get instead looks like ....

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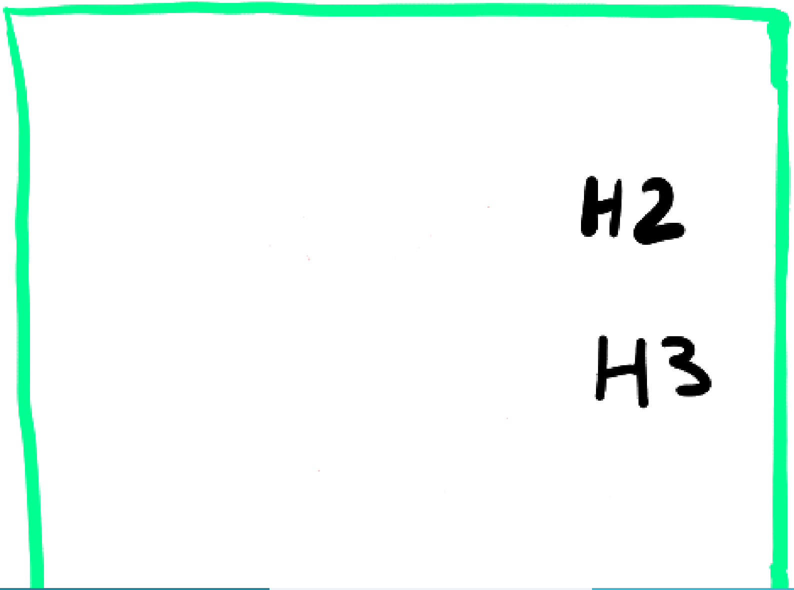

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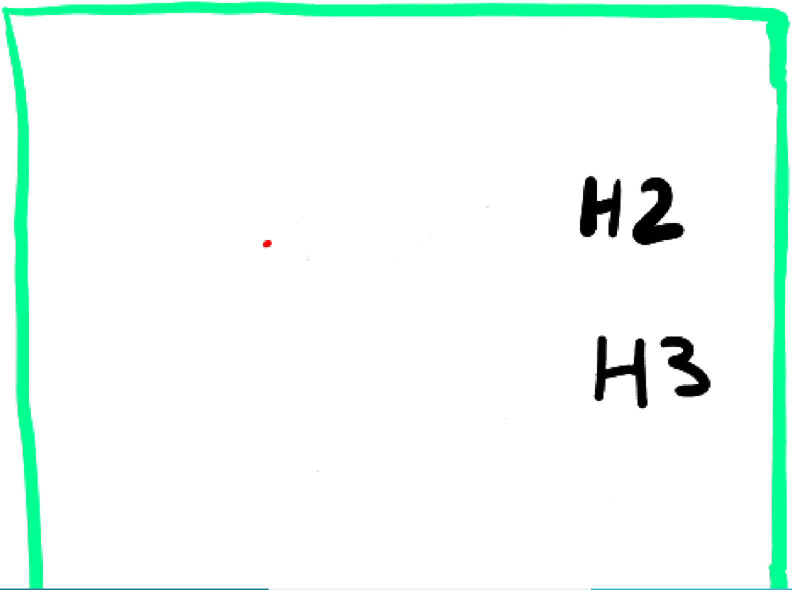

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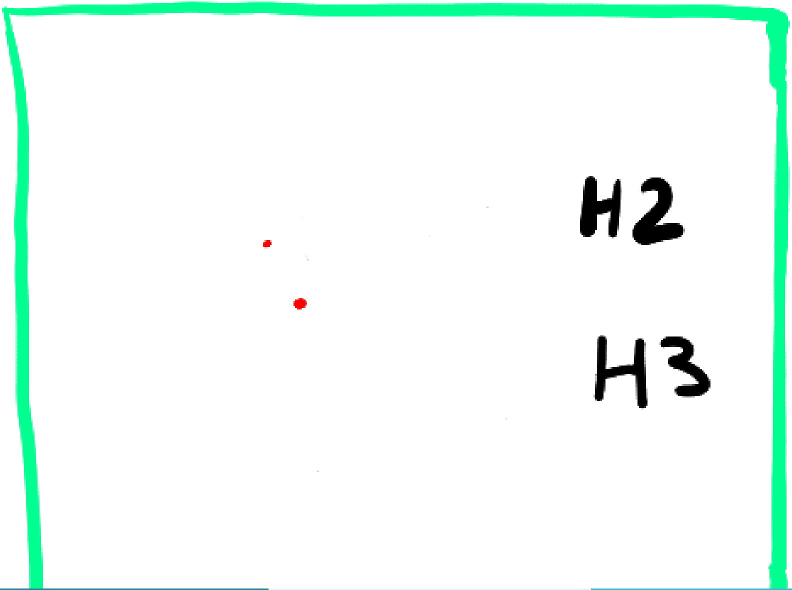

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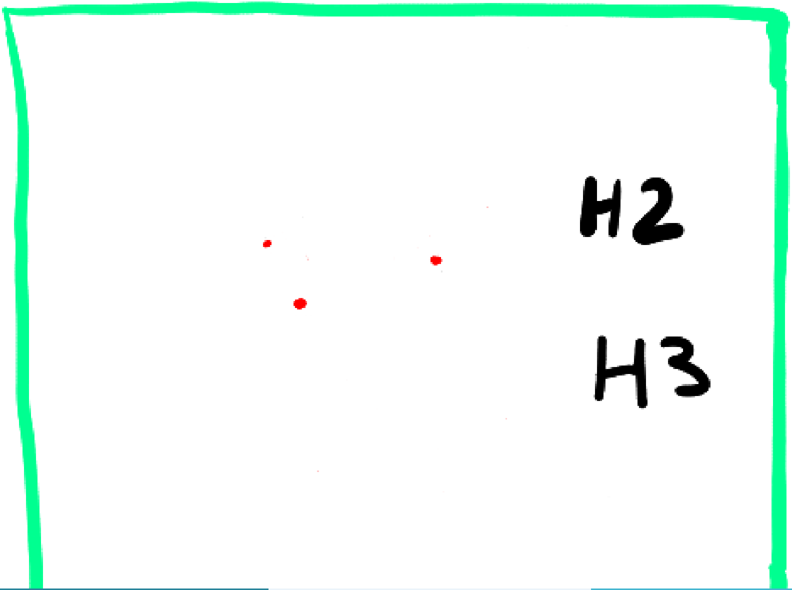

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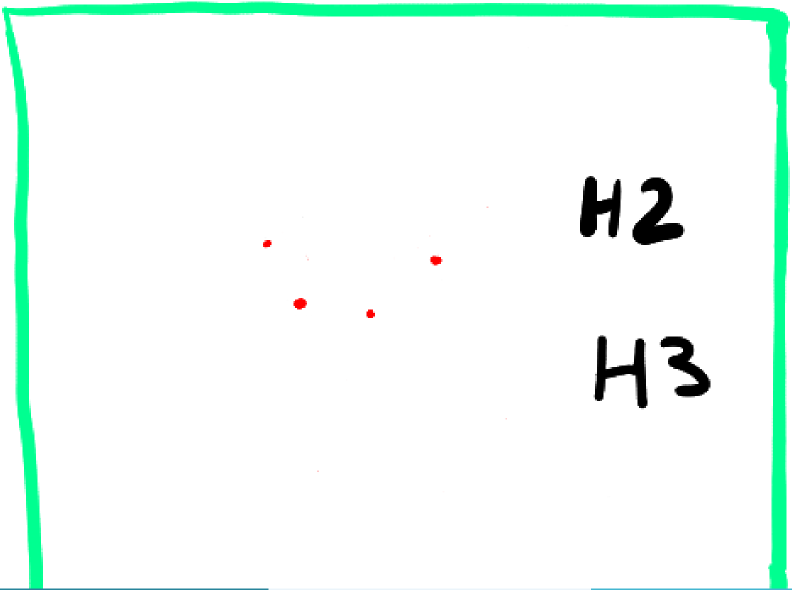

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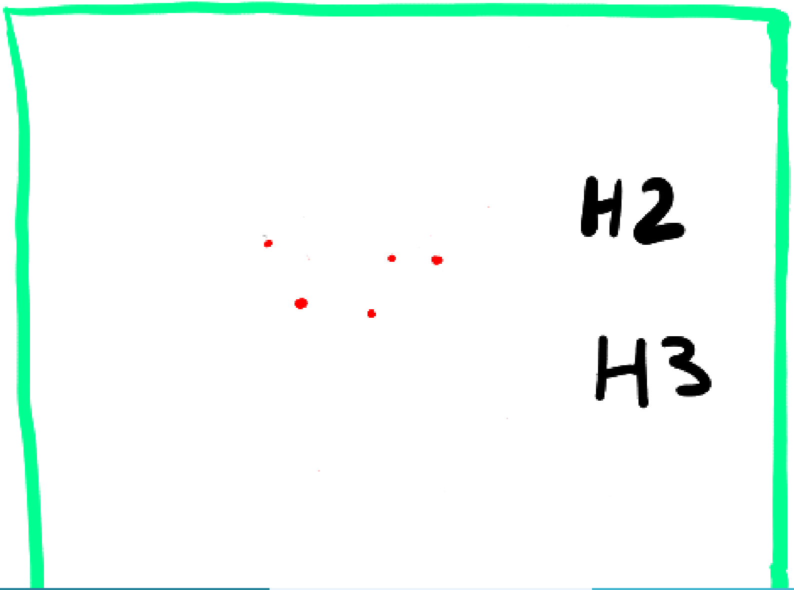

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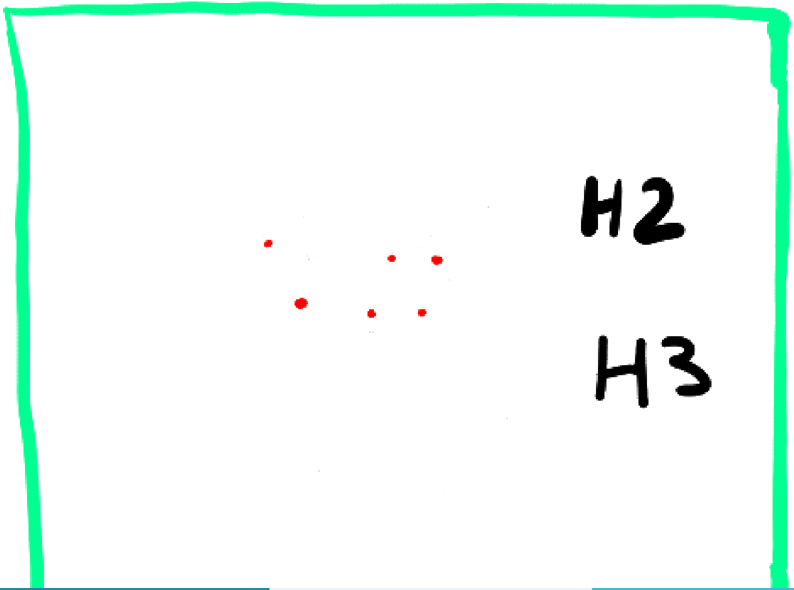

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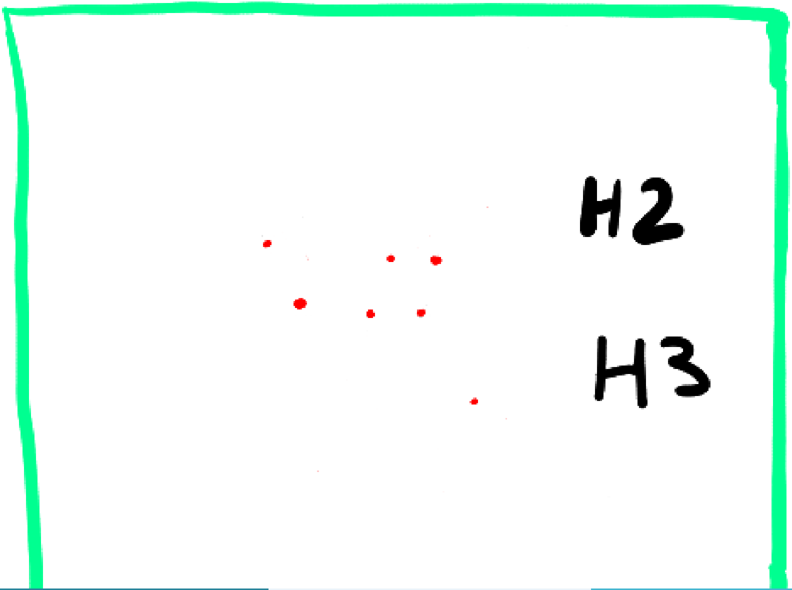

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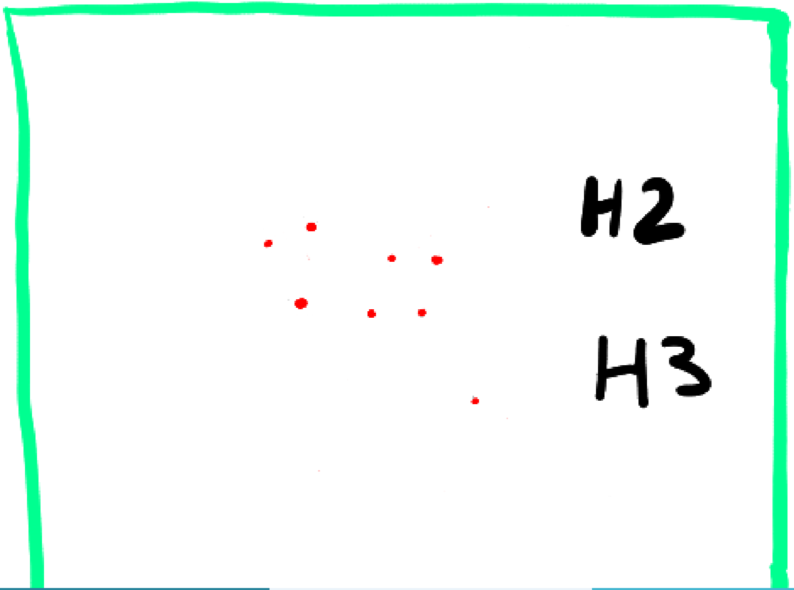

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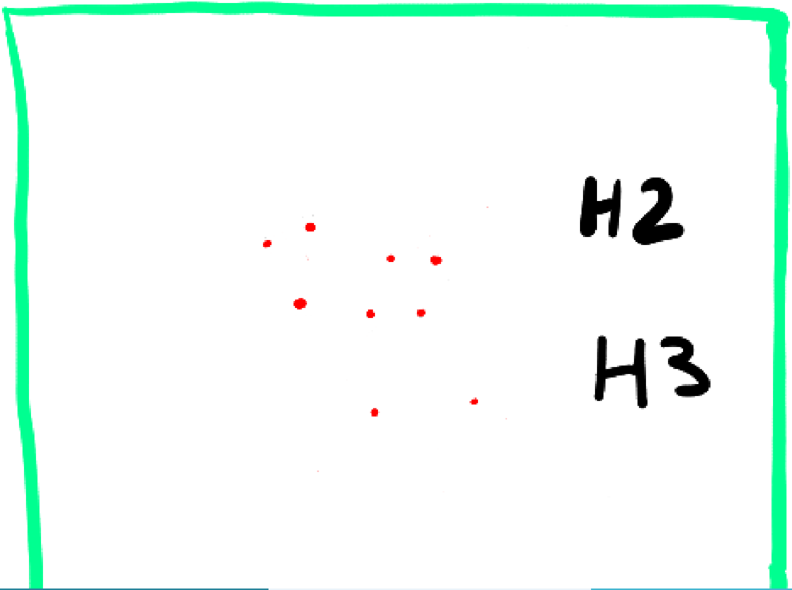

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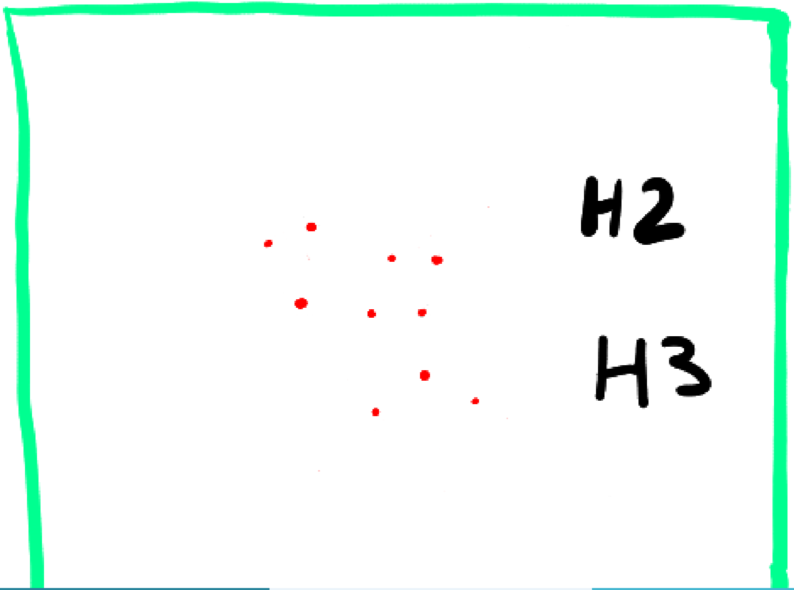

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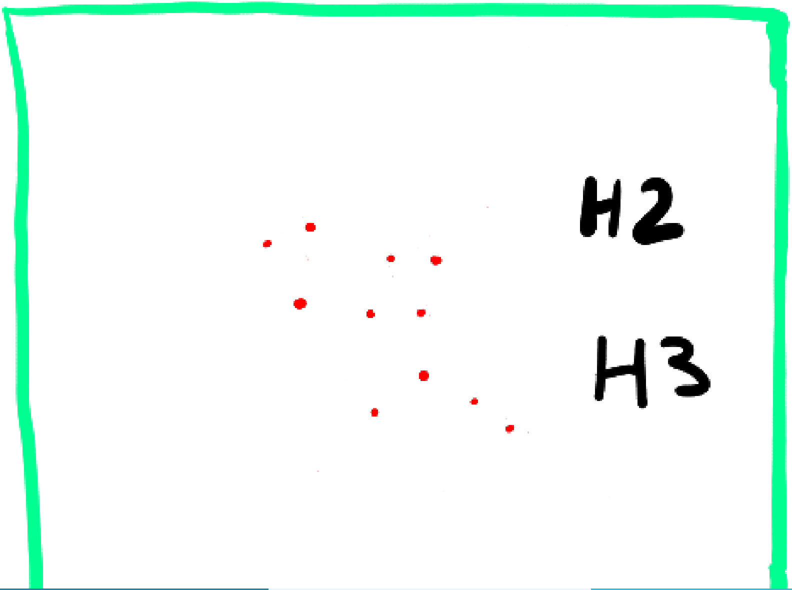

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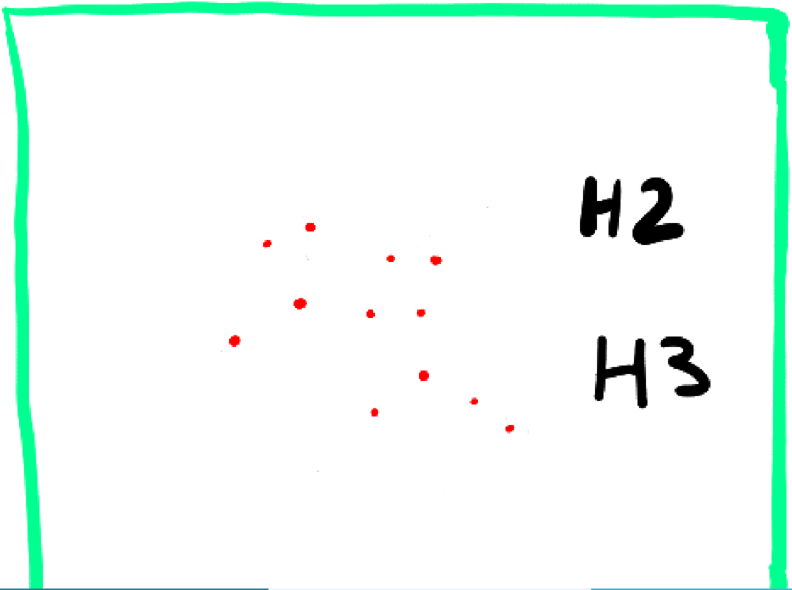

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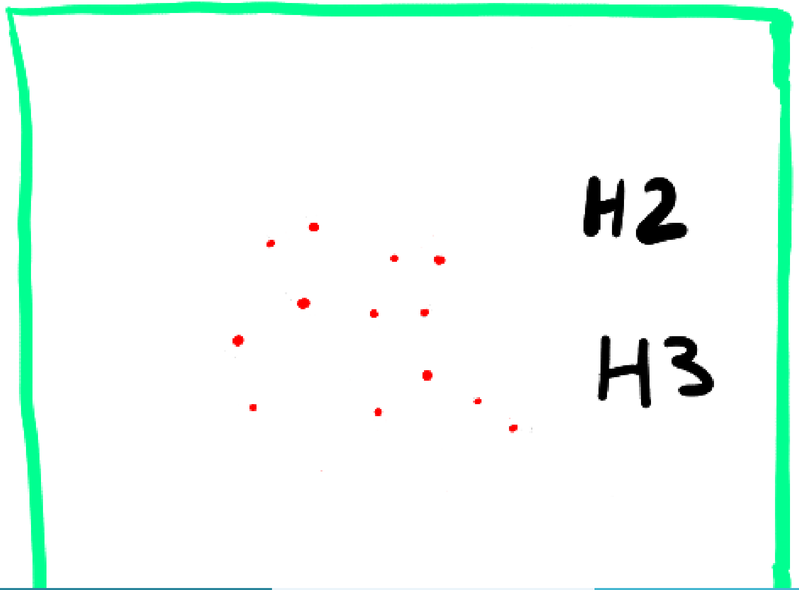

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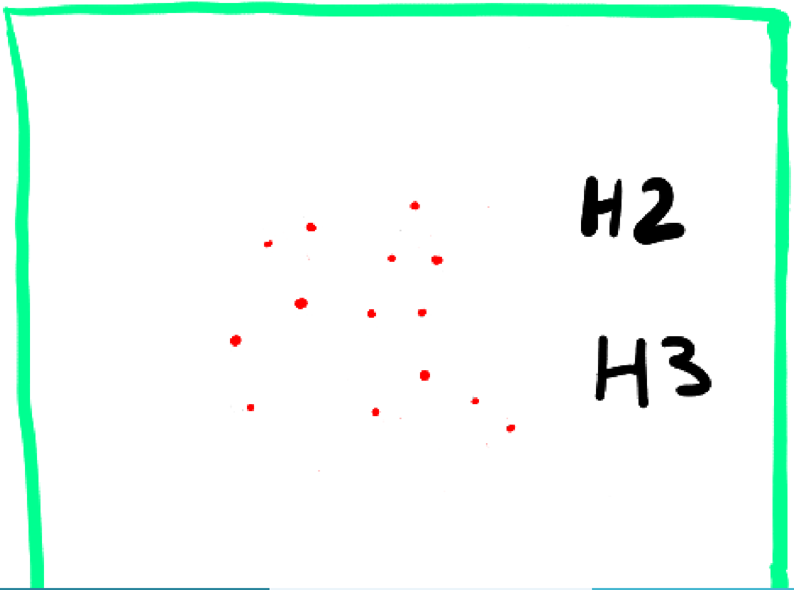

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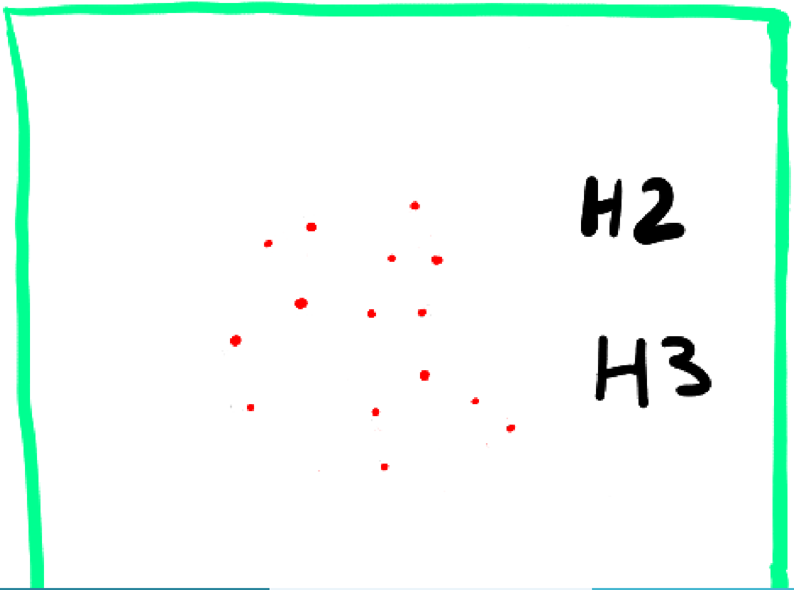
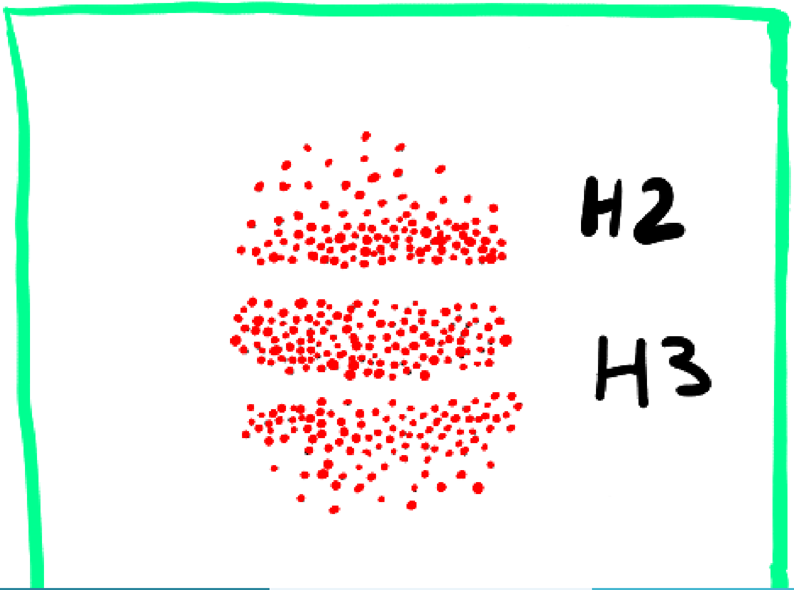
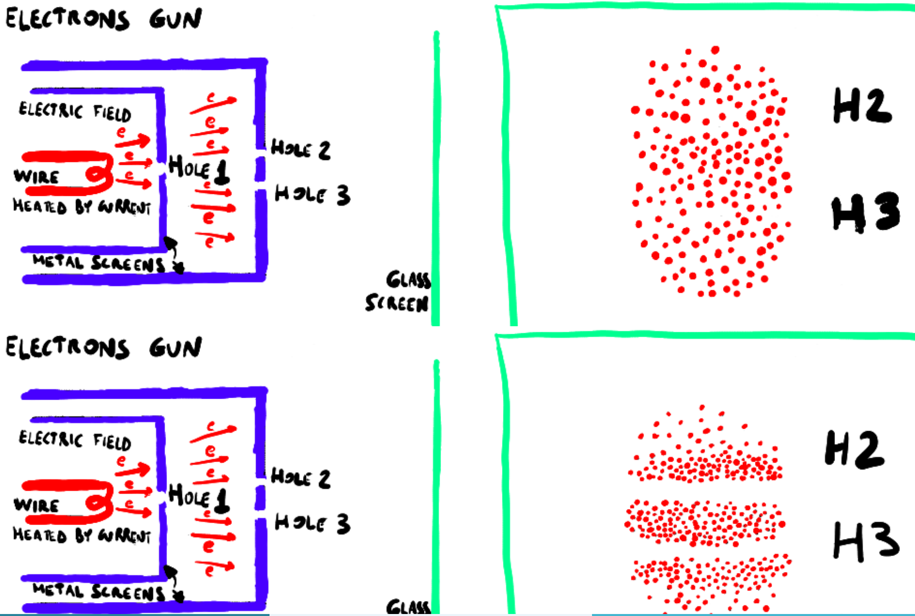


Both slits open ... Interference?? WAVES???



Both slits open: What we expect and what we get



Double slit experiment

Interference? Of what and with what? Waves?

If I shoot electrons one by one, what does each electron intefer with???

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Intrinsic randomness?

- Probabilities interference. The interference pattern will be consistent with the theoretical probabilities prescribed by Quantum Theory, Von Neumann axioms/ Born rule (VN-B).
- A local ignorance interpretation is not possible (Bell, Aspect...
- Non-local hidden variables possible (Pilot wave DeBroglie/Bohm).

Double slit experiment: What kind of Randomness?

Randomness and undefined positions before measurement

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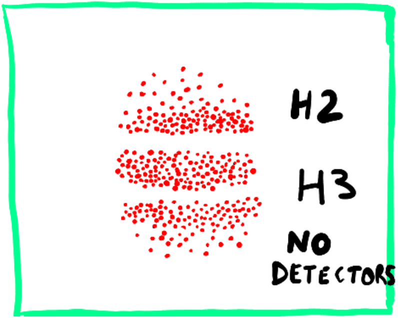
- **Is this what true randomness - not due to ignorance - should always look like?**
- Interference disappears **in both holes** if we place a detector near *just one hole*. Does the electron ”know” we are observing it?

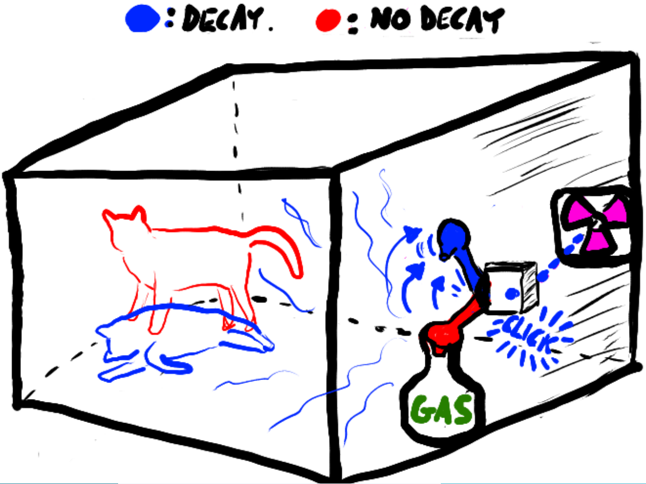
Randomness and Measurement

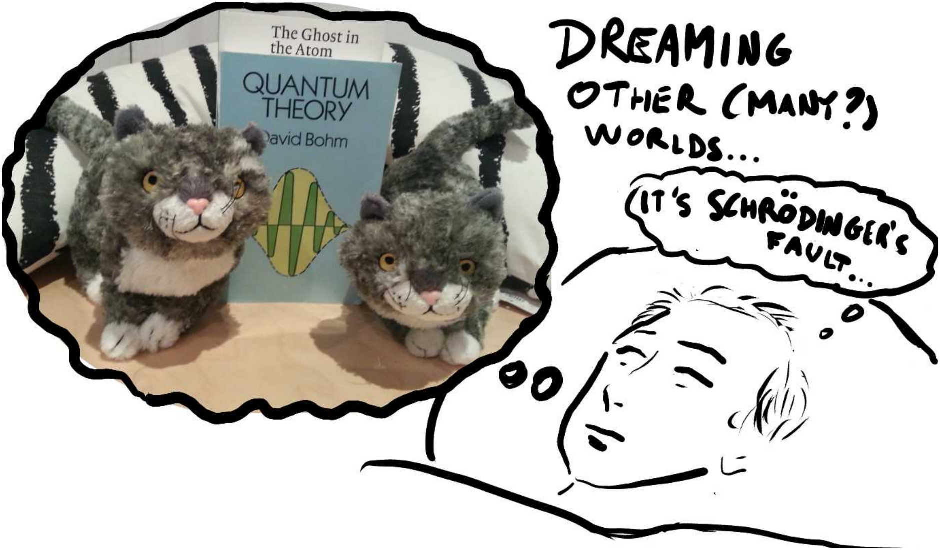
Randomness and interference? The role of partial measurement

Placing a detector near just one of the holes with both holes open, the picture goes back to the classical randomness (left), no interference. Right: no detector. A partial measurement produces a change from interfering to non-interfering random behaviour also in the distant hole.



Interference & no detectors: a particle does not have a defined (hidden) position before measurement. If this indeterminacy is amplified at macroscopic level through some device avoiding decoherence/partial "measurement", we have a cat who is simultaneously alert and asleep before we open ("measurement") the cat box.




The Ghost in the Atom
QUANTUM THEORY
David Bohm
DREAMING OTHER (MANY?) WORLDS...
(IT'S SCHRÖDINGER'S FAULT...

Quantum vs Classical Probabilities

Why this digression on QM?

1. **QM scale might show the "only" objective randomness** (well everything is a quantum system, including our macro scale, but...)

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3. ***QM probability axioms (Von Neumann/Born 1932) at odds with classical axiomatization (Kolmogorov 1933)*.**
4. Formally, VN-B is more general than K in a number of ways, reflecting non-commutative nature of QM (quantum probabs NOT sub-additive, $P(A \text{ or } B) > P(A) + P(B)$ may happen, not distributive, no Borel Cantelli...)

Classical and Quantum Probabilities

Quantum Probability rarely used in social sciences.

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We now turn to finance and financial derivatives in particular, and to signal processing, and we will work under Kolmogorov’s standard axioms for probability.

Options and Derivatives

Derivatives outstanding notional as of June 2011 (BIS) is estimated at **708 trillions USD** (US GDP 2011: 15 Trillions; World GDP: 79 Trillions)

708000 billions, 7.08×10^{14} USD (staggering, despite double counting)

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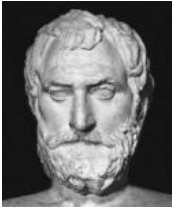


Father of philosophy, science and... derivatives?? A lot to answer for

Beginnings? 580 BC: Thales purchases options on the use of olive presses and makes a fortune when the (random) future olives crop turns out to be as abundant as he has predicted, with presses in demand. Precursors of modern theory include Bachelier (1900) and deFinetti (1931). Modern theory started by Black Scholes Merton (1973).

Derivative: Call Option paying $\max(S_{1year} - S_0, 0)$ in 1y

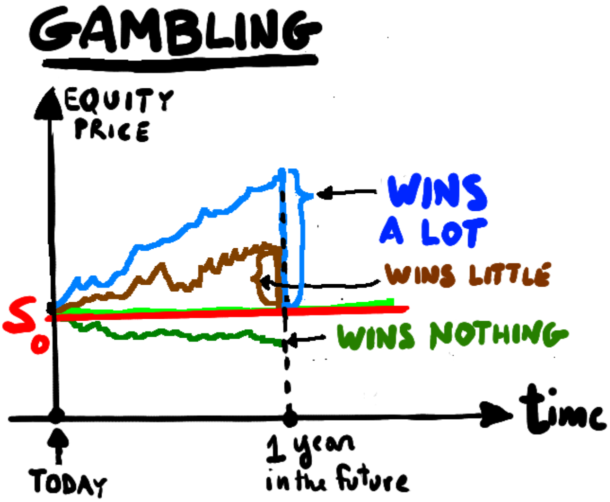


Figure: A Gamble on the growth of an equity stock in 1y. Call Option.

A basic derivative contract: Call Option

Gamble (call option) against bank on total growth of Stock S over 1y

We will receive from the bank, in one year:

$$\max(S_{1year} - S_0, 0)$$

S_{1year} is in the future and random for us at the time of the gamble.

- If the future price S_{1year} of the stock in 1y has grown and is larger than the value S_0 today, we receive from the bank the difference.
- If it has not grown, nothing happens.

Option price the bank will charge us?

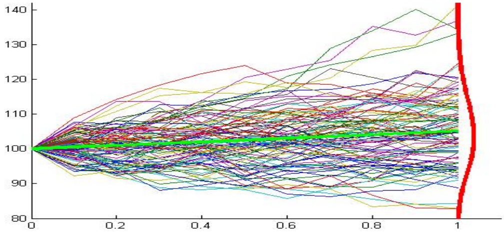
We'll be charged to enter this gamble, as we can only win or get into a draw. But what price should be charged? *Option pricing problem*

A basic derivative contract: Call Option

Randomness + dynamics:
stock price S_t follows SDE

$$dS_t = mS_t dt + \sigma S_t dW_t^P, \ \mathbb{P}$$
$$dS_t = \boxed{r} S_t dt + \sigma S_t dW_t^Q, \ \mathbb{Q}$$

(you're not seeing double!)



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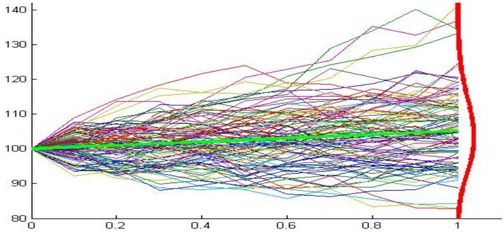

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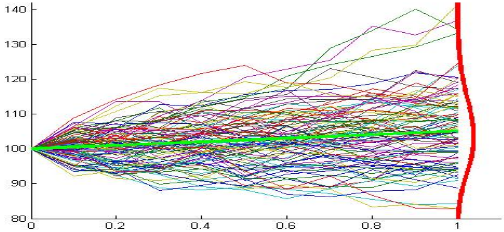

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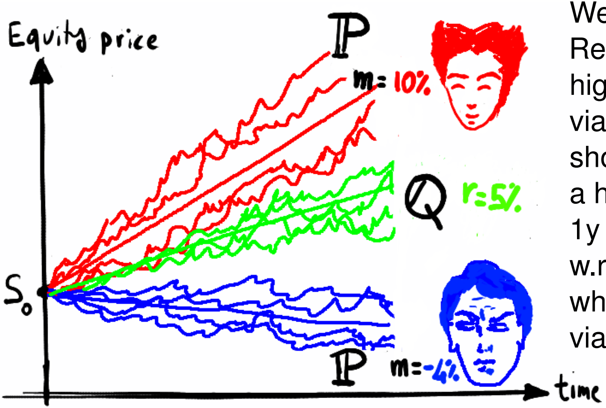
But expected value (mean) under which probability measure? $\mathbb{P}$ or $\mathbb{Q}$

BS formula **depends on the volatility** σ of the stock, and on the initial value S_0 today, but **does NOT depend on the real local mean/growth** m. This is because the expectation is UNDER THE PROBABILITY $\mathbb{Q}$, where the local mean/growth is the risk free rate r.



We live in world $\mathbb{P}$.
Red Investor, perceiving a high local mean/growth m via his statistics under $\mathbb{P}$, should be willing to pay a higher price for the 1y growth call option w.r.t. Blue Investor, who perceives a low m via his statistics under $\mathbb{P}$.
Right?

Wrong. Both have to pay the 1y growth gamble according to the green scenarios, with local mean/growth the risk free rate r, in the risk neutral world $\mathbb{Q}$. Volatility σ is a key input of the option price, but not m.

This ensures the market is arbitrage free, or a "fair game".

SDEs and Option Pricing: Black Scholes & Merton

Counterintuitive result?

Based on building a self financing trading strategy in S and cash up to 1y that perfectly mimics (replicates) the final payout $(S_{1y} - S_0)^+$. The initial price of this strategy does not depend on m.

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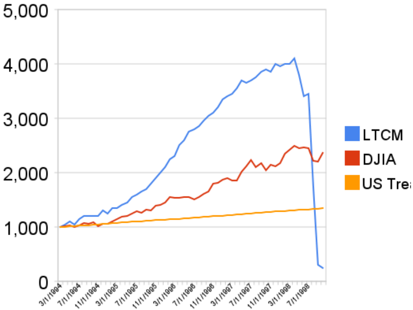
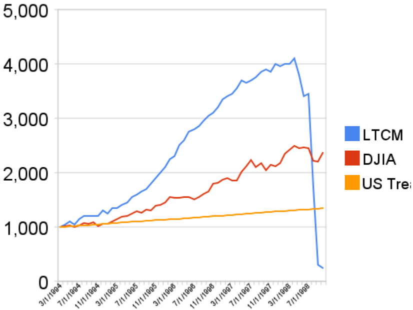
A mathematical result has contributed to create new markets

Avoiding m and using r makes derivatives valuation more objective, since m is very hard to estimate (if you could do that you would be a billionaire). This has contributed to derivatives growth worldwide, used today by banks and corporates for several purposes. However...



Sometimes the timing of the Nobel committee is funny, and we are not talking about the peace Nobel prize. Warning: anecdotal

1997: Nobel award.



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1997: Nobel award.

1998: the US Long-Term Capital Management hedge fund is bailed out after a huge loss. Merton and Scholes had been in the fund board. High use of leverage (derivatives). This leads us to...

Crisis

After Black Scholes 1973...

Market players introduced derivatives that may be much more complex functionals of underlying assets and events than the above call option

Gamble/speculate/hedge/protect on anything?

Derivatives on different sectors: Foreign Exchange Rates, Interest Rates, Default Events, Meteorology, Energy, population Longevity...

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Aggressive market participants extrapolating the basic theory

The initial Black Scholes theory (Nobel award 1997) has often been extrapolated beyond its limits to address new derivatives

I did work in all such areas, often criticizing simplistic market methodology and the excessive extrapolation of the basic theory, and introducing more appropriate and realistic models. A few examples:

Continuous time?

Markets operate at discrete time. In 1998 we investigated this: Given fixed discrete trading time grid (step can be a millisecond or even smaller) $\mathcal{T} = [0 = t_0, t_1, t_2, \ldots, t_n = T]$, I can define μ such that

$$dS_t = mS_t dt + \sigma S_t dW_t \quad and \quad dS_t = \mu(\mathcal{T}, t, S_t, v, \sigma)dt + vS_t dW_t$$

are **indistinguishable** by historical estimation (under $\mathbb{P}$) in $\mathcal{T}$.

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Indistinguishable underlying S, very different prices for $(S_T - K)^+$

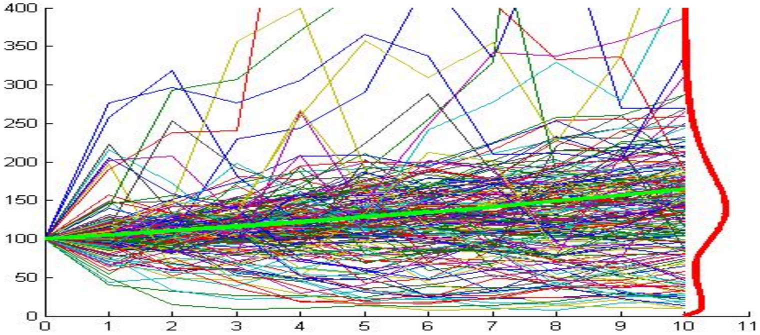
If we price options such as $(S_T - K)^+$, by choosing suitable μ and v, prices in the 2 models can be arbitrarily different (no-arbitrage bounds).

Removing continuous observations, the price becomes arbitrary

Should we be at ease employing continuous time models?

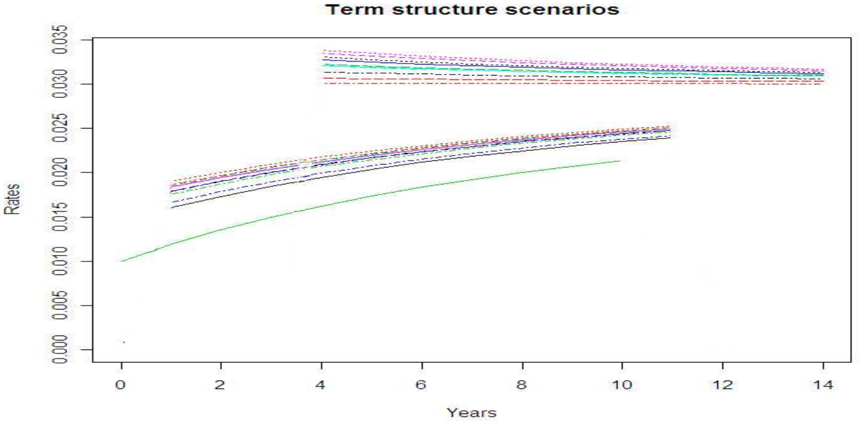
$dS = rSdt + \sigma SdW$? Richer dynamics ? Volatility smile

Patterns of prices for $(S_T - K)^+$ across T and $K \Rightarrow S$ could follow $dS_t = \mu(t, S_t)dt + \boxed{v(t, S_t)} S_t dW_t$. We proved $\exists!$ of S for a special v such that the law of S is a mixture of basic laws (option prices are linear combinations of basic prices). Market calibration across asset classes. Fat tails. Link with random volatility-markovian projection



Interest Rates: Term structure modeling

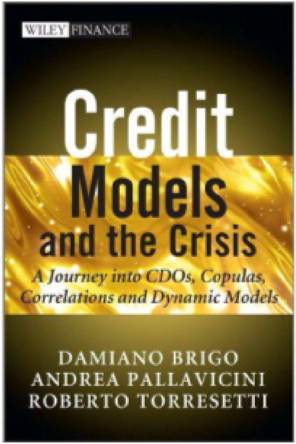
A number of financial derivatives, from simple bonds and swaps to exotics, reference interest rates. Term structure modeling hard: we model random dynamics of a whole (∞ dimensional) curve, and not just one variable. Now multiple curves even at $t = 0$ (OIS/LIBOR).

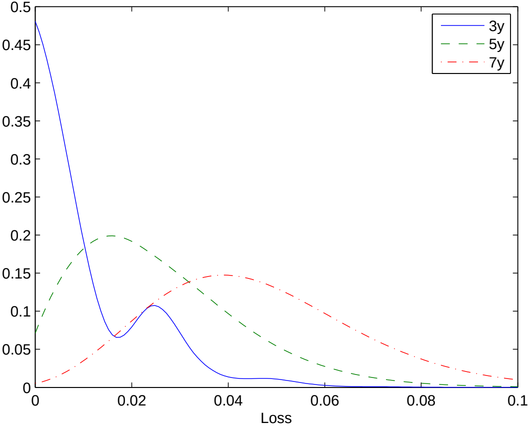


Credit Risk Modeling: Multiple Defaults

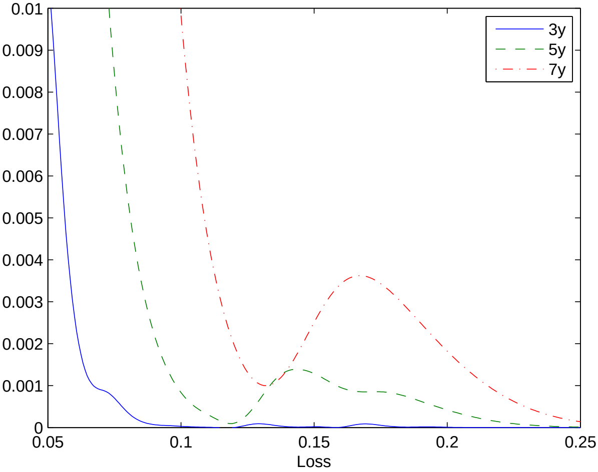
- Collateralized Debt Obligations (CDO) allow to trade on a portion (tranche) of the default loss of a pool of names (eg 125).
- Industry models simplistic: Gaussian copula/base correlation
  - static
  - deterministic spreads, no spread volatility
  - inconsistent even on single tranches
  - may imply resurrection of defaulted names
- Introduced no-arbitrage random dynamic loss model GPL
- Generalized Poisson. Sectors defaults. Clusters. Systemic risk.
- 1st model calibrated consistently across tranches and maturities.

Loss distribution of the calibrated GPL model, 2006.

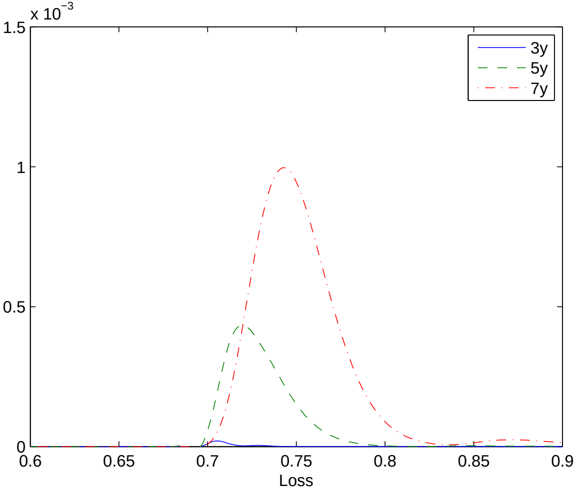




SEE ALSO THE MOVIE!!



Note also that CDOs had a lot of hostile press blaming quants and mathematicians. No journalist noticed this model published 2006. Notice the probability mode corresponding to a huge default in the pool. This is the real tail / systemic risk, not fat tailed distributions.



Valuation Adjustments: CVA, DVA, FVA... "ma VA?"[1]

No defaults?

Black Scholes assumes no default of the parties in the trade. *In one month of 2008 eight financials defaulted:* Fannie Mae, Freddie Mac, Lehman, Washington Mutual, 3 Icelandic banks, $\approx$ Merrill Lynch.

[1] "Ma va?" means "Really?!?" in Italian

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No funding costs: Borrowing and lending happen at the risk free rate r

In implementing our self financing trading strategy to replicate the option, cash can be borrowed or lent at the risk free rate r. Whether we hold a positive or negative amount of cash or risky S, interest is r.

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In implementing our self financing trading strategy to replicate the option, cash can be borrowed or lent at the risk free rate r. Whether we hold a positive or negative amount of cash or risky S, interest is r.

From 2002 I worked on removing the above unrealistic assumptions.

Credit risk not only (on CDS and) CDOs: most derivatives affected by a Credit Valuation Adjustment (CVA) and Funding VA. Margining costs and gap risk even under collateral or central clearing

[1] "Ma va?" means "Really?!?" in Italian

The end of Platonic valuation?

Industry: CVA, FVA... but risks not really separable. Size:

See banks press releases. Citigrup CVA 2009 third quarter: 2.5$ billions gain. Recently, JPMorgan 1.5$ billions FVA cost?

The end of Platonic valuation?

Industry: CVA, FVA... but risks not really separable. Size:

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Fundamental nonlinearities

In presence of collateral , credit risk and funding costs for the hedge both the payout and the pricing operator become *nonlinear*: pricing equations become recursive. Nonlinear PDEs and BSDEs.

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Industry: CVA, FVA... but risks not really separable. Size:

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Fundamental nonlinearities

In presence of collateral , credit risk and funding costs for the hedge both the payout and the pricing operator become *nonlinear*: pricing equations become recursive. Nonlinear PDEs and BSDEs.

Aggregation–dependence / asymmetry: The end of Platonic pricing?

The value of a portfolio of two assets is different from the sum of the values of the two assets. No "Platonic" $\mathbb{Q}$. *Pricing probability measure is product dependent: every trade has specific measure*. Price will also depend on trading entities. Operational implications enormous.

And finally... Signal Processing (Nonlinear filtering)

The filtering problem

Estimate a random signal from observations of this signal that are perturbed by further randomness (noise).

Submarines, spacecrafts, satellites orbits, Re-entry trajectories, target tracking, water level predictions, seismology, bioengineering, econometrics, *consistent mathematical finance under* $\mathbb{P}$ *and* $\mathbb{Q}$*...*

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A simple example: cubic sensor

The signal is a random walk: $dX_t = \sigma dW_t$

Observations are a cubic sensor plus random noise: $dY_t = X_t^3 dt + dV_t$

Problem: Estimate X_t from the history of Y up to the given time t.

Optimal filter for nolinear problems...

Nonlinearities make the stochastic analysis ∞ dimensional.

I developed a finite dimensional approximate solution for the infinite dimensional filtering problem using the differential geometric approach to statistics.

Probability distributions have a helpful (differential) geometry.

Optimal filter for nolinear problems...

Nonlinearities make the stochastic analysis ∞ dimensional.

I developed a finite dimensional approximate solution for the infinite dimensional filtering problem using the differential geometric approach to statistics.

Probability distributions have a helpful (differential) geometry.

This picture shows the ∞-dimensional optimal filter (heavy numerical methods for stochastic partial differential equations) vs a 4-dimensional projection filter for the cubic sensor, plus the local error.

Figure 4.15: Mean projection filter and mean optimal filter between 0 and 5.

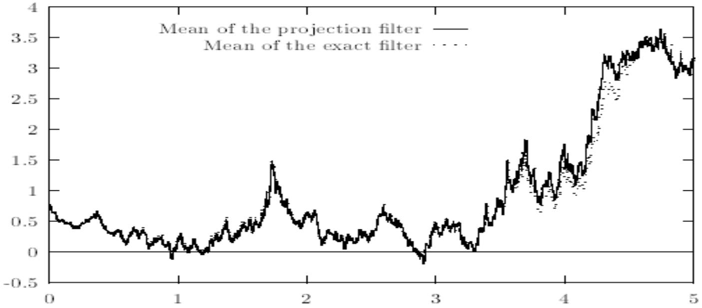
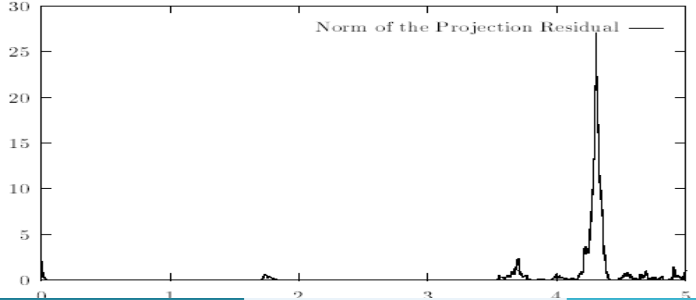


Figure 4.16: Projection residual between 0 and 5.



This is rocket science... finally!

Among filtering applications, the most well known might be...

NASA: Apollo 11. Moonlanding (1969)

"[...] The on-board computer that guided the descent of the Apollo 11 lunar module to the moon had a [21 dimensional] Kalman filter [and...] was also communicating with a system of four Doppler radar stations on Earth that were monitoring the module's position. [...] If [radar stations and onboard system] had disagreed too much [then] mission aborted."

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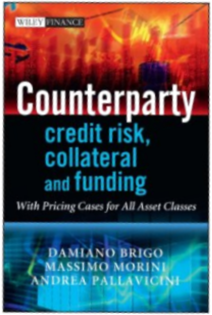
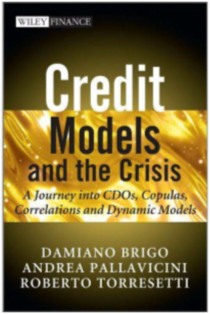
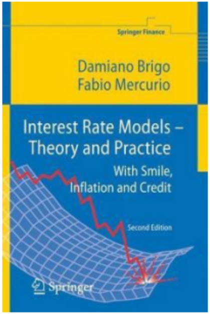
Having reached the stars, this looks like a good place to stop! However...

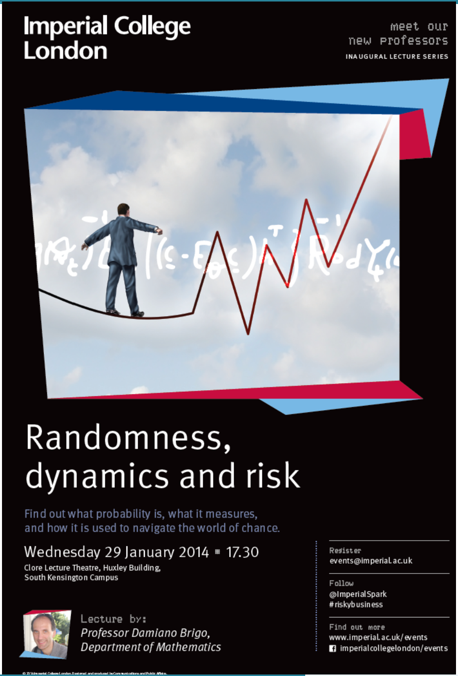
I was just looking for an excuse to show this picture!!

Lecture partly based on the following books (2001-2013) and about 70 papers (1995-2014)









Thank You for Your Attention!

References and further reading I

Gillies, D. (2000). Philosophical Theories of Probability. Routledge. *This book illustrates the logical, subjective, frequency, propensity1, propensity2, and intersubjective interpretations, among others. Recommended. It does not deal with Quantum probability.*

Quantum Mechanics is taught and presented in countless texts and books. A classic we recommend:

Bohm, D. (1989). Quantum Theory. Dover (reprint of the classic 1951 book). *This is a book that presents classic random-based QM in the context of the development of physics, motivating assumptions and ideas, and trying to develop as much intuition as possible. It is a very good book. Many books on QM start with a list of axioms on Hilbert spaces and self adjoint operators, without explaining why one makes some modeling assumptions, but this one looks also at the scientific cultural angle in introducing QM.*

References and further reading II

Interestingly Bohm, after writing this book on the classical probabilistic interpretation of QM in 1951 and receiving much praise for it from mainstream physicists, after a meeting with Einstein, decided to work on a deterministic interpretation of the theory, the Pilot Wave interpretation (1952) that had been started by De Broglie in the late twenties (1927 Solvay conference).
It is to be noted that Bohm contributed also to the advancement of Physics and QM in toto, with Bohm Diffusion in plasmas (1949), the Bohm version of the EPR experiment (that is the one that John Bell used to develop ideas on his famous inequality) and the Bohm-Aharonov effect (1959), and further Bohm's discovery of decoherence (1952). Bohm's revival and further results on pilot wave theory (1952 on) found much opposition that had more to do with unfortunate marketing of science (the Copenhagen interpretation group and especially Physicists of the caliber of

References and further reading III

Bohr, Heinsenberg, Pauli, Born, Oppenheimer, and many others...) than with its own deficiencies. Bohm was also unlucky in that he was targeted by McCarthysm. In 1949, the House Un-American Activities Committee summoned Bohm but he refused to give evidence against his colleagues on suspected communist activities. Princeton University suspended him despite Bohm having been acquitted. Einstein wanted Bohm to serve as his assistant in Princeton but the university refused. Bohm left for Brazil, University of Sao Paulo, and in the later years he moved a few times, ending up finally in London!
Feynman was an exception to the hostility Bohm had to endure, especially for his pilot wave theory. Feynman was a friend of Bohm and held him in high esteem, even though he seemingly did not like the Pilot Wave approach.

References and further reading IV

Bohm persevered notwithstanding the environment hostility and even threats. Many believed that it was impossible to have a deterministic or "hidden variables" theory for QM. "Hidden variables" hints at the idea that randomness is not fundamental in QM but can be explained by variables that we are not able to observe. It is a sort of ignorance interpretation, and interestingly if DeBroglie/Bohm are right we lose the only currently known potential source of authentic randomness in nature. We'd have nothing random left in current science. The reason why Bohm's theory was ignored is also because of some impossibility proofs for hidden variable theories that had been published many years earlier (1932) by, among others, John von Neumann.
But how could Bohm's deterministic but non-local theory in 1952 be right and consistent with experimental predictions of QM when von Neumann had shown in 1932 hidden variable theories to be

References and further reading V

impossible? von Neumann had kept almost everyone in awe due to his standing and reputation. Grete Hermann (1935) wrote about a problem with the proof of von Neumann, but went largely ignored. In 1966 John Bell published a refutation of von Neumann's claim, similar to Hermann's, and this time the rebuttal gained attention. Jeffrey Bub published a paper in 2010 claiming that von Neumann's proof was misunderstood, see also Belavkin reference below. However the pilot wave theory stands as proof that hidden variables are possible.
After the Pilot wave went ignored or opposed and even abused for years, finally John S. Bell noticed it and started promoting it, especially with his CERN report "On the impossible Pilot Wave", later collected in the (recommended) book "Speakable and Unspeakable in Quantum Mechanics". It is DeBroglie / Bohm's theory that prompted Bell to derive his famous inequality that some

References and further reading VI

scientists consider the most profound finding of all science (Henry Stapp, Berkeley's particle physicist: Bell's theorem is the most profound discovery of science). Bell died after being nominated but before receiving the Nobel prize for his famous inequality. In Bell's own words, from his 1982 report "On the impossible pilot wave": "... in 1952 I saw the impossible done. It was in papers by David Bohm. Bohm showed explicitly how parameters could indeed be introduced, into nonrelativistic wave mechanics, with the help of which the indeterministic description could be transformed into a deterministic one. More importantly, in my opinion, the subjectivity of the orthodox version, the necessary reference to the observer, could be eliminated... But why then had Born not told me of this pilot wave? If only to point out what was wrong with it? Why did von Neumann not consider it? More extraordinarily, why did people go on producing impossibility proofs, after 1952, and as recently as

References and further reading VII

1978? Why is the pilot wave picture ignored in text books? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show us that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate theoretical choice? [...] However that may be, long may Louis de Broglie continue to inspire those who suspect that what is proved by impossibility proofs is lack of imagination.".

According to mainstream interpretations, Bell showed with his inequality that nature would have to give up either realism or locality. Many interpreted Bell's inequality to say that the pilot wave theory, suggesting realism, was impossible, but this is not what Bell himself thought he had done. In fact the pilot wave theory is explicitly non-local. Bell thought his conclusions applied to mainstream interpretations of QM as well and that this nonlocality vs realism was not a problem just for Bohm's version.

References and further reading VIII

Summing up: In the lecture above I have worked under the Random interpretation of QM, but we need to keep in mind the pilot wave as an alternative, and in fact there are many other interpretations such as many worlds, GRW spontaneous collapse, advanced action/transactional...

The Pilot Wave version is being currently investigated by physicists such as Antony Valentini, who has been at Imperial College in the past. A good summary of the pilot wave theory is in the Encyclopedia in Stanford:

Goldstein, S. (2001). Bohmian Mechanics, Stanford Encyclopedia of Philosophy, 2nd Edition 2013
http://plato.stanford.edu/entries/qm-bohm/,

References and further reading IX

We conclude the part on Quantum Theory with the double slit experiment and Schroedinger's cat. The double slit experiment led Feynman to declare: "[the double slit experiment shows] a phenomenon which is impossible [...] to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery [...]." The version of the double slit experiment shown here is taken from Bell's book "Speakable and Unspeakable in QM". Schroedinger's Cat is a thought experiment meant to show that QM must be wrong: If one could transport the fundamental indeterminacy from the quantum to the classical level, we would have a cat that is at the same time alert and asleep (or, more brutally in Schroedinger's original version, alive and dead) until observed or "measured". The absurdity of this was meant to show that Quantum Mechanics as a theory is not complete or consistent, and there is a big measurement problem that is

References and further reading X

supposed to happen outside the key rules of the theory and is added ad hoc. In the Pilot Wave theory the impossibility to have quantum uncertain cats in practice is explained by decoherence. Recently, after much preparation larger and larger bodies have been shown to interfer in a double slit experiment, such as molecules, showing that if one is able to limit decoherence then the superposition shows up also for objects that are far larger than electrons. We are still far away from cats, though.

After this long digression on Quantum Theory, we go back to probability.

References and further reading XI

For probability theory associated to Quantum Mechanics and its relationship to standard probability a la Kolmogorov see

- Meyer, P. A. (1995). Quantum Probability for Probabilists. Lecture Notes in Mathematics, Vol. 1538, Springer. *This book (best read in French, since in the French version there is more commentary by Meyer) is important because written by one of the strongest classical probabilist ever, Paul Andre Meyer. Meyer is known for the Doob-Meyer decomposition and many other results. Another interesting volume is*

- Gudder, S. P. (2005). Stochastic Methods in Quantum Mechanics. Dover. Reprint of a classic of the seventies. *Gudder shows that Quantum probability (QP) is more general than classical probability (CP), that QP is non distributive, non subadditive, does not satisfy the Borel Cantelli lemma, etc. See also*

References and further reading XII

- Khrennikov, A. Interpretations of Probability, Walter de Gruyter, 1999, 2nd Ed 2009 *who argues that we need to assign different probability spaces to different quantum experiments, and*
- Belavkin, V. P. (2000). Quantum Probabilities and Paradoxes of the Quantum Century, in Inf. Dim. Anal. Quantum Probability and Related Topics, 3 (4) 577-610 *who argues that information theory and conditioning are key mathematical aspects that have not been properly used to correctly interpret quantum theory. He writes:*

References and further reading XIII

"The creators of QM, [...], were unable to find a consistent interpretation of it [...]. After inventing QM they spent much of their lives trying to tackle the Problem of Quantum Measurement, the greatest problem of quantum theory, not just of QM, or even of unified quantum field theory, which would be the same "thing in itself" as QM of closed systems without such interpretation. [...] The solution [...] can be found in the framework of Quantum Probability as a part of a unified mathematics rather than physics. Most [...] have a broad mathematical education, but it ignores just two crucial aspects – information theory and statistical conditioning. So they gave up this problem as an unsolvable – and it is indeed unsolvable in the traditional framework of mathematical physics." *The paper by*

References and further reading XIV

- Rau, J. (2009). On Quantum vs Classical Probability, arXiv.org *argues that in reality classical probability is not a subset of QP but rather the two probabilities intersect in a large set of principles, with some differences then where QP favours smoothness and CP favours decidability. He then goes on to make a (too?) bold conjecture, namely that beyond CP and its variant QP there is no other possible probability theory relevant to the physical world. A very recent paper by*
- Holik, F., Plastino, A., and Sáenz, M. (2013). A discussion on the origin of quantum probabilities. arXiv.org *uses Cox's reformulation of Kolmogorov's theory in propositional lattices. Recently, Cox approach has been adapted to non-boolean propositional lattices and has been used for Quantum probabilities. This provides a common framework (rather than Kolmogorov/Measure spaces vs*

References and further reading XV

von Neumann/Hilbert spaces) to compare the theories of QP and CP. In particular, the authors argue that the axioms of QP are completely driven by the non-boolean lattice structure of the quantum case so that in a way the event structure forces the definitions of QP, similarly to how the Kolmogorov rules follow from a boolean lattice structure. This provides a common ground for probability theories comparisons.

What do classical probabilists think of Quantum Probability? Are most classical probabilists ignoring it?

References and further reading XVI

Classical (above) and Quantum (below) Probabilities:

Sample Space	Events	Random Variables	Probability
Kolmogorov: Measure space Ω in $(\Omega, \mathcal{F}, P)$	σ-algebra $\mathcal{F}$ of measurable sets	measurable function $\Omega \to R$	Normalized measure P
Von Neumann: Hilbert space $\mathcal{H}$	Family of closed subspaces of $\mathcal{H}$	Self-adjoint operator on $\mathcal{H}$	Born rule

Paul–André Meyer (1934-2003), 1986:

"Although QM is essentially a probabilistic theory, the probability [theory] used by physicists has remained very rudimentary for quite some time, as opposed to their functional analysis, and "classical" probabilists could afford to ignore it. Today, this is no longer the case".

References and further reading XVII

This quick overview does no justice to the available authors and literature on QP. We refer the reader to references in the above articles for a comprehensive view.

Now we provide some references for readers interested in **stochastic differential equations.** We start with the book by

Friedman, A. (2003). Stochastic Differential Equations and Applications, Dover, reprint of the classic two volumes dated 1975-76. *Friedman is well known for his books on PDEs, but this is an excellent book on SDEs, despite some non-standard treatment of a few SDEs topics. A more standard good book for SDEs with brilliant exposition for the technically minded is*

References and further reading XVIII

Rogers, L.G.C, and Williams, D. (2000). Diffusions, Markov Processes and Martingales, Vol. 1 and 2, Cambridge University Press *(Solutions to SDEs are often called diffusion processes)*.

Chaos theory has been popularized much more than the theory of SDEs, so it is possible to read accounts of Chaos theory that are not overly technical. A standard popular reference is the book

Gleick, J. (1987). Chaos: Making a New Science, Viking Penguin.

Another good reference on chaos is again the Stanford Encyclopedia,

References and further reading XIX

Bishop, R.C. (2008). Chaos.
http://plato.stanford.edu/entries/chaos

An important point is that Chaos might amplify the fundamental randomness in QM to macroscopic levels via the nonlinear dynamics that is sensitive to initial conditions. This of course depends on whether QM is truly random (or pilot wave?) among other issues.

We now move to Nonlinear Stochastic Filtering.

References and further reading XX

The use of filtering in the first human moon landing (1969) has been reported in

Cipra, B. (1993). Engineers Look to Kalman Filtering for Guidance. SIAM News, Vol. 26, No. 5, August 1993

An extract is as follows:

'Even so, the new approach didn't catch on right away. Bucy recalls that "no one was very interested in the papers originally." But then researchers at NASA latched onto the Kalman filter as a way of dealing with problems in satellite orbit determination. Kalman filtering rapidly became a mainstay of aerospace engineering. It was used, for example, in the Ranger, Mariner, and Apollo missions of the 1960s. In particular, the on-board computer that guided the descent of the Apollo 11 lunar module to the moon

References and further reading XXI

had a Kalman filter. That computer was also communicating with a system of four Doppler radar stations on Earth that were monitoring the module's position. It was important that estimates from all sources be good: The Earth-based estimates were used to adjust the on-board system; if they had disagreed too much with the on-board estimates, the mission would have had to be aborted. It could have happened. According to William Lear, an aerospace engineer who was then at TRW in Redondo Beach, California, NASA contacted him about nine months before Apollo 11's scheduled launch because their Earth-based tracking program wasn't working. Lear, who now works for Draper Labs at the Johnson Space Center, wrote a 21-state Kalman filter program, which went into the Doppler radar system. The final check of the program, Lear recalls, was done the day before Armstrong, Aldrin, and Collins took off.'

References and further reading XXII

The part on the **Nonlinear Filtering Problem** is based on my PhD work (1993-96) with Bernard Hanzon (currently at Cork University, at the time at the Free University of Amsterdam) and Francois Le Gland (INRIA/IRISA), and on recent research with John Armstrong (King's College London).

As I am mentioning my PhD, **Special Thanks** here to Peter Spreij (currently at UVA) and Jan van Schuppen (CWI, Delft University) who helped me at the most difficult times during the PhD period and both taught me a lot, and not only in mathematics.

The main references on the projection filter: the idea started in 1987 by Bernard Hanzon:

References and further reading XXIII

Hanzon, B. A differential-geometric approach to approximate nonlinear filtering. In C.T.J. Dodson, Geometrization of Statistical Theory, pages 219 – 223, ULMD Publications, University of Lancaster, 1987
and

Hanzon, B., and Hut, R. (1991). New results on the projection filter. Working paper.

It was expanded and fully developed in 1993-1996 in the following works, with particular attention to exponential families:

D. Brigo, *Filtering by Projection on the Manifold of Exponential Densities*, PhD Thesis, Free University of Amsterdam, 1996.

References and further reading XXIV

- Brigo, D, Hanzon, B, LeGland, F, A differential geometric approach to nonlinear filtering: The projection filter, IEEE T AUTOMAT CONTR, 1998, Vol: 43, Pages: 247 – 252
- Brigo, D, Hanzon, B, Le Gland, F, Approximate nonlinear filtering by projection on exponential manifolds of densities, BERNOULLI, 1999, Vol: 5, Pages: 495 – 534
- Brigo, D. Diffusion Processes, Manifolds of Exponential Densities, and Nonlinear Filtering, In: Ole E. Barndorff-Nielsen and Eva B. Vedel Jensen, editor, Geometry in Present Day Science, World Scientific, 1999

More recently, variants based on mixture families are being investigated in

References and further reading XXV

Brigo, D. (2012). The direct L2 geometric structure on a manifold of probability densities with applications to Filtering. Available at arXiv.org

J. Armstrong, D. Brigo (2013). Stochastic filtering via L2 projection on mixture manifolds with computer algorithms and numerical examples. Available at arXiv.org

It is interesting to notice that stochastic filtering ideas have been used also in quantum theory. This is not so unnatural if one realizes that filtering is about conditional probabilities from observations and information. The late V. P. Belavkin (mentioned earlier) also did work in this sense and thought that filtering theory could contribute to QM.

References and further reading XXVI

The projection filter has a Quantum version introduced in Quantum ElectroDynamics:

van Hander, R., and Mabuchi, H. (2005). Quantum projection filter for a highly nonlinear model in cavity QED. J. Opt. B: Quantum Semiclass. Opt. 7 S226

We now move to derivatives and finance. We mentioned Bachelier and De Finetti among the forerunners.

References and further reading XXVII

- **Louis Bachelier** (1870 – 1946) (First to introduce Brownian motion W_t in Finance, First in the modern study of Options);
- **Bruno de Finetti** (1906 – 1985) (See also *Frank Ramsey*; Subjective interpret of probability; defines risk neutral measure very similarly to current theories: first to derive no arbitrage through inequalities constraints, discrete setting).
- Modern theory (1973 on) follows Nobel awarded **(Black,) Scholes and Merton** (+ Harrison and Kreps...)

References and further reading XXVIII

The Bachelier dissertation that started Brownian motion and first used Brownian motion in finance is

Bachelier, (1900). Théorie de la Spéculation. PhD dissertation, Sorbonne. This has been translated into English by Mark Davis, Imperial College London. Mark was first at Electrical Engineering and then became one of the founders of the Mathematical Finance group. The translation is co–authored with Alison Etheridge and has a preface by Nobel Awarded Paul Samuelson.

We mentioned Bruno De Finetti as one of the leading probabilist and statisticians of the past century. He worked in the industry and also in academia. He is well known for a number of results and for the subjective interpretation of probability. The way he characterized the No Dutch Book argument for consistency of probabilities is very similar to no arbitrage theory and the risk

References and further reading XXIX

neutral measure $\mathbb{Q}$. This had been done independently here in the UK by Frank Ramsey, who however passed away at the age of 26 before developing the theory further. De Finetti main work in this respect is

DeFinetti, B. (1931). Sul significato soggettivo della probabilitá. In Fundamenta Mathematicae, Warszawa, T. XVII, pp. 298–329.

DeFinetti's work relevance for current no arbitrage theory has been pointed out for example in

Nau, R. F. (2001). De Finetti was right: Probability does not exist. Theory and Decisions, 51: 89–124.

Nau writes: "In the 1970s the so-called golden age" of asset pricing theory – there was an explosion of interest among finance theorists in models of asset pricing by arbitrage. The key discovery of this period was the fundamental theorem of asset pricing [...]

References and further reading XXX

There are no arbitrage opportunities in a financial market if and only if there exists a probability distribution with respect to which the expected value of every assets future payoffs, discounted at the risk free rate, lies between its current bid and ask prices; and if the market is complete, the distribution is unique. This is just de Finetti's fundamental theorem of subjective probability, with discounting thrown in, although de Finetti is not usually given credit in the finance literature for having discovered the same result 40 years earlier.

Further discussion is available in

References and further reading XXXI

Pressacco, F., and Ziani, L. (2009). Bruno de Finetti forerunner of modern finance. In: Convegno di studi su Economia e Incertezza, Trieste, 23 ottobre 2009, Trieste, EUT Edizioni Universitá di Trieste, 2010, pp. 65–84.

The option pricing theory of Black, Scholes and Merton and subsequent formalization of no arbitrage by Harrison, Kreps, Pliska et al is accounted for in many academic texts. See for example

Bjork, T. (2004). Arbitrage theory in continuous time. 2nd Edition, Oxford University Press.

The paper that discusses the dangers in using continuous time as an approximation for discrete time is

References and further reading XXXII

D. Brigo and F. Mercurio (2000). Option pricing impact of alternative continuous time dynamics for discretely observed stock prices, Finance and Stochastics, Vol. 4, N. 2 (2000), pp. 147–160. Extended version (1998) available online SSRN.com and arXiv.org

The mixture dynamics volatility smile model alternative to the geometric brownian motion of Black Scholes has been developed in a number of papers:

D. Brigo, F. Mercurio, Displaced and Mixture Diffusions for Analytically-Tractable Smile Models, in: Geman, H., Madan, D.B., Pliska, S.R. (Editors), Mathematical Finance - Bachelier Congress 2000, Springer, Berlin (2001).

Brigo, D., Mercurio, F., Rapisarda, F., Smile at Uncertainty, Risk Magazine (2004), May issue.

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- Brigo, D., Mercurio, F., and Sartorelli, G., Alternative Asset Price Dynamics and Volatility Smile, Quantitative Finance, Vol 3, N. 3. (2003) pp. 173-183
- D. Brigo and F. Mercurio, Analytical pricing of the smile in a forward LIBOR market model, Quantitative Finance, Vol. 3, No. 1 (2003).
- D. Brigo and F. Mercurio, Lognormal-mixture dynamics and calibration to market volatility smiles, International Journal of Theoretical and Applied Finance, Vol. 5, No. 4 (2002), 427-446.

This is just one of the countless models developed by academic research and by the industry to generalize the Geometric Brownian Motion in Black Scholes, *although it is one of the most tractable and flexible in terms of parameterization. It is also fully*

References and further reading XXXIV

rigorous, as one can show existence and uniqueness of the strong solution for the related SDE.

Term structure modeling and interest rate dynamics is discussed in several papers and in the book

D. Brigo and F. Mercurio, Interest-Rate Models: Theory and Practice with Smile, Inflation and Credit, Springer Verlag, 2001, second edition 2006.

Despite the book becoming a field reference and the main reference worldwide for term structure modeling in derivatives markets, the theory is currently inadequate as multiple curves and credit-liquidity effects started affecting interest rates in a way that is not consistently accounted for by most available theories. A paper that tries to develop a consistent new theory with all these effects is

References and further reading XXXV

A. Pallavicini and D. Brigo (2013). Interest-Rate Modelling in Collateralized Markets: Multiple curves, credit-liquidity effects, CCPs. SSRN.com and arXiv.org

The Credit (CDO) crisis, the models limitations and proposed solutions (dated 2006) are best summarized in the small book

D. Brigo, A. Pallavicini and R. Torresetti, Credit Models and the Crisis: A journey into CDOs, Copulas, Correlations and Dynamic Models. Wiley, 2010
referencing several papers in the period 2006-2010. This is also the story of how media and a few mainstream journalists misunderstood the role of financial modeling and of modeling more generally. An online report "Credit models and the crisis or: How I learned to stop worrying and love the CDOs" is also available on arXiv and SSRN. See also

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Brigo, D. and Morini, M. (2011). No-Armageddon Measure for Arbitrage-Free Pricing of Index Options in a Credit Crisis, Mathematical Finance, 21 (4), pp 573–593

for the related issues on Credit Index Options.

The new or formerly neglected risks including credit risk, funding liquidity risk, collateral gap risk, CCPs, initial margining, re-hypothecation are discussed in the book

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D. Brigo, M. Morini and A. Pallavicini (2013). Counterparty Credit Risk, Collateral and Funding – With Pricing Cases for all Asset Classes. Wiley

and in several recent working papers and publications. See arXiv or SSRN for a complete list. I started researching Credit Valuation Adjustments back in 2002, with the first papers published in 2004. See also

D. Brigo (2012). Counterparty Risk FAQ: Credit VaR, CVA, DVA, Closeout, Netting, Collateral, Re-hypothecation, Wrong Way Risk, Basel, Funding, and Margin Lending. Available at SSRN.com and arXiv.org

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