Implied Expected Tranched Loss Surface from CDO Data

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Abstract

We explain how the payoffs of credit indices and tranches are valued in terms of expected tranched losses (ETL). ETL are natural quantities to imply from market data. No-arbitrage constraints on ETL's as attachment points and maturities change are introduced briefly. As an alternative to the temporally inconsistent notion of implied correlation we consider the ETL surface, built directly from market quotes given minimal interpolation assumptions. We check that the kind of interpolation does not interfere excessively. Instruments bid/asks enter our analysis, contrary to Walker's (2006) earlier work on the ETL implied surface. By doing so we find less and very few violations of the no-arbitrage conditions. The ETL implied surface can be used to value tranches with nonstandard attachments and maturities as an alternative to implied correlation.

JEL classification code: G13.

Keywords: expected tranche loss, loss surface, implied correlation, CDO, Tranches, interpolation.

1 Introduction

We consider market payoffs for credit indices and their tranches. We observe that the net present values of these payoffs can be characterized through a set of key quantities given by expected tranched losses (ETL). We then explain exactly how the payoffs are valued in terms of ETL and consider natural no-arbitrage constraints on ETL's as tranche attachment points and maturities vary.

The notion of ETL surface across maturity and tranche attachments is as close as a model independent notion of implied dependence as possible, since it focuses on one of the most direct market objects embedded in market quotes.

Rather than going through implied correlation, based on the arbitrary assumption of a Gaussian copula connecting defaults across names and leading to inconsistencies in the temporal axis, one considers directly quantities entering the valuation formula and implies them from market quotes given minimal interpolation assumptions. To make sure that interpolation does not interfere excessively we carry out the calibration through two different interpolation techniques (linear and splines).

Our results remind of Walker (2006)'s earlier work and of the formal analysis of the properties of expected tranched loss in connection with no arbitrage in Livesey and Schlögl (2006).

As we explain more in detail in the conclusions, while in our framework the bid/asks of the instruments enter the target function we aim at minimizing in order to imply the surface, in Walker's (2006) framework the instruments NPV's must be exactly zero. By including bid/asks as we do the no-arbitrage constraints are satisfied across the vast majority of dates, in particular we find less violations of the no-arbitrage condition than in Walker's (2006).

The method appears to be helpful as a first model-independent procedure to deduce implied expected loss surfaces from market data, allowing one to check basic no-arbitrage constraints in the market quotes. It is of immediate use to value tranches with nonstandard attachments and maturities, although excessive *extra*polation is to be avoided.

2 Market quotes

The most liquid multi-name credit instruments available in the market are credit indices and CDO tranches (e.g. iTraxx, CDX).

2.1 Credit indices

The index is given by a pool of CDS on the names $1, 2, \ldots, M$, typically $M = 125$. Each name has the same initial notional $1/M$, so that the pool total notional is one. The index default leg consists of protection payments corresponding to the defaulted names of the pool. Each time one or more names default the corresponding loss increment is paid to the protection buyer, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

In exchange for loss increase payments, a periodic premium (spread) with rate S is paid from the protection buyer to the protection seller, until final maturity T_b. This premium is computed on a notional that decreases each time a name in the pool defaults, and decreases of an amount corresponding to the notional of that name (without taking out the recovery).

We denote with L_t the portfolio cumulated loss; In case the time is one of the grid dates T_i below we write $L_i = L_{T_i}$. Assuming the recovery rate for each name to be R, we have that at each default in the pool the notional decreases by $(1 - R) \cdot 1/M$, since part of the defaulted notional is recovered. The number of defaulted names scaled by M ("default rate") is expressed as

$$\frac{L_t}{1-R}$$

since

$$L_t = (1-R)\frac{\text{number of defaulted names by } t}{M}.$$

The discounted payoff of the two legs of the index is given as follows:

$$\text{Def}_{\text{Ind}} = \int_0^T D(0,t) dL_t$$

$$\text{Prem}_{\text{Ind}} = \sum_{i=1}^{b} D(0,T_i)\text{Spr}_{\text{Ind}} \int_{T_{i-i}}^{T_i} \left(1 - \frac{L_t}{1-R}\right) dt$$

where in general $D(s,u)$ is the deterministic discount factor (and thus zero coupon bond) at time s for maturity u, and Spr is the index spread. The integral on the right hand side of the premium leg is the outstanding notional on which the premium is computed for the index.

The value of the index spread Spr_{Ind} that balances the npv of the two legs can be written as.

$$\text{Spr}_{\text{Ind}} = \frac{E\left[\sum_{i=1}^{b} D(0,T_i)\left(L_i - L_{i-1}\right)\right]}{E\left[\sum_{i=1}^{b} (T_i - T_{i-1}) D(0,T_i)\left(1 - \frac{L_i + L_{i-1}}{2(1-R)}\right)\right]}, \tag{1}$$

E denoting the risk neutral expectation. Notice the approximation we have introduced in the computation of the integrals involved in determining the average Outstanding Notional and Default Leg NPV in each period. For a thorough exposition of CDS pricing and the accuracy of different approximations to the relevant integrals see O'Kane and Turnbull (2003).

2.2 CDO tranches

Synthetic CDOs with maturity T_b are contracts involving a protection buyer, a protection seller and an underlying pool of names. They are obtained by putting together a collection of Credit Default Swaps (CDS) with the same maturity on different names, $1, 2, ..., M$, tipically $M = 125$, and then "tranching" the loss L_t of the resulting pool at two points A and B with $0 \leq A \leq B \leq 1$. The tranched loss reads

$$L_t^{A,B} := \frac{1}{B-A}\left[(L_t - A)1_{\{A<L_t\leq B\}} + (B-A)1_{\{L_t>B\}}\right]$$

Once enough names have defaulted and the loss has reached A, the count starts. Each time the loss increases the corresponding loss change re-scaled by the tranche thickness $B - A$ is paid to the protection buyer, until maturity arrives or until the total pool loss exceeds B, in which case the payments stop (this is the default or protection leg).

In exchange for loss payments, a periodic premium with rate $\mathrm{Spr}_{\mathrm{A,B}}$ is paid from the protection buyer to the protection seller (premium leg). This premium is computed on a notional that decreases of the same amounts as the tranched loss increases. The notional surviving at each time is called "outstanding notional". In some of the most risky tranches the premium can be partially payed at the contract starting date, as an upfront amount $U^{A,B}$.

More formally, discounted payoff of the default leg can be written as

$$\int_0^T D(0,t) dL_t^{A,B}$$

In the premium leg, the premium rate $\mathrm{Spr}_{\mathrm{A,B}}$, fixed at time $T_0 = 0$, is paid periodically, say at times $T_1, T_2, \ldots, T_b = T$. Part of the premium can be paid at time $T_0 = 0$ as an upfront $U_0^{A,B}$. The premium leg discounted payoff can be written as

$$\mathrm{Prem}_{\mathrm{A,B}} = U_0^{A,B} + \mathrm{Spr}_{\mathrm{A,B}} \sum_{i=1}^{b} (T_i - T_{i-1}) D(0, T_i) \int_{T_{i-i}}^{T_i} \left(1 - L_i^{A,B}\right) dt.$$

We solve for the $\mathrm{Spr}_{\mathrm{A,B}}$ that sets to 0 the tranche NPV and obtain

$$\mathrm{Spr}_{\mathrm{A,B}} = \frac{E\left[\sum_{i=1}^{b} D(0,T_i)\left(L_i^{A,B} - L_{i-1}^{A,B}\right)\right]}{E\left[\sum_{i=1}^{b} (T_i - T_{i-1}) D(0,T_i)\left(1 - \frac{L_i^{A,B} + L_{i-1}^{A,B}}{2}\right)\right]} \tag{2}$$

The above expression can be easily recast in terms of the upfront premium $U_0^{A,B}$ for tranches that are quoted in terms of upfront fees. Notice again the approximation we have introduced in the computation of the integrals involved in determined the average Outstanding Notional and Default Leg NPV in each period.

The tranches that are quoted on the market refer to standardized pools. Let us consider for example the DJ i-TRAXX index, referring to the most liquid $M = 125$ names in the European CDS market. Standard attachment points are used. For the DJ-iTraxx Europe, the traded tranches are: an equity tranche, responsible for all losses between 0% and 3%, then other mezzanine and senior tranches covering 3%-6%, 6%-9%, 9%-12% and 12%-22%. For the main US index, the DJ CDX NA the tranche sizes are different: 0%-3%, 3%-7%, 7%-10%, 10%-15% and 15%-30%.

The market quotes either the periodic premiums rate $\text{Spr}_{\text{A,B}}$ of these tranches or their upfront premium rate $U_0^{A,B}$ for maturities $T = 3y, 5y, 7y, 10y$. The equity tranche is quoted by the upfront amount needed to make it fair when a running spread of 500bps is taken as periodic spread in the premium leg.

3 Index and Tranche NPV as a function of Expected Tranche Loss (ETL)

The tranches and the index pay their spread on dates $T_1, T_2, ..., T_b$, expressed as year fractions (we call the start date $T_0 = 0$). We assume a constant recovery of 40%.

The NPV of the premium and default legs of the index can be rewritten as:

$$\begin{aligned}
\text{NPVPrem}_{\text{Ind}} &= \text{Spr}_{\text{Ind}}\text{Annuity}_{\text{Ind}} \\
\text{Annuity}_{\text{Ind}} &= \sum_{i=1}^{N}(T_i - T_{i-1})D(0,T_i)\left(1 - \frac{E[L_i] + E[L_{i-1}]}{2(1-R)}\right) \qquad (3)\\
\text{NPVDef}_{\text{Ind}} &= \sum_{i=1}^{N} D(0,T_i)\left[E[L_i] - E[L_{i-1}]\right]
\end{aligned}$$

where the notation for the index annuity, the index premium leg, the index spread and the index default leg is self evident as before.

The NPV of the premium and default leg of the tranche with attachment A and detachment B is:

$$\text{NPVPrem}_{\text{A,B}} = \begin{cases} \text{U}_{\text{A,B}} + 0.05 \cdot \text{Annuity}_{\text{A,B}}, & A = 0 \\ \text{Spr}_{\text{A,B}}\text{Annuity}_{\text{A,B}}, & A > 0 \end{cases} \qquad (4)$$

$$\text{Annuity}_{\text{A,B}} = \sum_{i=1}^{N}(T_i - T_{i-1})D(0,T_i)\left(1 - \frac{E[L_i^{A,B}] + E[L_{i-1}^{A,B}]}{2}\right) \qquad (5)$$

$$\text{NPVDef}_{\text{A,B}} = \sum_{i=1}^{N} D(0,T_i)\left[E[L_i^{A,B}] - E[L_{i-1}^{A,B}]\right] \quad (6)$$

$$E\left[L_i^{A,B}\right] = \frac{E\left[\max(L_i - A, 0)\right] - E\left[\max(L_i - B, 0)\right]}{B-A} \quad (7)$$

$$= \frac{B}{B-A}E\left[\min(L_i, B)\right] - \frac{A}{B-A}E\left[\min(L_i, A)\right]$$

If we have for a given date the ETL's throughout the entire capital structure (all (A,B)'s) then given the expected recovery we can back-out the expected portfolio loss.

To span the entire capital structure we need a set of k tranches with attachments A_j and detachments B_j with $j = 1, ..., k$ where $A_1 = 0$, $B_k = 1$ and $A_{i+1} = B_i$. The expected portfolio loss is then the summation of the ETL multiplied by the tranche depth (detachment minus attachment):

$$\sum_{i=1}^{k} L_t^{A_i,B_i}(B_i - A_i) = L_t \quad \Rightarrow \sum_{i=1}^{k} E[L_t^{A_i,B_i}](B_i - A_i) = E[L_t] \quad (8)$$

Given the expected portfolio loss $E[L_t]$ and the recovery rate R we can compute the expected portfolio default rate as $E[L_t]/(1-R)$. Thus to compute the npv of the premium leg of the index we need the ETL throughout all the capital structure, in other words we need a set of adjacent (A,B) tranches spanning 0% - 100%.

The standardized iTraxx tranches have detachments (3%, 6%, 9%, 12%, 22%). To price each of these 5 tranches we need the ETL on all payment dates. To price the index we would also need the ETL of the 22%-100% tranche. To price the 3y tranches and index (6 market quotes = 5 tranches + 1 index) we will be looking for the 6 unknown 3y ETL's that will set the npv of the instruments as close as possible to 0. To compute the npv of the tranches and index we also need the ETL on all payment dates with maturity shorter than 3y: these will be obtained interpolating for each tranche between time 0 (by definition this will be 0) and the 3y unknown ETL to be found. Once the 3y nodal ETL's matching the data are found, to price the 5 years tranches and index we will need also the ETL between 3 and 5 years. For each tranche this will be obtained by interpolation between the expected tranche loss at 3y and the 6 unknown ETL's at 5y to be found. We then iterate the procedure.

We call $f(t,h,k)$ the ETL at time t of the tranche with attachment h and detachment k (to simplify the notation we will often identify the seniority of the tranche in the capital structure of the CDO only through the detachment point k, writing $f(t,h,k) = f(t,k)$ when h is clear from the context).

	0% − 3%	3% − 6%	6% − 9%	9% − 12%	12% − 22%	22% − 100%
$t = 0$	0	0	0	0	0	0
...						
$t = 3$	$f(3y, 3\%)$	$f(3, 6)$	...			$f(3, 100)$
...						
$t = 5$	$f(5y, 3\%)$	$f(5, 6)$	...			$f(5, 100)$
...						
$t = 7$	$f(7y, 3\%)$	$f(7, 6)$	...			$f(7, 100)$
...						
$t = 10$	$f(10y, 3\%)$	$f(10, 6)$	...			$f(10, 100)$

The set of $4 \cdot 10 \cdot 6 = 240$ f's (one for each quarterly payment date and for each tranche) created by interpolating the $4 \cdot 6 = 24$ basic nodal $f(t, k)$'s (one for each market maturity date and for each tranche) will be used to set the npv of the instruments as close to 0 as possible whilst maintaining the constraints (10) below. For the generic tranche the npv of the premium and default leg expressed in terms of $f(t, k)$ is obtained by Equations (4-6) by substituting

$$E[L_i^{A,B}] = f(T_i, A, B).$$

The npv of the premium and default leg of the index expressed in terms of $f(t, k)$ is instead given by (3) where we substitute

$$E[L_i] = \sum_{j=1}^{k} f(T_i, k_j)(k_j - k_{j-1}) \tag{9}$$

(recall that $f(T_i, k_j) = f(T_i, k_{j-1}, k_j)$). We will require that the ETL $f(t, k)$ is non-decreasing in t and non-increasing in k, both requirements being natural given that the loss of a pool should be non-decreasing in time and that the tranched losses re-scaled by the tranche thickness across adjacent attachment-detachment intervals decreases as we move to larger intervals. These requirements ensure also that (9) is increasing in T. Being $f(t, k)$ the expectation of a quantity bounded between 0 and 1, we will also require that $f(t, k)$ lies in the $[0, 1]$ interval. The set of constraints will thus be:

$$\begin{cases} 0 \leq f(t, h, k) \leq 1 \\ f(T_i, h, k) \geq f(T_{i-1}, h, k) \\ f(t, k_{j-1}, k_j) \leq f(t, k_{j-2}, k_{j-1}) \end{cases} \tag{10}$$

We will further check a posteriori the condition that the ETL profile for equity tranche losses (attachment-detachment $A - B$ with $A = 0$) implicit in our f table be convex with respect to the detachment B. Indeed, through a Breeden-Litzenberger (1978) like result, the second derivative of the expected equity tranche loss with respect to the detachment B is related to the opposite of the risk neutral loss density, and as such must be negative.

To find equity ETL remember the link between tranches and equity tranches: Invert

$$f(t, k_{j-1}, k_j) = \frac{k_j}{k_j - k_{j-1}} f(t, 0, k_j) - \frac{k_{j-1}}{k_j - k_{j-1}} f(t, 0, k_{j-1})$$

to find the equity ETL and check concavity on these.

Given a set of 24 nodal $f(t, k)$ satisfying the constraints in (10) and given an interpolation method (linear or spline for example) we will get the 240 $f(t, k)$ we need to compute the npv of the tranche and indices for all maturities. At this point we can calculate the theoretical spread that would set exactly to 0 the npv of the instrument if inserted in the premium leg:

$$\mathrm{Spr}_{\mathrm{Theoretical}} = \frac{\mathrm{NPVDefaultLeg}}{\mathrm{Annuity}} \tag{11}$$

This theoretical spread is a function of the nodal points f we take in the expected tranched loss surface. If the difference between this theoretical spread (function of f) and the mid market quoted spread is smaller than half the bid-ask spread for all instruments, then we have found a set of $f(t, k)$ satisfying the constraints set forth in (10) whilst pricing all instruments within the bid-ask spread. [1]

$$\mathrm{Mispr}_{\mathrm{BidAsk}} = \frac{\mathrm{Spr}_{\mathrm{Theoretical}} - \mathrm{Spr}_{\mathrm{Mid}}}{\mathrm{BidAsk}_{\mathrm{Spread}}/2} \tag{12}$$

Our objective function will be the minimization of the sum of the squared standardazed mispricings (12). If using the sum of the squared mispricings will yield a solution for which some of the instruments are priced outside the bid-ask spread ($\mathrm{Mispr}_{\mathrm{BidAsk}} > 1$) then we will try minimizing the sum of even powers of the standardized mispricing (12) for exponents $4, 6$ and 8. In case of persistence of instruments priced outside the bid-ask spread, we take the solution for which the maximum absolute standardized mispricing is smallest.

4 Numerical Results

Our sample goes from 13-nov-03 to 14-jun-06 for the CDX and from 21-jun-04 to 23-May-06 for iTraxx. From Table (1) we note that, except for the iTraxx pool with a linear interpolation, in all other cases we find a solution where the theoretical spread exceeds the bid ask spread by less than one fifth (0.4/2) the bid ask range. In the case of the iTraxx pool with a linear interpolation we find only one date where all instruments cannot be priced within the bid ask range: in this case the theoretical spread is outside the bid ask spread by **less** than one third (0.6/2) the bid ask range.

[1] For the equity tranche in (11) we do not need to divide by the outstanding notional, since the market quotes directly the upfront amount.

	CDX		**ITRAXX**	
sample	from 13-nov-03	to 14-jun-06	from 21-jun-04	to 23-may-06
interpolation	linear	spline	linear	spline
Number of dates	616	616	473	473
% $\mathbf{Mispr}_{\mathbf{BidAsk}}$ > 1	1.0%	2.6%	0.2%	1.3%
% $\mathbf{Mispr}_{\mathbf{BidAsk}}$ > 1.2	0.8%	0.2%	0.2%	0.2%
% $\mathbf{Mispr}_{\mathbf{BidAsk}}$ > 1.4	0.0%	0.0%	0.2%	0.0%
% $\mathbf{Mispr}_{\mathbf{BidAsk}}$ > 1.6	0.0%	0.0%	0.0%	0.0%

Table 1: Percentage of sample repriced outside the bid-ask range

In Figure 1 are plotted the CDX and iTraxx ETL's for the 5y and 10y tranches. It can be clearly noticed the higher perceived riskiness of the CDX universe: despite the higher attachments the ETL is on average higher.

For some dates in our samples we had no quoted market spreads for particular maturities. More specifically, the 3y and 7y tranches are available only from 20-may-05 and 6-may-05 for the CDX and iTraxx respectively. In this cases the unknowns are reduced to be the ETL's on 5 and 10 years maturities only. From Figure 1 we note that were we had only the 5y and 10y tranches available the calibrated 5y and 10y ETL's were much more volatile.

5 Comparison with other approaches and Conclusions

Our results can be reconciled with Walker (2006)'s earlier work. Related work and a formal analysis of the properties of expected tranched loss in connection with no arbitrage is also in Livesey and Schlögl (2006). In both Walker and our approaches, given a set of tranche and index spreads on a set of maturities, one looks for a set of expected tranched losses f satisfying the same box constraints and monotonicity constraints. While in our framework the bid/asks of the instruments enter the target function we aim at minimizing, in Walker (2006)'s framework the instruments npv's must be exactly zero: npv's are treated as a set of linear equality constraints. Including bid/asks the no-arbitrage constraints are satisfied across the vast majority of dates, we find less violations of the no-arbitrage condition than in Walker's (2006). The method appears to be quite powerful as a first *model-independent* procedure to deduce implied expected loss surfaces from market data, allowing one to check basic no-arbitrage constraints in the market quotes. To price non-standard tranches with attachments or maturities within the observed market range (see for example a $4\% - 10\%$ tranche with $4y$ maturity) we can easily resort to the implied surface in a model independent way. We overcome the inconsistency of having different expected loss profiles on the same intervals that would be typically occurring if implied correlation had been used. In Figure 2 we see for example the ETL for the $(0, 3\%)$ equity tranche associated with the $3y$, $5y$ and $10y$ implied correlations for that tranche. As one can see the $[0, 3y]$ expected loss coming from the $5y$ implied correlation quote is different from the $[0, 3y]$ expected loss coming from the 3y correlation quote, and so on. The model independent ETL surface method avoids this inconsistency practically by construction.

Explicit dynamical models for the loss are an active area of research, see for example Brigo, Pallavicini and Torresetti (2006) where a family of models allowing for tranches and index calibration is considered, or Brigo, Pallavicini and Torresetti (2007) for such a model that is also consistent with single names defaults (top down approach).

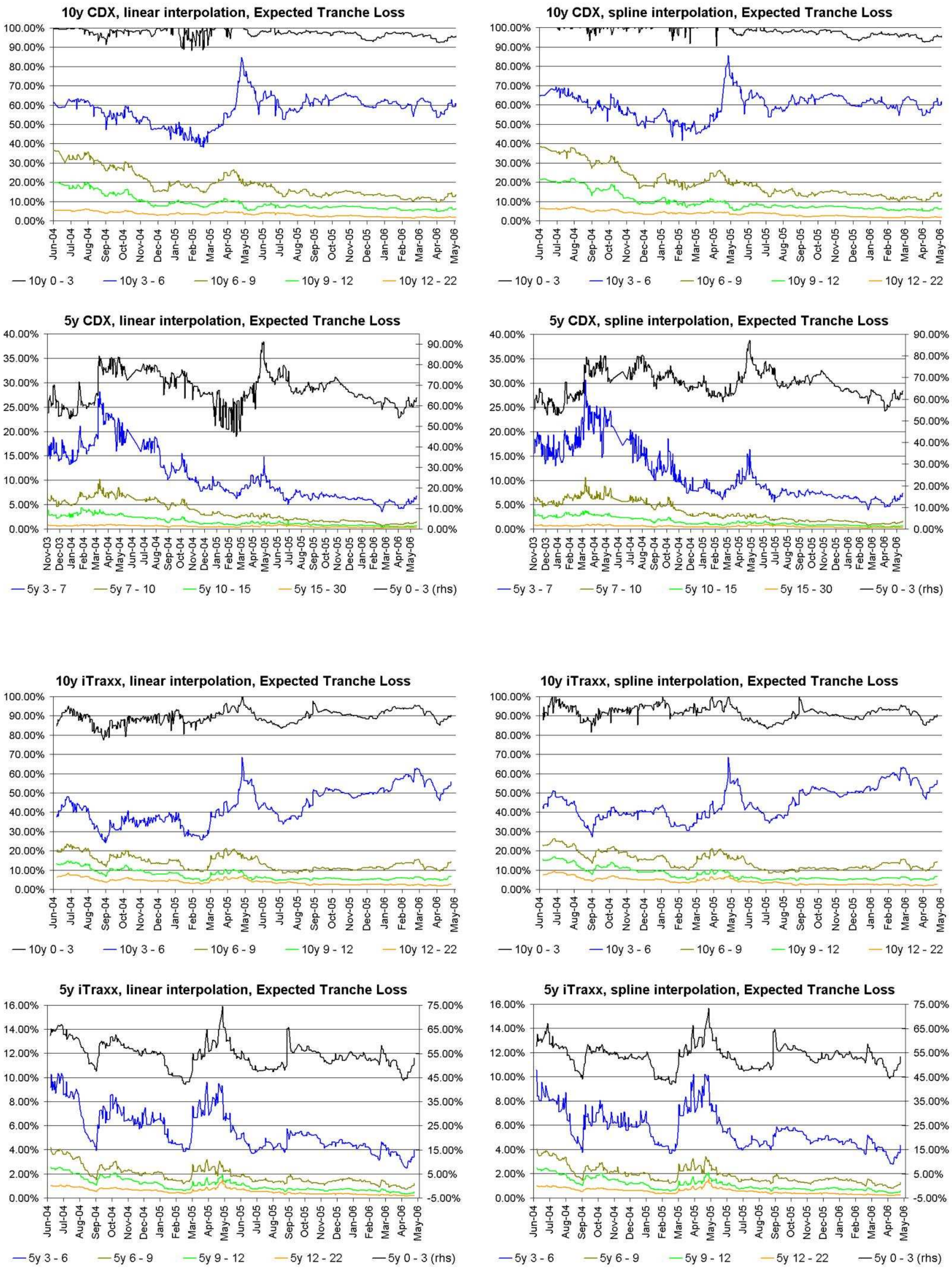

Figure 1: Expected Tranched Loss, CDX and iTraxx, 5y and 10y tranches

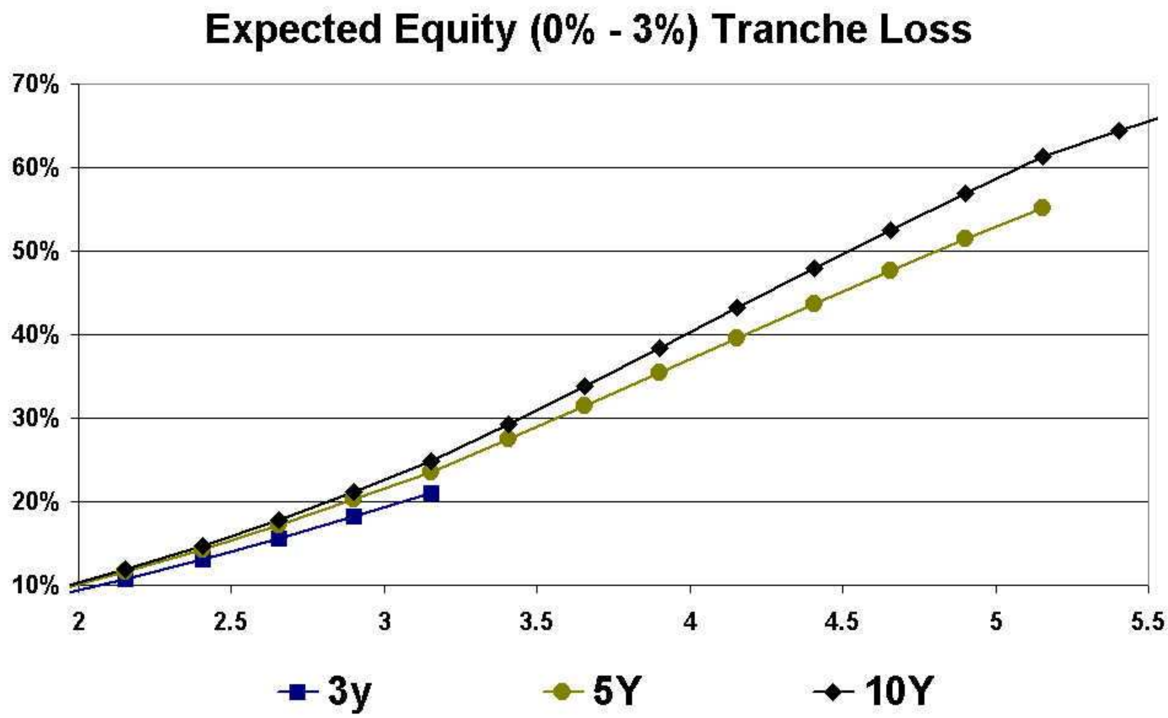

Figure 2: Inconsistency in Expected Tranched Losses coming from implied correlation. We plot the time evolution of Expected tranche losses for $(0, 3\%)$ resulting from implied correlation calibrated to the CDX equity tranches on April 26, 2006. Implied correlation is obtained through inversion of the homogeneos finite pool gaussian copula model price.

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