Calibration of CDO Tranches with the Dynamical Generalized-Poisson Loss Model *

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Abstract

In the first part we consider a dynamical model for the number of defaults of a pool of names. The model is based on the notion of generalized Poisson process, allowing for more than one default in small time intervals, contrary to many alternative approaches to loss modeling. We illustrate how to define the pool default intensity and discuss recovery assumptions. The models are tractable, pricing and simulation are straightforward, and consistent calibration to quoted index CDO tranches and tranchelets for several maturities is feasible, as we illustrate with numerical examples. In the second part we model directly the pool loss and we introduce extensions based on piecewise-gamma, scenario-based or CIR random intensities, leading to richer spread dynamics, investigating calibration improvements and stability.

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Part I
Introduction and CDO calibration

In this paper we consider a simple dynamical model for the loss distribution of a pool of names. This model aims at addressing the direct loss modeling in a simple and tractable way. Our model focuses on three points:

1. Tractability: the loss distribution should be known analytically;

2. Calibration: The calibration of market information, currently quoted index CDO tranches and tranchelets for several maturities, should be possible and realistic numerical examples should be given;

3. Pricing: the pricing of correlation products depending on the loss distribution dynamics should be feasible in a reasonable amount of time, possibly by simulation.

We adopt a "homogeneous pool" framework, in that we assume that all that matters in determining the loss distribution is knowledge of the number of defaulted names rather than knowledge of the specific defaulted names themselves. This means in particular that the recovery rate associated with any loss has to be a function of the number of defaults in the pool rather than a function of the specific defaulted names.

The basic idea of our approach here, following Pallavicini (2006), consists in modeling the number of defaults as a linear combination of independent Poisson processes with different intensities or, in other terms, as a generalized Poisson process. The jumps of different processes represent defaults with different multiplicity. As a sketchy example, we may assume that each time the first process in the summation jumps there is a single default, while each time the second process jumps there are two defaults, and so on. We may calibrate to market data both the multiplicity coefficients and the intensities in the single Poisson processes. The resulting model is called Generalized-Poisson Loss (GPL) dynamical model.

The idea of modeling financial variables as a linear combination of Poisson processes has been explored before, for example Babbs and Webber (1994) use this idea to model interest rates. Here we find that this kind of approach lends itself naturally to modeling the loss distribution of the pool of names. The basic GPL model is able to fit satisfactorily all maturities except the last one, and in the second part of the paper we consider all the maturities in a more general formulation of the model.

Our model is based on a simple idea. Different frameworks with loss dynamics have been proposed recently by Bennani (2005), Schönbucher (2005), Di Graziano and Rogers (2005), Elouerkhaoui (2006), and Sidenius et al. (2005). We aim at a completely specified and manageable model, rather than at an abstract framework, and we present detailed calibration results, with numerical outputs. Modeling the aggregate loss directly rather than obtaining it from single default models with a dependence structure constitutes the "top-down" approach. See for example Errais, Giesecke and Goldberg (2006) and references given therein, where a random thinning technique is also suggested to zoom from the aggregate loss to single defaults. More promisingly, our model seems to be related to insurance shock models leading to a "bottom-up" approach where single name default dependence is represented through a Marshal-Olkin copula (see for example Lindskog and McNeil (2003)). We pursue this relationship and show consistency with single names and cluster dynamics, leading to the GPCL model, in Brigo, Pallavicini and Torresetti (2007).

In this first part of our work we focus on the basic model formulation and calibration. The model is based on the default counting process and the loss is derived as a byproduct. In the second part we consider a loss based version of the model as long as stochastic intensity extensions, spread dynamics, recovery assumptions, option payoffs and calibration stability.

1 Market quotes

The most liquid multi-name credit instruments available in the market are credit indices and CDO tranches (e.g. DJi-TRAXX, CDX). Recently, credit index options have been considered as well. We discuss indices and tranches in the following, while we address index options in the second part of the paper.

1.1 Credit indices

The index is given by a pool of CDS on the names $1, 2, \ldots, M$, typically $M = 125$, each with notional $1/M$ so that the total pool has unitary notional. The index default leg consists of protection payments corresponding to the defaulted names of the pool. Each time one or more names default the corresponding loss increment is paid to the protection buyer, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

In exchange for loss increase payments, a periodic premium with rate S is paid from the protection buyer to the protection seller, until final maturity T_b. This premium is computed on a notional that decreases each time a name in the pool defaults, and decreases of an amount corresponding to the notional of that name (without taking out the recovery).

We denote with $\bar{L}_t$ the portfolio cumulated loss and with $\bar{C}_t$ the number of defaulted names up to time t re-scaled by M. Thus, $0 \leq \bar{L}_t \leq \bar{C}_t \leq 1$. The discounted payoff of the two legs of the index is given as follows:

$$\text{DefLeg}(0) := \int_0^T D(0,t) d\bar{L}_t$$

$$\text{PremiumLeg}(0) := S_0 \sum_{i=1}^{b} D(0,T_i) \int_{T_{i-i}}^{T_i} (1 - \bar{C}_t) dt$$

where $D(s,t)$ is the discount factor (often assumed to be deterministic) between times s and t. The integral on the right hand side of the premium leg is the outstanding notional on which the premium is computed for the index. Often the premium leg integral involved in the outstanding notional is approximated so as to obtain

$$\textsc{PremiumLeg}(0) = S_0 \sum_{i=1}^{b} \delta_i D(0,T_i)(1-\bar{C}_{T_i})$$

where $\delta_i = T_i - T_{i-1}$ is the year fraction.

Notice that, differently from what will happen with the tranches (see the following section), here the recovery is not considered when computing the outstanding notional, in that only the number of defaults matters.

The market quotes the value of S_0 that, for different maturities, balances the two legs. If one has a model for the loss and the number of defaults one may impose that the loss and number of defaults in the model, when plugged inside the two legs, lead to the same risk neutral expectation (and thus price) when the quoted S_0 is inside the premium leg, leading to

$$S_0 = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)d\bar{L}_t\right]}{\mathbb{E}_0\left[\sum_{i=1}^{b} \delta_i D(0,T_i)(1-\bar{C}_{T_i})\right]} \tag{1}$$

1.2 CDO tranches

Synthetic CDO with maturity T are contracts involving a protection buyer, a protection seller and an underlying pool of names. They are obtained by putting together a collection of Credit Default Swaps (CDS) with the same maturity on different names, $1, 2, ..., M$, typically $M = 125$, each with notional $1/M$, and then "tranching" the loss of the resulting pool between the points A and B, with $0 \leq A < B \leq 1$.

$$\bar{L}_t^{A,B} := \frac{1}{B-A}\left[(\bar{L}_t - A)1_{\{A<\bar{L}_t\leq B\}} + (B-A)1_{\{\bar{L}_t>B\}}\right]$$

Once enough names have defaulted and the loss has reached A, the count starts. Each time the loss increases the corresponding loss change re-scaled by the tranche thickness $B-A$ is paid to the protection buyer, until maturity arrives or until the total pool loss exceeds B, in which case the payments stop.

The discounted default leg payoff can then be written as

$$\textsc{DefLeg}(0;A,B) := \int_0^T D(0,t)d\bar{L}_t^{A,B}$$

Again, one should not be confused by the integral, the loss $\bar{L}_t^{A,B}$ changes with discrete jumps. Analogously, also the total loss $\bar{L}_t$ and the tranche outstanding notional change with discrete jumps.

As usual, in exchange for the protection payments, a premium rate $S_0^{A,B}$, fixed at time $T_0 = 0$, is paid periodically, say at times $T_1, T_2, \ldots, T_b = T$. Part of the premium can be paid at time $T_0 = 0$ as an upfront $U_0^{A,B}$. The rate is paid on the "survived" average tranche notional, leading to the following discounted payoff for the premium leg

$$\text{PremiumLeg}(0; A, B) := U_0^{A,B} + \sum_{i=1}^{b} D(0, T_i) S_0^{A,B} \int_{T_{i-1}}^{T_i} (1 - \bar{L}_t^{A,B}) dt.$$

If we assume that the payments are made on the notional remaining at each payment date T_i, rather than on the average in $[T_{i_1}, T_i]$, the premium leg can be written as

$$\text{PremiumLeg}(0; A, B) = U_0^{A,B} + S_0^{A,B} \sum_{i=1}^{b} \delta_i D(0, T_i)(1 - \bar{L}_{T_i}^{A,B})$$

where $\delta_i = T_i - T_{i-1}$ is the year fraction.

When pricing CDO tranches, one is interested in the premium rate $S_0^{A,B}$ that sets to zero the risk neutral price of the tranche. The tranche value is computed taking the (risk-neutral) expectation (in $t = 0$) of the discounted payoff consisting on the difference between the default and premium legs above. We obtain

$$S_0^{A,B} = \frac{\mathbb{E}_0\Big[\int_0^T D(0,t) d\bar{L}_t^{A,B}\Big] - U_0^{A,B}}{\mathbb{E}_0\Big[\sum_{i=1}^{b} \delta_i D(0, T_i)(1 - \bar{L}_{T_i}^{A,B})\Big]} \tag{2}$$

The above expression can be easily recast in terms of the upfront premium $U_0^{A,B}$ for tranches that are quoted in terms of upfront fees.

The tranches that are quoted on the market refer to standardized pools. Let us consider for example the DJi-TRAXX index, referring to the most liquid $M = 125$ names in the European CDS market. This index can indeed be traded in terms of leveraged tranches, in the same way as standard CDO's are traded. The fundamental variable on which we now have to concentrate is the total portfolio loss. Tranches with different seniorities are traded in the market. The main difference with respect to generic CDO's is that now tranches are standardized. That means that standard attachment points are used. For the DJ-iTRAXX Europe, the traded tranches are: an equity tranche, responsible for all losses between 0% and 3%, then other mezzanine and senior tranches covering 3%-6%, 6%-9%, 9%-12% and 12%-22%. For the main US index, the DJ CDX NA the tranche sizes are different: 0%-3%, 3%-7%, 7%-10%, 10%-15% and 15%-30%.

The market quotes either the periodic premiums rate $S_0^{A,B}$ of these tranches or their upfront premium rate $U_0^{A,B}$ for maturities $T = 3y, 5y, 7y, 10y$. Tranches with low detachment points ($B \leq 3\%$) are usually quoted in terms of the upfront premium, while tranches with higher detachment points are quoted in terms of the periodic premium. The equity tranche is quoted by the upfront amount needed to make it fair when a running spread of 500bp is taken as periodic spread in the premium leg.

2 Modeling framework and model definition

The no-arbitrage expressions for the quoted spread of credit indices, given by equation (1), and of CDO tranches, given by equation (2), show that the only information we can infer from market quotes are expected quantities, while we lack direct information about dependencies across single names. In particular credit indices depend both on expected portfolio cumulated loss and on expected number of defaults, while CDO tranches depend only on expected tranched portfolio cumulated loss.

These market data suggest to model loss-related quantities, i.e. portfolio cumulated loss and number of defaults, directly as fundamental objects, rather than patching single default models through a copula.

Since both the loss and the default counting process are jump processes, we now specify some technical assumptions on jump processes. We assume our jump processes to be right continuous with left limits. With the notation dX_t, where X_t is such a jump process, we actually mean the jump size of process X at time t if X jumps at t, and zero otherwise, or, in other terms, pathwise, $dX_t = X_t - X_{t^-}$, where in general we define $X_{t^-} := \lim_{h\downarrow 0} X_{t-h}$.

2.1 Non-arbitrage constraints

The portfolio cumulated loss ($\bar{L}_t$) and the re-scaled number of defaults ($\bar{C}_t$) cannot be independently modelled, since they are coupled by the forward realization of the recovery rate (R_t) at default dates, which we could call "instantaneous recovery".

$$\bar{L}_T = \int_0^T [1 - R_{s^-}(\bar{C}_{s^-})] d\bar{C}_s.$$

Notice that, in general, the recovery rate R_t depends on the number of defaults and possibly on other random sources. However, here we assume R_t to be possibly stochastic only via the number of defaults.

As a first approach we choose to model directly the number of defaults and to introduce an average recovery rate ($\mathrm{R}\textsc{ec}$) defined by

$$\mathrm{R}\textsc{ec}(T) := \frac{\int_0^T R_{s^-}(\bar{C}_{s^-}) d\bar{C}_s}{\int_0^T d\bar{C}_s}$$

to model the portfolio cumulated loss, as given through direct substitution by

$$\bar{L}_T = \bar{C}_T (1 - \mathrm{R}\textsc{ec}(T)) \tag{3}$$

The recovery rate is not arbitrary, but it must be constrained in order to ensure that the resulting dynamics is arbitrage free.

In order to ensure an arbitrage-free dynamics, the portfolio cumulated loss and the (re-scaled) number of defaults must be non-decreasing processes taking values in the $[0, 1]$ interval, the increments of the former always smaller or equal than the increments of the latter. In the following we further assume, for the sake of simplicity, that $\bar{L}_0 = 0$ and $\bar{C}_0 = 0$ too, i.e. no defaults have occurred before the initial time 0.

Any choice of the function $R_s(\cdot)$ which is bounded in the interval $[0, 1]$ for all times s leads to a mean recovery rate (Rec) ensuring the no-arbitrage constraint by construction. A simple prescription is to take a constant value. In the following we select $R_t = 40\%$ for any time t, namely Rec = 40%.

Remark 2.1. *As an alternative formulation we can consider the portfolio cumulated loss as the fundamental object to model. In this case we have to reformulate the non-arbitrage constraint by taking into account that the recovery rate goes to zero for maximum loss. We address this issue, and discuss further recovery models, in the second part of the paper.*

2.2 The basic GPL dynamical model

The basic Generalized Poisson Loss (GPL) model can be formulated as follows. Consider a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ ($\mathbb{Q}$ is the risk neutral measure, the related expectation conditional on $\mathcal{G}_t$ is denoted by $\mathbb{E}_t$) supporting a number n of independent Poisson processes $N_1, \ldots, N_n$ with time-varying intensities $\lambda_1, \ldots, \lambda_n$. Define the stochastic process

$$Z_t := \sum_{j=1}^{n} \alpha_j N_j(t),$$

for integers $\alpha_1, \ldots, \alpha_n$, and model the number of defaults in the pool of names as Z_t. One possible choice is $\alpha_j = j$, so that in this case

$$Z_t = N_1(t) + 2N_2(t) + 3N_3(t) + \ldots + nN_n(t).$$

If N_1 jumps there has been just one default (idiosyncratic risk), if N_n jumps there are n defaults and the whole pool defaults one shot (systemic risk), otherwise for other N_i's we have intermediate situations (contagion, sectors, etc). This model explicitly contemplates the possibility of multiple defaults in small time intervals, contrary for example to Schönbucher (2005) and Errais, Giesecke and Goldberg (2006). Multiple defaults are allowed for example also in Putyatin et al. (2006). Notice also that limiting ourselves to some values of α can be interpreted in turn as the missing α's being there with zero intensity. So in this sense the real parameters of the model are the intensities and one can always think of the α's as all being there.

It was recently brought to our attention that a similar approach considering dL rather than L is pursued by Elouerkhaoui (2006), who links the resulting loss process to a model consistent with single names connected through a Marshall-Olkin copula. See also the earlier work of Lindskog and McNeil (2003).

A drawback of our model is that the number of defaults in time may increase without limit. If our pool contains M names, we may then consider as actual number of defaults

$$C_t := \min(Z_t, M) = Z_t 1_{\{Z_t < M\}} + M 1_{\{Z_t \geq M\}}$$

In turn the re-scaled number of defaults can be defined as

$$\bar{C}_t := \frac{1}{M} C_t$$

while the portfolio cumulated loss is given by substituting $\bar{C}_t$ into equation (3).

If Z_t has a known distribution, the distribution of C_t can be easily derived as a byproduct. Indeed,

$$\mathbb{Q}(C_t \leq x) = 1_{\{x<M\}} \mathbb{Q}(Z_t < x) + 1_{\{x \geq M\}}.$$

The related density (defined on integer values since the law is discrete) is

$$p_{C_t}(k) = p_{Z_t}(k) 1_{\{k<M\}} + \mathbb{Q}(Z_t \geq M) 1_{\{k=M\}}.$$

The distribution of Z_t (and thus of C_t) is directly known through its characteristic function. Indeed, compute the characteristic function of Z_t,

$$\varphi_{Z_t}(u) := \mathbb{E}_0[\,\exp(iuZ_t)\,] = \int_0^\infty \exp(iux) p_{Z_t}(x) dx$$

i.e. the Fourier transform of the density p_{Z_t} of the random variable Z_t. We have easily, thanks to independence of different N_i's,

$$\varphi_{Z_t}(u) = \prod_{j=1}^n \mathbb{E}_0[\,\exp(iu\alpha_j N_j(t))\,] = \prod_{j=1}^n \varphi_{N_j(t)}(\alpha_j u),$$

where now $\varphi_{N_j(t)}$ is the characteristic function of the Poisson process N_j. Since we know the characteristic function of the Poisson law, we may write

$$\varphi_{Z_t}(u) = \exp\left[\sum_{j=1}^n \Lambda_j(t) \left(e^{i\alpha_j u} - 1\right)\right] \tag{4}$$

where we define $\Lambda_j(t) := \int_0^t \lambda_j(v) dv$ to be the cumulated intensity at time t. The density of Z_t can be obtained as the inverse Fourier transform of $\varphi_{Z_t}(u)$.

If one wishes to avoid Fourier methods there are alternative possibilities for finding the law of Z_t. Indeed, given the vector $\alpha = [\alpha_1, \ldots, \alpha_n]$ of possible jump sizes for Z_t, for any possible value k of Z_t define the sets

$$A_k^\alpha := \{[m_1, \ldots, m_n] : m_1, \ldots, m_n \in \mathbb{N} \cup \{0\}, \quad \alpha_1 m_1 + \ldots + \alpha_n m_n = k\}.$$

The set A_k^α is the set of all possible values of the constituent Poisson processes $[N_1, ..., N_n]$ leading to Z_t having the value k. It is a finite set, given non-negativeness of the m's. Determining A_k^α amounts to solving a linear Diophantine equation in dimension n, a problem for which integer programming algorithms are available. Once the A_k^α have been determined, we compute the law of Z_t as follows:

$$\mathbb{Q}(Z_t = k) = \sum_{m \in A_k^\alpha} \mathbb{Q}(N_1(t) = m_1, \ldots, N_n(t) = m_n) = \sum_{m \in A_k^\alpha} \prod_{j=1}^n \mathbb{Q}(N_j(t) = m_j) \tag{5}$$

where terms in products are known from the Poisson law and we have used independence of the N_j's.

2.3 Default intensity

An important feature of loss models is to link default intensities jumps to loss dynamics, so that the default intensity decreases, as long as loss increases, and it is equal to zero when the whole portfolio has defaulted.

Let us consider the compensator A_t of the default-counting point process C_t, namely the nondecreasing predictable process that added to a local martingale gives C_t itself (Doob-Meyer decomposition), satisfying

$$\mathbb{E}_t[\, C_T - A_T \,] = C_t - A_t,$$

see for example Giesecke and Goldberg (2005). A can be computed as

$$\begin{aligned} A_T := \lim_{h \downarrow 0} \int_0^T \frac{\mathbb{E}_t[\, C_{t+h} - C_t \,]}{h} dt &= \lim_{h \downarrow 0} \int_0^T \frac{\mathbb{E}_t\big[\min(Z_{t+h} - Z_t, M - Z_t) 1_{\{Z_t < M\}}\big]}{h} dt \\ &= \lim_{h \downarrow 0} \int_0^T \frac{\mathbb{E}_t\Big[\sum_{j=1}^n \min(\alpha_j, M - Z_t) 1_{\{Z_t < M\}} 1_{\{Z_{t+h} - Z_t = \alpha_j\}}\Big]}{h} dt \\ &= \lim_{h \downarrow 0} \int_0^T \frac{\sum_{j=1}^n \min(\alpha_j, M - Z_t) 1_{\{Z_t < M\}} \mathbb{E}_t\big[1_{\{Z_{t+h} - Z_t = \alpha_j\}}\big]}{h} dt \end{aligned}$$

so that, with a final calculation,

$$A_T = \int_0^T \sum_{j=1}^n \min(\alpha_j, (M - Z_{t^-})^+) \lambda_j(t) dt \tag{6}$$

where we have taken the left limit in the integrand to ensure its left-continuity (and hence predictability).

If A_t is absolutely continuous, as in our case, its density is known as the intensity of the process C_t, and is given by

$$h_C(t) = \sum_{j=1}^n \min(\alpha_j, (M - Z_{t^-})^+) \lambda_j(t). \tag{7}$$

The intensity h goes to zero when the whole portfolio has defaulted, as expected. Further, if all the amplitudes α_j are greater than zero, as with the choice $\alpha_j = j$, the intensity h_C jumps whenever the default-counting process C jumps.

Remark 2.2. *A possible generalization of the GPL model which can show a wider class of default intensities can be obtained if we redefine C_t in a more general fashion as*

$$C_t := \Psi(Z_t)$$

where $\Psi : \mathbb{R}_0^+ \to [0,1]$ is non-decreasing and deterministic. The default intensity is

$$h_C(t) = \sum_{j=1}^n (\Psi(Z_{t^-} + \alpha_j) - \Psi(Z_{t^-})) \lambda_j(t)$$

We plan to address this generalization in future works.

Remark 2.3. *The default intensity h_C in the basic GPL model is a stochastic object only through its dependence on the process Z_t. However, it is possible to extend the GPL model by considering the Poisson intensities λ_j as stochastic processes, e.g. following a Gamma or CIR process. In this case the default intensity h_C acquires a new source of stochasticity. We address such extensions in the second part of the paper, in relationship with credit index options.*

2.4 Equivalent formulation as Generalized Poisson Process

The process Z_t can also be characterized as a so called generalized Poisson process (GPP, hence the name GPL for the loss model). A generalized Poisson process has the same properties as a Poisson process with the exception of the possibility to allow for multiple jumps. A (time-homogeneous) GPP J_t is defined as a process with stationary independent increments, where the increments of J_t may amount to positive integer values $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$. The probability to have a jump of size α_k given that there has been at least one jump of any positive size satisfies

$$\lim_{h \to 0} \mathbb{Q}\{J_{t+h} - J_t = \alpha_k | J_{t+h} - J_t \geq \alpha_1\} = p_k. \tag{8}$$

Also, the probability of having no jumps up to time t and to have at least one jump in arbitrarily small times is

$$\mathbb{Q}(J_t = 0) = \exp(-\lambda t), \quad \lim_{h \to 0} \mathbb{Q}\{J_h > 0\}/h = \lambda,$$

exactly as for the standard Poisson process. But, differently from the generalized Poisson process satisfying (8), the standard Poisson process N_t satisfies

$$\lim_{h \to 0} \mathbb{Q}\{N_{t+h} - N_t \geq 2 | N_{t+h} - N_t \geq 1\} = 0.$$

Now let us go back to Z_t. In case we take time homogeneous Poisson processes N_j with constant intensities λ_j, our process above for Z_t is the same as a GPP J_t with the same α's and multiple jump probabilities p_i and intensity λ given by

$$p_i = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \qquad \lambda = \sum_{j=1}^n \lambda_j. \tag{9}$$

In other terms, our linear combination of Poisson processes is the same as a generalized Poisson process allowing for multiple jumps with given probabilities. The two processes do not coincide only as far as marginal distributions (or characteristic functions) are concerned, but share the same finite dimensional distributions, and are thus the same process from the process law point of view.

2.5 Equivalent formulation as Compound Poisson Process

One more way of looking at our process is the compound Poisson process. Indeed, at any time t our process Z_t has the same characteristic function as a particular compound Poisson process. Consider the following compound Poisson process

$$X_t = \sum_{j=1}^{N_t} Y_j,$$

where N is a standard Poisson process with intensity λ and the Y_j's are i.i.d random variables, independent of N, and with distribution given by

$$Y_j \sim \begin{cases} \alpha_1 & \lambda_1/(\sum_{k=1}^n \lambda_k) \\ \alpha_2 & \lambda_2/(\sum_{k=1}^n \lambda_k) \\ \vdots & \\ \alpha_n & \lambda_n/(\sum_{k=1}^n \lambda_k) \end{cases}$$

If, as before, we define λ as in (9), then the compound Poisson process X_t has the same characteristic function, at all times t, as our process Z_t for the default counting function. The finite dimensional distributions of the two processes coincide as well, so that substantially Z_t and X_t are the same process. This is easily checked by writing the finite dimensional distributions in terms of independent increments, while recalling that both Z_t and X_t have stationary independent increments. Finally, we notice that also Di Graziano and Rogers (2005) in some of their formulations obtain a compound Poisson process for the loss distribution.

Remark 2.4. *The marginal distributions of compound Poisson processes can be explicitly calculated in closed form if the jump amplitude Y has a discrete-valued distribution, since it is possible to find a relationship, known as Panjer recursion, between the probability densities $p_{X_t}(n)$ and $p_{X_t}(n-1)$. By following Hess et al. (2002), we can write for $k \in [1, n]$ and in the case $\alpha_j = j$*

$$p_{X_t}(0) = \exp(-\lambda t), \quad p_{X_t}(k) = \frac{1}{k} \sum_{j=1}^{k} j \lambda_j t p_{X_t}(k-j)$$

If we expand the recursion for each k, we get equation (5), previously obtained by general considerations.

3 Numerical results

The GPL model is calibrated to the market quotes observed on March 6, 2006. Deterministic discount rates are listed in Table 1, while tranche data and DJi-TRAXX fixings are listed in Table 2. All calibrations assume $R = 40\%$.

The model parameters fixed by the calibration procedure are the amplitudes α_i with $i = 1 \ldots n$, which can assume only positive integer values, and the cumulated intensities $\Lambda_i(T)$, which are real non-decreasing piecewise linear functions in the tranche maturity.

The optimal values for the amplitudes α are selected, by adding non-zero amplitudes one by one, as follows:

1. set $\alpha_1 = 1$ and calibrate Λ_1;

2. add the amplitude α_2 and find its best integer value by calibrating the cumulated intensities Λ_1 and Λ_2, starting from the previous value for Λ_1 as a guess, for each value of α_2 in the range $[1, 125]$,

3. repeat the previous step for α_i with $i = 3$ and so on, by calibrating the cumulated intensities $\Lambda_1, \ldots, \Lambda_i$, starting from the previously found $\Lambda_1, \ldots, \Lambda_{i-1}$ as initial guess, until the calibration error is under a pre-fixed threshold or until the intensity Λ_i can be considered negligible.

The objective function f to be minimized in the calibration is the squared sum of the errors shown by the model to recover the tranche and index market quotes weighted by market bid-ask spreads:

$$f(\alpha, \Lambda) = \sum_i \epsilon_i^2, \quad \epsilon_i = \frac{x_i(\alpha, \Lambda) - x_i^{\text{Mid}}}{x_i^{\text{Bid}} - x_i^{\text{Ask}}} \tag{10}$$

where the x_i, with i running over the market quote set, are the index values S_0 for DJi-TRAXX index quotes, and either the index periodic premiums $S_0^{A,B}$ or the upfront premium rates $U^{A,B}$ for the DJi-TRAXX tranche quotes.

3.1 Calibration of CDO indices and tranches

As a first calibration example we consider data coming from standard DJi-TRAXX tranches up to a maturity of seven years with constant recovery rate of 40%. The calibration procedure selects five Poisson processes as listed in Table 4. The 18 market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread.

One possible comparison of our implied loss distribution according to the GPL model is with the implied loss distribution according to Hull and White's (2005) "perfect copula" approach. The comparison makes sense on a single maturity, being the perfect copula approach inherently static[1], contrary to our dynamical model. If we compare the implied loss distribution resulting from the calibration of the five year index and tranche quotes with the perfect copula approach as reformulated in Torresetti et al. (2006a), we find a qualitative pattern similar to the pattern we have in Figure 1. Indeed, notice in particular the large portion of mass concentrated near the origin, the subsequent modes when moving along the loss distribution for increasing values, and the bumps in the far tail. These features are common to both approaches, and multiple modes occur also with different methods, see for example Albanese et al. (2005). In our GPL models the bumps in the tails of the loss distributions, which seem to be necessary in order to be able to recover the market quotes, are obtained thanks to the multiple jumps components contributing to the loss distribution. In particular, the components with higher α's are giving rise to the little bumps in the far tail of the loss distribution.

[1] See Walker (2006) and Torresetti et al. (2006b) on how to improve the perfect copula approach to add constraints ensuring an arbitrage-free dynamics across maturities.

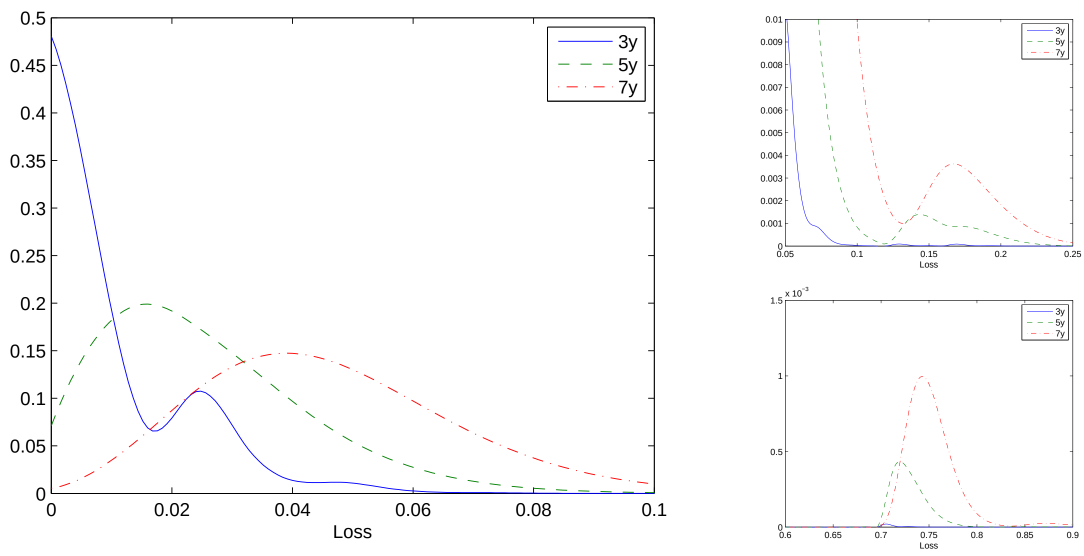

Figure 1: Loss distribution of the basic GPL model for three different maturities drawn as a continuous line. The right-side plots are drawn with different scales to zoom on some fine-grain details of the distribution.

If we repeat the calibration and add the ten year maturity, the calibration errors grow and not all market quotes can be recovered within the market bid-ask spread. In particular, equity and mezzanine tranches of the ten year maturity set are quite out of the bid-ask spread. We see a solution to this problem in the second part of the paper, where we also repeat the calibration at different dates finding that the calibration parameters are quite stable.

3.2 Calibration of CDO tranchelets

The market quotes also non-standard tranches, which are quoted over the counter. An interesting case is given by the so called "tranchelets", namely DJi-TRAXX tranches with attachment and detachment points possibly smaller than 3%. On the first of march 2006 we obtain market quotes for a set of tranchelets with maturity of five and seven years (see Table 3).

We calibrate the market data with constant recovery rate of 40%. The calibration procedure selects five Poisson processes as listed in Table 5. The 18 market quotes used by the calibration procedure are recovered, but within an error that is occasionally larger than the bid-ask spread.

4 Pricing

The GPL model belongs in the "top-down" approach to credit models, in that we can directly price only products whose payoff depends on the loss distribution rather than on the single default events. Indices, CDO tranches, forward start CDO tranches and tranche options are among such products, whereas, for example, CDO squared are not. However, for products such as CDO squared a random thinning procedure can be considered as a possibility to consistently "zoom" on single name defaults. This possible extension is under investigation and would be based on our earlier expression (7) for the pool default intensity in the GPL model. Alternatively, our model seems to be related to a "bottom-up" approach where single name default dependence is represented through a Marshal-Olkin copula.

Now we hint at pricing products based on the loss distribution such as tranche options, forward start tranches, and so on, with the calibrated model. This task is simple, given knowledge of the marginal and transition distributions for the constituent Poisson processes. Indeed, if we have a payoff or additive portion of a payoff depending on the loss at one single maturity, we simply sample one-shot the independent Poisson laws of the constituent processes N_j at maturity, add them up using the related multiplicity coefficients α_j, plug the resulting loss in the payoff portion and average over scenarios. This procedure is substantially maintained also under possible random intensities. Alternatively, we may decide to use the inverse Fourier transform of the known characteristic function of the terminal distribution to obtain the loss density and then integrate numerically the payoff against this density. This approach avoids simulation.

If a payoff is path dependent on the loss we still may simulate the independent increments of the independent constituent Poisson processes N_j among the relevant instants. Given independence this can be realized by sampling known independent Poisson laws. Once this has been done, we obtain the constituent processes at every relevant time by adding up their increments, and then we obtain the loss at any time by simply adding the constituent processes times their multiplicity coefficients α. Then we plug each temporal path of the loss distribution in the payoff and average over scenarios. This procedure is substantially maintained also under possible random intensities. Simulation is thus easy and based on the ability to sample from a Poisson law.

5 Conclusions and Second Part

We have introduced a dynamical model for the loss distribution of a pool of names. The model builds on the notion of generalized Poisson process, is tractable and can be consistently calibrated to quoted index CDO tranches and tranchelets for several maturities. We have illustrated consistent calibration of the basic model to market CDO index data with different maturities. We have further explained that pricing and in particular simulation with the resulting model is easy.

The second part of the paper shows the extended versions of the GPL models which include direct loss modeling rather than number of defaults, recovery rate specifications and stochastic intensities. We also address issues concerning calibration stability. Further, we plan to analyze credit index options within the extended version of the model, showing which extensions of the GPL model may account for index spread volatility.

Part II

Calibration stability and spread dynamics extensions

In this second part of the paper we address more advanced issues concerning direct loss modeling with the Generalized Poisson Loss (GPL) model introduced in part one (Brigo, Pallavicini and Torresetti (2006a)), and propose enhanced versions of the basic model.

Our basic GPL model can be improved in several respects. First, we review the payoffs of credit index options and leveraged super-senior CDO tranches as fundamental examples motivating the subsequent extensions of the model. Then we introduce the stochastic intensity versions of the basic GPL process to be used later for loss modeling. We introduce explicit stochastic intensities maintaining analytical tractability, leading to the gamma , piecewise gamma, scenario and CIR GPL processes. Then we explain how one can model directly the loss dynamics rather than the number of defaults, as we did instead in the first part, and introduce general recovery assumptions, discussing the link between recovery and pool intensities. We explicitly write the index spread in terms of intensities explaining how this is obtained in our models, with possible benefits of the stochastic intensity extensions. The same benefits would apply in valuation of forward start CDO tranches and tranche options. We finally focus on calibration results and stability when the *loss based* GPL model is used with some minimalist recovery assumptions.

6 Market quotes

We briefly review the payoffs of credit indices and of CDO tranches. A detailed discussion is present in the first part of the paper. Then, we discuss the payoff of credit index options and leveraged super-senior CDO tranches as fundamental examples motivating the subsequent extensions of the model.

6.1 Credit indices and CDO tranches

Let us denote by $\bar{C}(t)$ the number of defaults by t divided by the number of names in the pool. Let us denote the related portfolio cumulated loss by t as $\bar{L}(t)$. We may write the following general expression at initial time 0 for the credit index spread quoted by the market:

$$S_0 := \frac{\mathbb{E}_0\left[\int_0^{T_b} D(0,t)d\bar{L}_t\right]}{\mathbb{E}_0\left[\sum_{i=1}^{b}\delta_i D(0,T_i)(1-\bar{C}_{T_i})\right]} \tag{11}$$

where $\{T_i : i = 1 \dots b\}$ is the premium leg time-structure, $T_0 = 0$ and $\delta_i = T_i - T_{i-1}$ is the year fraction. We are assuming that no default has happened before trade date, i.e.

$\bar{C}_0 = 0$ and $\bar{L}_0 = 0$. The denominator in (11) is the spot index risky-duration or annuity. The forward annuity at time T is defined as

$$\Theta_T := \mathbb{E}_T\left[\sum_{i=1}^{b} \delta_i D(T, T_i)(1 - \bar{C}_{T_i}) 1_{\{T_i > T\}}\right]$$

It is useful to introduce the forward index that ignores defaults occurred before the index fixing in determining the default leg. Such assumption is known as "knock-out" feature. The forward index with and without knock-out feature, respectively, is

$$S_T = \frac{1}{\Theta_T}\mathbb{E}_T\left[\int_T^{T_b} D(T,t)d\bar{L}_t\right], \quad \bar{S}_T = \frac{1}{\Theta_T}\left(\mathbb{E}_T\left[\int_T^{T_b} D(T,t)d\bar{L}_t\right] + \bar{L}_T\right) \qquad (12)$$

Moving from the index to its tranches, the spread quoted by the market for CDO tranches at the initial time 0 is

$$S_0^{A,B} := \frac{\mathbb{E}_0\left[\int_0^{T_b} D(0,t)d\bar{L}_t^{A,B}\right] - U_0^{A,B}}{\mathbb{E}_0\left[\sum_{i=1}^{b} \delta_i D(0,T_i)(1 - \bar{L}_{T_i}^{A,B})\right]} \qquad (13)$$

and the above expression can be easily recast in terms of the upfront premium $U_0^{A,B}$ for tranches that are quoted in terms of upfront fees. The analogous forward tranche spread is defined similarly to the case of the index.

Remark 6.1. (Recovery: splitting loss and number of defaults information). *As noticed en passant also in the first part, from (13) we see that the spread tranche quotes contain information only on the loss $\bar{L}$ (both numerator and denominator), and thus they do not allow us to discriminate between the loss and the number of defaults or default rate $\bar{C}$. The only market quantity allowing us to do so is the index spread, (11), where we have information on both $\bar{L}$ (numerator) and $\bar{C}$ (denominator). But even with (11) we can derive only very stylized features of expected recovery rates and not sharp recovery dynamics, as is reflected in our definition (23) below.*

6.2 Credit index options

Credit markets quote few data on credit index derivatives. In particular (premium-) receiver (put) and payer (call) options on entering a credit index contract at a given strike value are traded with a certain liquidity.

A simplified version of the option payoff, which avoids cash-settling features usually considered in market practice (see e.g. Pedersen (2003)), is given for the put option (premium receiver) by

$$\Pi_{\text{PUT}}(T,T;K) := \left(K\Theta_T - \mathbb{E}_T\left[\int_T^{T_b} D(T,t)d\bar{L}_t\right]\right)^+$$

and for the call option (premium payer) by

$$\Pi_{\text{CALL}}(T,T;K) := \left(\mathbb{E}_T\left[\int_T^{T_b} D(T,t)d\bar{L}_t\right] + \bar{L}_T - K\Theta_T\right)^+$$

Notice that the receiver option depends only on the loss in $[T, T_b]$ whereas the payer option incorporates the losses $\bar{L}_T$ occurred up to expiry date T ("front-end protection"). By substituting the credit index spread into the payoff expressions, we obtain the call and put option price under the risk-neutral measure:

$$\begin{aligned} \Pi_{\text{PUT}}(0,T;K) &:= \mathbb{E}_0\big[\, D(0,T)\Theta_T(K - S_T)^+ \,\big] \\ \Pi_{\text{CALL}}(0,T;K) &:= \mathbb{E}_0\Big[\, D(0,T)\Theta_T\left(\bar{S}_T - K\right)^+ \,\Big] \end{aligned} \tag{14}$$

6.3 Leveraged super-senior CDO tranches

Let us define the forward tranched risky-duration on a premium leg time-structure starting at $t_0 \geq T$ and ending at $t_f > T$ for a senior tranche (which usually does not require an upfront payment) as

$$\Theta_T^{A,B}(t_0, t_f) := \mathbb{E}_T\left[\sum_{i=1}^{b} \delta_i D(T, T_i)(1 - \bar{L}_{T_i}^{A,B}) 1_{\{t_0 < T_i \leq t_f\}}\right]$$

Let us define the value of a forward tranche entered at a strike premium $K \ll 1$ on a premium leg time-structure starting at t_0 and ending at t_f as

$$\textsc{Tranche}(T; K, A, B, t_0, t_f) := \mathbb{E}_T\left[\int_T^{T_b} 1_{\{t_0 < t \leq t_f\}} D(0,t) d\bar{L}_t^{A,B}\right] - K\Theta_T^{A,B}(t_0, t_f)$$

Let us define the value of a leveraged super-senior CDO tranche with stochastic trigger time τ as

$$\begin{aligned} \text{LSS}(0; K, A, B) &:= \mathbb{E}_0\Big[\textsc{Tranche}(0; S_0^{A,B}, A, B, 0, \tau)\Big] \\ &+ \mathbb{E}_0\Big[D(0,\tau)\min(K, \textsc{Tranche}(\tau; S_0^{A,B}, A, B, \tau, T_b))\Big] \end{aligned} \tag{15}$$

In some of the simplest prototypical LSS tranches, the trigger time τ is the time where the loss goes the first time above a pre-specified safety level K, and the index spread is not involved directly. The trigger ensures the position unwind before the default leg becomes too severe for the protection seller. This way the investor (protection seller) may try and obtain a premium on the notional 1 by risking default payments only on a much smaller K. In more sophisticated versions, the trigger time τ depends also on the index spread, requiring a realistic model also for the spread. We comment on the stochastic index spread extensions of the GPL process below.

7 Model definition

The GPL model was already defined in the first part of the paper as a dynamical model for the number of defaults. Here, we extend it as a generic dynamical model either for portfolio cumulated loss or for the number of defaults. Further, we show how to dynamically model the recovery rate and the default intensity taking into account the kind of information that is available in the market.

7.1 The underlying GPL dynamics

Consider the GPL process Z_t, which is defined as

$$Z_t := \sum_{j=1}^{n} \alpha_j N_j(t) \tag{16}$$

where N_j, with $j = 1 \dots n$, are (conditional on the intensity) time-inhomogeneous Poisson counting processes with possibly stochastic intensity $\lambda_j(t)$ and deterministic integer-valued positive amplitudes α_j. Under stochastic intensity the N_j are actually Cox processes. We deal with stochastic intensity here, since in the first part we assumed deterministic intensity.

The characteristic function of the Z_t process is

$$\varphi_{Z_t}(u) = \mathbb{E}_0\left[\, e^{iuZ_t} \,\right] = \mathbb{E}_0\left[\, \mathbb{E}_0\left[\, e^{iuZ_t} | \Lambda_1(t) \dots \Lambda_n(t) \,\right] \,\right]$$

where $\Lambda_j(t) := \int_0^t \lambda_j(s)\, ds$, with $i = 1 \dots n$, are the cumulated intensities of each Poisson process. Now, we substitute Z_t obtaining

$$\varphi_{Z_t}(u) = \mathbb{E}_0\left[\prod_{j=1}^{n} \mathbb{E}_0\left[\, e^{iu\alpha_j N_j(t)} | \Lambda_1(t) \dots \Lambda_n(t) \,\right] \right] = \mathbb{E}_0\left[\prod_{j=1}^{n} \varphi_{N_j(t)|\Lambda_j(t)}(u\alpha_j) \right]$$

which can be directly calculated since the characteristic function $\varphi_{N_j(t)|\Lambda_j(t)}$ of each Poisson process, given its intensity, is known in closed form, leading to

$$\varphi_{Z_t}(u) = \mathbb{E}_0\left[\exp\left(\sum_{j=1}^{n} \Lambda_j(t)(1 - e^{iu\alpha_j}) \right) \right] \tag{17}$$

The marginal distribution p_{Z_t} of the process Z_t can be directly computed at any time via inverse Fourier transformation of the characteristic function of the process. The characteristic function $\varphi_{Z_t}(u)$ can be explicitly calculated for some relevant choices of Poisson cumulated intensities distributions, as we see below.

The GPL process, which is a linear combination of Poisson process, has independent increments. This property allows to explicitly calculate the transition probability of the GPL process too. Indeed, we obtain

$$\begin{aligned} \pi_Z(s, x; t, y) &:= \mathbb{Q}\left\{\, Z_t = y \;\middle|\; Z_s = x \,\right\} = \mathbb{Q}\left\{\, Z_t - Z_s = y - x \;\middle|\; Z_s = x \,\right\} \\ &= \mathbb{Q}\left\{\, Z_t - Z_s = y - x \,\right\} = p_{Z_t - Z_s}(y - x) \end{aligned}$$

where $s < t$ and $p_{Z_t - Z_s}$ is the marginal distribution of the $Z_t - Z_s$ process. This result is useful when computing forward expectations for option pricing.

The intensity of the process Z_t can be defined in term of the density $h_Z(t)$ of its predictable compensator $A_Z(t)$,

$$A_Z(t) := \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \mathbb{E}_s[\, Z_{s+h} - Z_s \,] = \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \sum_{j=1}^{n} \alpha_j \mathbb{E}_s[\, N_j(s+h) - N_j(s) \,] = \sum_{j=1}^{n} \alpha_j \Lambda_j(s),$$

$$h_Z(t) = \sum_{j=1}^{n} \alpha_j \lambda_j(t). \tag{18}$$

Notice that the intensity of the GPL process does not depend on the process itself, but only on the intensities of the constituent Poisson processes.

In the first part we have seen that if the Poisson cumulated intensities $\Lambda_j(t)$ are deterministic, the characteristic function of the process Z_t is known in closed form and is simply (17) without expectation. In this case the GPL intensity $h_Z(t)$ is deterministic.

7.1.1 The Gamma intensity GPL model

Assume now that the cumulated intensities $\Lambda_j(t)$ are distributed at any time t according a Gamma distribution, i.e.

$$\Lambda_j(t) \sim \Gamma(k_j(t), \theta_j)$$

where $k > 0$ is the shape parameter and $\theta > 0$ is the scale parameter in the Gamma distribution. We take different $\Lambda_j(t)$ to be independent as j changes.

The Gamma choice is convenient because it does not alter the tractability of the basic model. Indeed, we can still compute the characteristic function of the process Z_t in closed form as

$$\varphi^{\mathrm{G}}_{Z_T}(u) = \prod_{j=1}^{n} \left[\left(1 + \theta_j\left(1 - e^{i\alpha_j u}\right)\right)\right]^{-k_j(T)}$$

Since the constituent Poisson intensities are stochastic and independent of each other, also the GPL intensity $h_Z(t)$ is stochastic and distributed according to a linear combination of different and independent Gamma distributed random variables.

Remark 7.1. *The Gamma distribution assumption for $\Lambda_j(t) \sim \Gamma(k_j(t), \theta_j)$ at every time t is consistent with a Gamma process assumption for $\Lambda_j(t)$. Consider indeed a Gamma process with independent stationary increments, each increment between any two instant $s < t$ distributed as*

$$\Lambda_j(t) - \Lambda_j(s) \sim \Gamma(k_j(t) - k_j(s), \theta_j).$$

By taking the limit case we see that

$$\lambda_j(t)dt \sim \Gamma\left(dk_j(t), \theta_j\right).$$

Thanks to independence of increments and to the fact the the sum of two independent $\Gamma(k_1, \theta)$ and $\Gamma(k_2, \theta)$ is $\Gamma(k_1 + k_2, \theta)$ we have that $\Lambda_j(t)$ can be simulated at discrete instants by means of sums of independent Gamma random variables.

Finally we notice that the Gamma distribution of $\Lambda(t)$ is controlled by both parameters k and θ. In particular, recall that, omitting the index,

$$\mathbb{E}[\Lambda(t)] = k(t)\theta, \quad \mathrm{Var}[\Lambda(t)] = k(t)\theta^2.$$

Thus in this first extension we have parameters $k(T)$ to control the mean of the cumulated intensity in time T for each number of jumps in the loss but only one parameter θ to modify/calibrate the variance. In the next subsection we improve the model in this respect.

7.1.2 The piecewise-Gamma intensity GPL model

A further example is obtained by extending the Gamma case by letting the θ's vary over time, thus allowing for a larger control of the variance. Assume each θ_j to be piecewise constant w.r.t. times T_i, with $i = 1 \dots b$. Define $\Lambda_j(t)$ through its independent increments, distributed as

$$\Lambda_j(T_i) - \Lambda_j(T_{i-1}) \sim \Gamma(k_{i,j}, \theta_{i,j})$$

where $T_0 = 0$.

We have lost the gamma distribution of $\Lambda(T)$, since the fact that the increments of Λ are Gamma with different θ's renders their sums not Gamma. However, we still know the distribution of the sum, thanks to independence of the increments, and again we know the characteristic function of the loss in closed form as

$$\varphi_{Z_t}^{\mathrm{GPW}}(u) = \prod_{j=1}^{n} \prod_{h=1}^{b} \left[\left(1 + \theta_{h,j}\left(1 - e^{i\alpha_j u}\right)\right) \right]^{-k_{h,j}}$$

As in the Gamma case the GPL intensity (18) is stochastic.

7.1.3 The scenario GPL model

A different extension is as follows. By taking scenarios on the intensities we may easily extend our basic GPL model. In this model we assume the intensities in all the components to take different scenarios with different probabilities. Indeed, assume now that the (possibly time varying) intensities λ are indexed by a random variable I taking values $1, 2, \dots, m$ with (risk-neutral) probabilities $q_1, q_2, \dots, q_m$: λ_j^I is then a random intensity for the j-th Poisson process, depending on I. The related Poisson process is denoted by N_j^I, and the extended GPL process is denoted by

$$Z^I(t) = \sum_{j=1}^{n} \alpha_j N_j^I(t).$$

I is assumed to be independent of the exponential random variables triggering the jumps of the Poisson processes. Conditional on $I = i$, the intensities of the processes $N_1^I, \dots, N_n^I$ are $\lambda_1^i, \dots, \lambda_n^i$. This formulation does not spoil analytical tractability. Indeed, the characteristic function is now computed easily through iterated expectation:

$$\begin{aligned} \varphi_{Z_t}^{\mathrm{MIX}}(u) &= \mathbb{E}_0\left[\,\mathbb{E}_0\left[\exp(iuZ_t^I) \mid I\,\right]\,\right] = \sum_{k=1}^{m} q_k \,\mathbb{E}_0\left[\,\mathbb{E}_0\left[\exp(iuZ_t^I) \mid I = k\,\right]\,\right] \\ &= \sum_{k=1}^{m} q_k \exp\left[\sum_{j=1}^{n} \Lambda_j^k(t)\left(e^{i\alpha_j u} - 1\right)\right]. \end{aligned}$$

Any discounted payoff Π that can be priced under the basic GPL model is easily priced under the mixture extension:

$$\mathbb{E}_0[\,\Pi\,] = \mathbb{E}_0[\,\mathbb{E}_0[\,\Pi | I\,]\,] = \sum_{k=1}^{m} q_k \,\mathbb{E}_0\left[\,\Pi \mid I = k\,\right]$$

We obtain the usual linear combination of prices under each basic GPL model implied by each single intensity scenario.

7.1.4 The GPL-CIR model

A different and possibly more interesting extension is the CIR- Generalized Poisson (CIR-GPL) model

$$Z_t^{\mathrm{CIR}} = \sum_{j=1}^{n} \alpha_j N_j^{\mathrm{CIR}}(t), \qquad d\lambda_j(t) = k_j(\theta_j - \lambda_j(t))dt + \sigma_j \sqrt{\lambda_j(t)} dW_j,$$

with $2k_j\theta_j > \sigma_j^2$, and where the intensities of multiple defaults with different sizes follow different independent CIR processes.

The characteristic function of Z can be computed again in closed form, the calculation being similar to the bond price formula for the CIR interest rate model. Alternatively, jump diffusion JCIR intensities can be considered, maintaining tractability.

7.2 Loss dynamics

The GPL process (Z_t) can be considered as a driving process for the market relevant quantities, namely the cumulated portfolio loss $(\bar{L}_t)$ and the re-scaled number of defaults $(\bar{C}_t)$. The underlying GPL process Z is non-decreasing and takes arbitrarily large values, given large enough times. The portfolio cumulated loss and the re-scaled number of defaults processes are non-decreasing, but limited to the interval $[0,1]$. Thus, we consider the deterministic non-decreasing function $\Psi : \mathbb{N} \cup \{0\} \to [0,1]$ and we define the process Y_t, either the loss or the counting process, as

$$Y_t := \Psi(Z_t)$$

In the first part of the paper we use the GPL process to drive the re-scaled number of defaults, i.e. $\bar{C}_t := \Psi_{\bar{C}}(Z_t) := \min(Z_t/M, 1)$, where $M > 0$ is the number of names in the portfolio, while in this second part we follow the other way round by modelling directly the cumulated portfolio loss, i.e. $\bar{L}_t := \Psi_{\bar{L}}(Z_t) := \min(Z_t/M', 1)$, where $1/M'$, with $M' \geq M > 0$, is the minimum jump for the loss process. The quantity that is not modelled directly between $\bar{C}_t$ and $\bar{L}_t$ can be obtained from the one modelled directly through explicit assumptions on the recovery rate.

Remark 7.2. *If we model the loss process with the GPL model, i.e.* $\bar{L}_t := \Psi(Z_t)$*, the loss is bounded within the interval [0,1] by construction, but there is still the possibility that the loss jumps more than* M *times, where* M *is the number of names in the portfolio. If this is the case, we may check a-posteriori that the probability of such events is negligible. This happens in all our examples.*

The marginal distribution of the Y_t process can be calculated by "tabulation" from the discrete distribution of Z:

$$p_{Y_t}(y) = \sum_{z \in \Psi^{-1}(\{y\})} p_{Z_t}(z)$$

The intensity of Y_t can also be directly computed

$$h_Y(t) = \sum_{j=1}^{n} (\Psi(Z_{t^-} + \alpha_j) - \Psi(Z_{t^-}))\lambda_j(t) \tag{19}$$

Notice that the intensity of the process Y_t, in general, depends on the underlying GPL process Z. This behaviour is similar to the loss feedback present in the default intensities of Hawkes process, as shown in Giesecke and Goldberg (2005).

Example 7.3. *Consider the map $\Psi(x) := \min(x, M)$, where M is a positive constant*

$$Y_t := \Psi(Z_t) = \min(Z_t, M) = Z_t 1_{\{Z_t < M\}} + M 1_{\{Z_t \geq M\}}$$

Since Z_t has a known distribution, the distribution of Y_t can be easily derived as a byproduct. Indeed,

$$\mathbb{Q}\{ Y_t \leq y \} = \mathbb{Q}\{ Z_t < y \} 1_{\{y<M\}} + 1_{\{y \geq M\}}$$

The related density (defined on integer values since the law is discrete) is

$$p_{Y_t}(y) = p_{Z_t}(y) 1_{\{y<M\}} + \mathbb{Q}\{ Z_t \geq M \} 1_{\{y=M\}}$$

(in this case $\Psi^{-1}(\{k\}) = k$ for $k < M$ and $\Psi^{-1}(\{M\}) = \{M, M+1, M+2, \ldots\}$).

The transition probability for Y_t can be calculated as well, by using the independence of the Poisson's increments. We get for $s < t$:

$$\begin{aligned} \pi_Y(s, x; t, y) &= \pi_Z(s, x; t, y) 1_{\{x \leq y < M\}} + \mathbb{Q}\left\{ Z_t \geq M \;\middle|\; Z_s = x \right\} 1_{\{x \leq y = M\}} \\ &= p_{Z_t - Z_s}(y - x) 1_{\{x \leq y < M\}} + \mathbb{Q}\{ Z_t - Z_s \geq M - x \} 1_{\{x \leq y = M\}} \end{aligned}$$

The intensity of Y_t can also be directly computed, as explicitly done in the first part of the paper, where Y is used as default counting process, and is given by

$$h_Y(t) = \sum_{j=1}^{n} \min(\alpha_j, (M - Z_{t^-})^+)\lambda_j(t)$$

7.3 Recovery dynamics and intensities

The cumulated loss process $\bar{L}_t$ and the re-scaled number of defaults $\bar{C}_t$ must satisfy at any time t the arbitrage-free constraint

$$d\bar{L}_t = d\bar{C}_t(1 - R_t) \tag{20}$$

where we define R_t as the "recovery rate at default", assuming it is a $\mathcal{G}_t$-adapted and left continuous (and hence predictable) process taking values in the interval $[0, 1]$.

Remark 7.4. *The arbitrage-free constraint (20) leaves us with the freedom of defining only two processes among $\bar{L}_t$, $\bar{C}_t$ and R_t. The more natural approach would be modeling explicitly $(\bar{L}_t, R_t)$, obtaining $\bar{C}_t$, or modeling explicitly $(\bar{C}_t, R_t)$, obtaining $\bar{L}_t$. However, if we choose to model both $\bar{L}_t$ and $\bar{C}_t$ and to infer the recovery, we have to ensure that the resulting process R_t obtained by "inverting" (20) is indeed predictable.*

The recovery rate can be expressed also in terms of the intensities of the loss and default rate processes. By taking the expectation on both sides of equation (20) conditional on $\mathcal{G}_t$, we obtain

$$R_t = 1 - \frac{h_{\bar{L}}(t)}{h_{\bar{C}}(t)} \tag{21}$$

where the expression can be further expanded, by using equation (19), if we introduce the GPL process to define $\bar{L}_t$ and/or $\bar{C}_t$.

Equation (21) shows that the recovery rate at default is directly related to the intensities of both the loss and the default rate processes. Thus, the choice for the intensity dynamics does induce a dynamics for the recovery rate.

Example 7.5. *Consider the maps*

$$\Psi_{\bar{C}}(Z_t) := \min\left(\frac{Z_t}{M}, 1\right) \quad \text{and} \quad \Psi_{\bar{L}}(Z_t) := (1-\beta)\min\left(\frac{Z_t}{M}, 1\right),$$

where $0 < \beta \leq 1$ and $M > 0$. This choice ensures that R_t is $\mathcal{G}_t$-predictable and it corresponds to the default rate model presented in the first part of the paper.

In this second part of the paper we avoid the simultaneous specification of a dynamics for both the cumulated portfolio loss and the re-scaled number of defaults, since the market data mostly depend on the values of the expected loss. Thus, we choose to drive the loss process by means of a GPL model and, then, we take a minimalist assumption for the recovery rate, as described in Section 8 below. This will be sufficient to calibrate all available market data. Also, notice that recovery swaps, as stated in Albanese et al. (2005), are not liquid enough to be used to extract valuable information on recovery dynamics.

7.4 Spread dynamics

The valuation of credit index forward contracts or options requires the calculation of the forward index spread S_t given by equation (12), which in turn depends on the default intensity. Consider, for instance, the case of deterministic interest rates

$$S_T = \frac{1}{\Theta_T}\mathbb{E}_T\left[\int_T^{T_b} D(T,t)d\bar{L}_t\right] = \frac{\int_T^{T_b} D(T,t)\,\mathbb{E}_T[\,h_{\bar{L}}(t)\,]\,dt}{\sum_{i=1}^b \delta_i D(T,T_i)\left(1-\bar{C}_T - \int_T^{T_i}\mathbb{E}_T[\,h_{\bar{C}}(t)\,]\right)1_{\{T_i>T\}}}$$

where $h_{\bar{L}}(t)$ is the default intensity of the cumulated portfolio loss process and $h_{\bar{C}}(t)$ is the default intensity of the re-scaled default counting process.

The dynamics of the index S_t (spread dynamics) can be introduced within the GPL model by modelling the default intensities $h_{\bar{L}}(t)$ and $h_{\bar{C}}(t)$, either by explicitly adding stochasticity to the Poisson intensities $\lambda_j(t)$, e.g. the Gamma , scenario or CIR extensions of the GPL model seen above or, indirectly, by choosing particular deterministic maps Ψ transforming Z, leading to appropriate loss or default counting processes.

8 Numerical Results

In the first part of the paper we considered the calibration of the default-rate based GPL model to credit index and CDO tranche quotes. In this second part we do the same but we resort to the loss based GPL model. Since these products depend only on the expectation of the portfolio cumulated tranched loss ($\bar{L}_t$) and of the re-scaled number of defaults ($\bar{C}_t$), we avoid to directly introduce stochasticity either on the process intensities or on the recovery rate. This stochasticity would help for modeling spread dynamics for tranche or index options pricing, for example.

We map the GPL process Z by setting the quantity $1/M'$ as minimum jump for the cumulated loss process $\bar{L}_t$, with $M' \geq M > 0$, where M is the number of names in the portfolio, usually $M = 125$. Further, we try to specify as little as possible of the recovery dynamics, since the credit index quotes contain only little information about it and tranches no information at all, see also Remark 6.1.

The GPL model specification is:

$$Z_t := \sum_{j=1}^{n} \alpha_j N_j(t) \quad \text{and} \quad \bar{L}_t := \Psi_{\bar{L}}(Z_t) := \min\left(\frac{Z_t}{M'}, 1\right) \tag{22}$$

where $M' \geq M > 0$ and each Poisson mode N_j has a deterministic piecewise-constant intensity $\lambda_j(t)$.

We do not characterize completely the re-scaled default counting process $\bar{C}_t$, but we give only its expectation values. This is done because calibration payoffs depend on $\bar{C}$ only via said expectation:

$$\mathbb{E}_0\left[\,\bar{C}_t\,\right] := \frac{1}{1-\mathcal{R}}\,\mathbb{E}_0\left[\,\bar{L}_t\,\right] \quad \text{with} \quad 0 \leq \mathcal{R} < 1 - \mathbb{E}_0\left[\,\bar{L}_{T_b}\,\right] \tag{23}$$

where the range of definition of the constant $\mathcal{R}$ is taken in order to ensure that at each time t the expected value of the re-scaled number of defaults is greater, or equal to, the cumulated portfolio loss, and that both be smaller or equal to one. As a direct consequence we avoid to introduce an explicit dynamics for the recovery rate too.

In the following we take $\mathcal{R} = 30\%$ as reference value for the recovery rate in the DJi-TRAXX Europe market for spot and forward contracts. The quality of our calibration below is not altered if we select a value $\mathcal{R} = 40\%$ resembling the recovery typically used in simplified quoting mechanisms in the market. In terms of the mean recovery $\mathcal{R}$ the credit index spread can be recast in the following form

$$S_T = \frac{(1-\mathcal{R})\,\mathbb{E}_T\left[\,\int_T^{T_b} D(T,t)\bar{L}_t\,\right]}{\mathbb{E}_T\left[\,\sum_{i=0}^{b} \delta_i D(T,T_i)(1-\mathcal{R}-\bar{L}_{T_i})1_{\{T>T_i\}}\,\right]}$$

8.1 Calibration procedure

The model parameters fixed by the calibration procedure are the amplitudes $\alpha_j \in \{m \in \mathbb{N} : m \leq M'\}$ with $j = 1 \ldots n$, and the cumulated intensities $\Lambda_j(T)$, which are real non-decreasing piecewise linear functions in the tranche maturity.

The optimal values for the amplitudes α are selected in the following way:

1. Fix the minimum jump size to $1/M'$ by choosing the integer $M' \geq M > 0$.
2. Find the best integer value for α_1 by calibrating the cumulated intensity Λ_1 for each value of α_1 in the range $[1, M']$,
3. Add the amplitude α_2 and find its best integer value by calibrating the cumulated intensities Λ_1 and Λ_2, starting from the previous value for Λ_1 as a guess, for each value of α_2 in the range $[1, M']$,
4. Repeat the previous step for α_i with $i = 3$ and so on, by calibrating the cumulated intensities $\Lambda_1, \ldots, \Lambda_i$, starting from the previously found $\Lambda_1, \ldots, \Lambda_{i-1}$ as initial guess, until the calibration error is under a given threshold or until the intensity Λ_i can be considered negligible.
5. Check a-posteriori that the probability to have more than M jumps is negligible and that the value of $\mathcal{R}$ is within the arbitrage-free range given in (23).

The objective function f to be minimized in the calibration is the squared sum of the errors shown by the model to recover the tranche and index market quotes weighted by market bid-ask spreads:

$$f(\alpha, \Lambda) = \sum_i \epsilon_i^2, \quad \epsilon_i = \frac{x_i(\alpha, \Lambda) - x_i^{\text{Mid}}}{x_i^{\text{Bid}} - x_i^{\text{Ask}}} \tag{24}$$

where the x_i, with i running over the market quote set, are the index values S_0 for DJi-TRAXX index quotes, and either the index periodic premium rates $S_0^{A,B}$ or the upfront premium rates $U^{A,B}$ for the DJi-TRAXX tranche quotes.

8.2 Calibration results

The GPL model is calibrated to the market quotes observed weekly from May 6, 2005 to October 18, 2005. All calibrations assume $\mathcal{R} = 30\%$. We try as minimum loss jumps the values 2bp, 10bp and 50bp corresponding, respectively, to M' equal to 5000, 1000 and 200.

The behaviour of the mean calibration error for the three different choices of M' is quite similar and within about one bid-ask spread. Also the values of the Poisson amplitudes are quite stable across the calibration dates. Indeed, in six months we observe at most four changes in their values, as shown in Table 8.

Consider, as a first example, the calibration date May 13, 2005. Tranche data and DJi-TRAXX fixings are listed in Table 6. We list in Table 9 the calibration result and the values of the calibrated parameters. The calibration errors are very low for all maturities.

Consider, as a second example, the calibration date October 11, 2005. Tranche data and DJi-TRAXX fixings are listed in Table 7. We list in Table 10 the calibration results and the values of the calibrated parameters. The calibration errors show that the ten

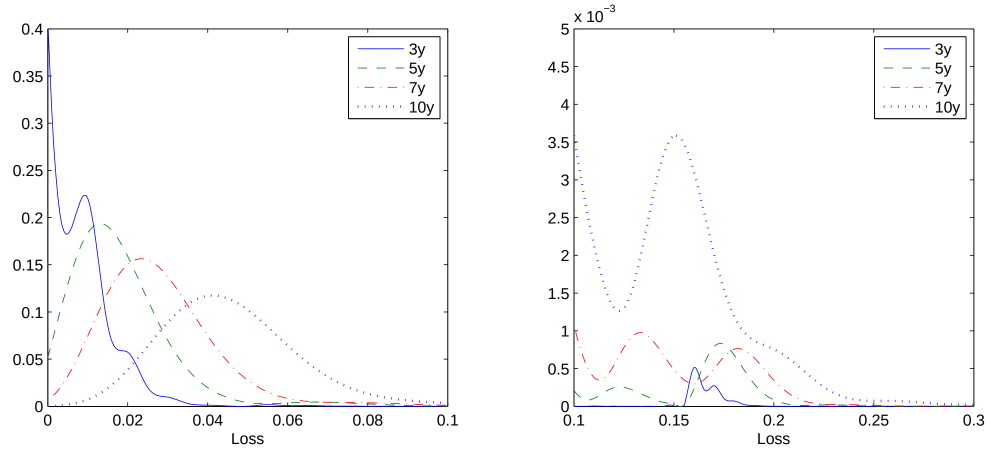

Figure 2: Loss distribution evolution of the GPL model with minimum jump size of 50bp at all the quoted maturities up to ten years, drawn as a continuous line.

year equity tranche is not correctly priced. We find such mispricing in many calibration examples, in particular after October 2005.

If we decrease the minimum loss jump size $1/M'$, we observe that the calibration error does not decrease significatively, in particular the difference between the $M' = 1000$ calibration and the $M' = 5000$ one is small. For instance the mispricing on the ten year tranches as the minimum jump size decreases is given in Table 11.

Remark 8.1. *As the minimum jump size decreases, the loss distribution becomes noisier, due to the presence of small amplitudes. Further, very small modes, appearing when the minimum jump size is as small as a few basis points, may violate the requirement that the loss process jumps less than M times (see remark 7.2). We tried also calibrations with M' less than* 200*, i.e. with minimum loss jump greater than* 50*bp. In this case the calibration error grows quickly. Indeed, the minimum jump size, in this case, becomes greater than the typical portfolio loss given when one name defaults.*

The loss distribution implied by the GPL model is multi-modal and the probability mass moves towards larger loss values as the maturity increases, as already noticed in the first part of the paper. These features are shared by different approaches. For instance, static models, such as perfect copula approach by Hull and White (2005) or Torresetti et al (2006a), or the implied expected tranched-loss surface by Walker (2006) or Torresetti et al. (2006b), predict multi-modal loss distributions. The evolution of the implied loss distribution is shown in Figure 2.

The credit dynamic correlation model by Albanese et al. (2005) shows implied loss distributions whose modes tend to group, as the maturity increases, leading to a distribution approaching normality. The GPL model reproduces this behaviour as shown in Figure 3.

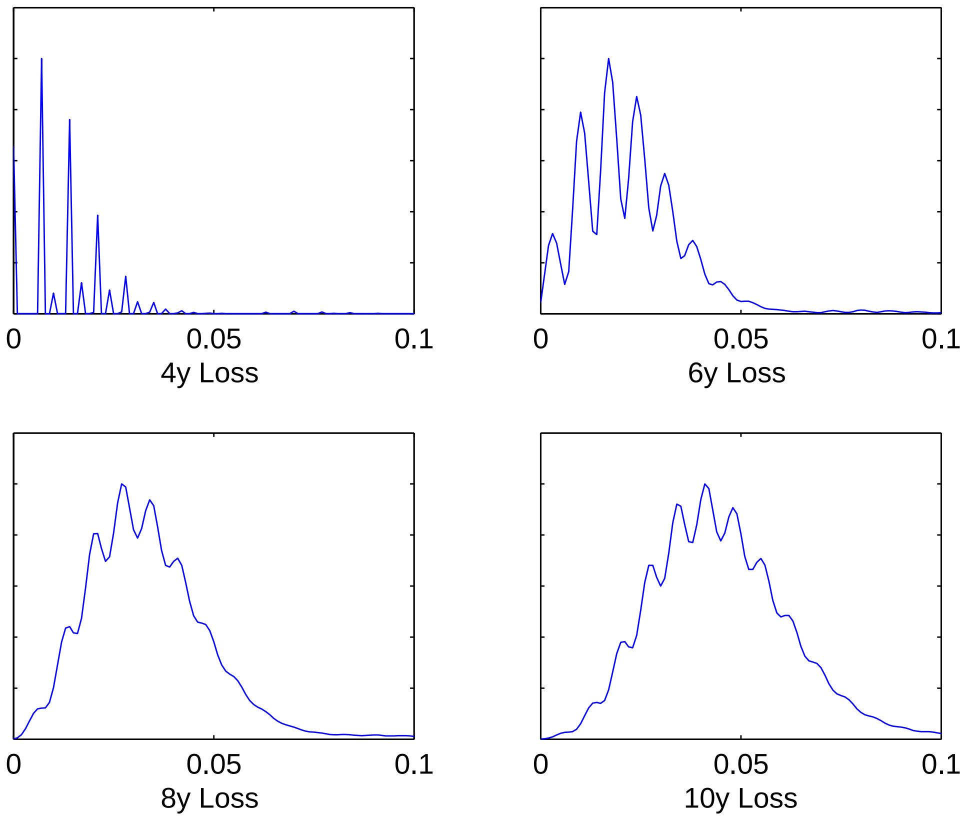

Figure 3: Probability density of the cumulate portfolio loss process with minimum loss jump size of 10bp for $4y$, $6y$, $8y$ and $10y$ maturities drawn as a continuous line on calibration date October,11 2005.

9 Conclusions

In this second part of the paper we introduced the stochastic intensity versions of the basic GPL process. We introduced explicit stochastic intensities maintaining analytical tractability, leading to the Gamma , piecewise Gamma, scenario and CIR GPL processes. Then we explained how one can model directly the loss dynamics rather than the number of defaults, as we did instead in the first part. We introduced general recovery assumptions, illustrating the link between recovery and pool intensities. We introduced the index spread in terms of intensities explaining how this is obtained in our models, with possible benefits of the stochastic intensity extensions. A similar approach holds also for tranche spreads, which can be helpful when dealing with tranche options. Finally we focused on calibration results and stability when the *loss based* GPL model is used with some minimalist recovery assumptions. Further work concerns the extension of the calibration to index options, and examples of valuation of tranche options, leveraged super senior tranches and other correlation payoffs. Also, consistency with single name is to be investigated either through random thinning or the Marhsal-Olkin copula.

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A Appendix: Inputs and Numerical Results

Calibration to 2006 data

Date	Rate	Date	Rate	Date	Rate	Date	Rate
22-Mar-06	2.58%	19-Sep-08	3.31%	18-Sep-13	3.61%	21-Mar-11	3.50%
21-Jun-06	2.72%	19-Dec-08	3.34%	18-Dec-13	3.62%	20-Jun-11	3.51%
20-Sep-06	2.84%	23-Mar-09	3.36%	20-Mar-14	3.63%	19-Sep-11	3.52%
20-Dec-06	2.95%	22-Jun-09	3.38%	19-Jun-14	3.64%	19-Dec-11	3.53%
22-Mar-07	3.04%	21-Sep-09	3.40%	18-Sep-14	3.65%	20-Mar-12	3.54%
21-Jun-07	3.11%	21-Dec-09	3.42%	18-Dec-14	3.66%	19-Jun-12	3.56%
20-Sep-07	3.17%	22-Mar-10	3.44%	20-Mar-15	3.68%	18-Sep-12	3.57%
20-Dec-07	3.21%	21-Jun-10	3.45%	19-Jun-15	3.69%	18-Dec-12	3.58%
25-Mar-08	3.25%	20-Sep-10	3.47%	18-Sep-15	3.70%	20-Mar-13	3.59%
20-Jun-08	3.28%	20-Dec-10	3.48%	18-Dec-15	3.71%	19-Jun-13	3.60%

Table 1: EUR zero-coupon continuously-compounded spot rates (ACT/365).

	Att-Det	**Maturities**		
		3y	5y	7y
Index		20(1)	35(1)	48(1)
Tranche	0-3	500(20)	2655(25)	4825(25)
	3-6	7.50(2.50)	67.50(1.00)	225.50(2.50)
	6-9	1.25(0.75)	22.00(1.00)	51.00(1.00)
	9-12	0.50(0.25)	10.50(1.00)	28.50(1.00)
	12-22	0.15(0.05)	4.50(0.50)	10.25(0.50)

Table 2: DJi-TRAXX index and tranche quotes in basis points on March 6, 2006, along with the bid-ask spreads. Index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium. See section 1.

	Att-Det	Maturities	
		5y	7y
Index		35(1)	48(1)
tranchelet	0-1	6100(200)	7400(300)
	1-2	1085(70)	5025(300)
	2-3	393(45)	850(60)
Tranche	0-3	2600(50)	4788(50)
	3-6	71.00(2.00)	210.00(5.00)
	6-9	22.00(2.00)	49.00(2.00)
	9-12	10.00(2.00)	29.00(2.00)
	12-22	4.25(1.00)	11.00(1.00)

Table 3: DJi-TRAXX index, tranche and tranchelets OTC quotes in basis points on March 1, 2006. Index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium. The five year tranchelets with 2% and 3% detachment points and the seven year tranchelets with 3% detachment point are quoted through the periodic premium, whereas the other tranchelets are quoted as an upfront premium. See section 1.

	Att-Det	Maturities		
		3y	5y	7y
Index		-0.4	-0.2	-0.9
Tranche	0-3	0.1	0.0	-0.7
	3-6	0.0	0.0	0.7
	6-9	0.0	0.0	-0.2
	9-12	0.0	0.0	0.0
	12-22	0.0	0.0	0.2

$\boldsymbol{\alpha}$	$\boldsymbol{\Lambda(T)}$		
	3y	5y	7y
1	0.535	2.366	4.930
3	0.197	0.266	0.267
16	0.000	0.007	0.024
21	0.000	0.003	0.003
88	0.000	0.002	0.007

Table 4: Left side: calibration error calculated with respect to the bid-ask spread (i.e. ϵ_i in (10)) for tranches quoted by the market (see Table 2). Right side: cumulated intensities, integrated up to tranche maturities, of the basic GPL model. Each row corresponds to a different Poisson component with jump amplitude α. Recovery rate is 40%.

	Att-Det	Maturities	
		5y	7y
Index		-0.8	-2.1
tranchelet	0-1	1.1	-1.4
	1-2	1.7	-0.6
	2-3	-0.1	-0.4
Tranche	0-3	0.1	0.4
	3-6	-1.9	0.2
	6-9	0.4	0.6
	9-12	2.8	0.9
	12-22	-0.4	-1.5

$\boldsymbol{\alpha}$	$\boldsymbol{\Lambda(T)}$	
	5y	7y
1	0.834	3.336
2	1.070	1.070
13	0.008	0.015
21	0.004	0.013
104	0.002	0.007

Table 5: Left side: calibration error calculated with respect to the bid-ask spread for tranches quoted by the market (see Table 3). Right side: cumulated intensities, integrated up to tranche maturities, of the basic GPL model. Each row corresponds to a different Poisson component with jump amplitude α. Recovery rate is 40%.

Calibration to 2005 data

	Att-Det	Maturities			
		3y	5y	7y	10y
Index		38(4)	54(1)	65(3)	77(2)
Tranche	0-3	2060(100)	4262(118)	5421(384)	6489(124)
	3-6	72(10)	173(68)	398(40)	590(20)
	6-9	28(6)	57(6)	141(17)	188(15)
	9-12	13(2)	31(5)	72(20)	87(15)
	12-22	3(1)	21(3)	42(13)	60(10)

Table 6: DJi-TRAXX index and tranche quotes in basis points on May 13, 2005, along with the bid-ask spreads. Index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium. See section 1.

	Att-Det	Maturities			
		3y	5y	7y	10y
Index		23(2)	38(1)	47(1)	58(1)
Tranche	0-3	762(26)	137(26)	4862(76)	5862(74)
	3-6	20(10)	95(1)	200(3)	515(10)
	6-9	7(6)	28(1)	43(2)	100(4)
	9-12		12(2)	27(4)	54(5)
	12-22		7(1)	13(2)	23(3)

Table 7: DJi-TRAXX index and tranche quotes in basis points on October 11, 2005, along with the bid-ask spreads. Index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium. See section 1.

50bp	**Poisson's Amplitudes**						
Date	**1**	**2**	**3**	**4**	**5**	**6**	**7**
06-May-05	0.50%	1.50%	4.00%	6.00%	9.50%	39.50%	92.50%
02-Sep-05	0.50%	1.00%	4.00%	5.50%	12.50%	39.00%	100.00%
11-Oct-05	0.50%	1.00%	5.50%	11.00%	14.50%	16.00%	96.00%

10bp	**Poisson's Amplitudes**						
Date	**1**	**2**	**3**	**4**	**5**	**6**	**7**
06-May-05	0.10%	1.50%	4.60%	5.90%	9.60%	39.60%	53.00%
05-Aug-05	0.20%	1.10%	1.40%	8.10%	11.30%	49.00%	62.40%
11-Oct-05	0.10%	0.70%	1.00%	6.30%	11.50%	14.50%	93.70%

2bp	**Poisson's Amplitudes**						
Date	**1**	**2**	**3**	**4**	**5**	**6**	**7**
06-May-05	0.02%	1.50%	5.26%	9.64%	17.58%	39.64%	99.78%
12-Aug-05	0.38%	1.06%	1.14%	7.38%	12.24%	41.34%	99.80%
03-Oct-05	0.02%	0.98%	1.16%	7.52%	9.74%	43.34%	65.16%
11-Oct-05	0.16%	0.68%	1.00%	6.30%	10.98%	14.46%	94.90%

Table 8: Values of the Poisson's amplitudes α, normalized to 1, for different values of the minimum loss jump $1/M'$. Only the calibration dates between 6 May 2005 and 18 October 2005 where the α values change are listed.

	Att-Det	Maturities			
		3y	5y	7y	10y
Index		0.0	-0.1	0.3	0.0
Tranche	0-3	0.0	0.1	0.2	-0.2
	3-6	0.0	0.0	-0.2	0.0
	6-9	0.0	0.0	-0.3	0.1
	9-12	-0.1	0.1	-0.1	0.4
	12-22	0.0	0.0	-0.2	-0.3

α	$\Lambda(T)$			
	3y	5y	7y	10y
1	1.955	3.726	4.464	7.694
3	0.000	0.062	0.305	0.305
8	0.016	0.033	0.011	0.011
12	0.004	0.013	0.026	0.026
19	0.006	0.006	0.017	0.017
72	0.000	0.009	0.026	0.049
185	0.000	0.002	0.002	0.008

Table 9: Left side: calibration error calculated with respect to the bid-ask spread for tranches quoted by the market on May 13, 2005. Right side: cumulated intensities, integrated up to tranche maturities, of the GPL model with $M' = 200$. Each row corresponds to a different Poisson component with jump amplitude α. Recovery rate is 30%.

	Att-Det	Maturities			
		3y	5y	7y	10y
Index		0.0	0.0	0.1	0.1
Tranche	0-3	-0.1	0.1	-1.2	2.1
	3-6	-0.1	-0.1	0.3	-1.0
	6-9	0.0	-0.1	0.3	0.9
	9-12		0.4	-0.8	-0.8
	12-22		0.0	0.0	0.0

α	$\Lambda(T)$			
	3y	5y	7y	10y
1	0.441	2.498	4.466	7.555
2	0.435	0.435	0.435	0.671
11	0.004	0.023	0.023	0.023
22	0.000	0.001	0.006	0.030
29	0.000	0.000	0.001	0.001
32	0.001	0.004	0.004	0.004
192	0.000	0.001	0.005	0.011

Table 10: Same as Table 9 on October 11, 2005 (the three year maturity quotes lack two tranches).

	Att-Det	Maturity 10y		
		50bp	**10bp**	**2bp**
Tranche	0-3	2.1	1.8	1.8
	3-6	-1.0	-1.0	-1.0
	6-9	0.9	0.9	0.9
	9-12	-0.8	-0.9	-0.8
	12-22	0.0	0.2	0.0

Table 11: Calibration error given by the GPL model for different minimum loss jump size $1/M'$ calculated with respect to the bid-ask spread for tranches with a maturity of ten years quoted by the market on October 11, 2005. Recovery rate is 30%.