

# COUNTERPARTY RISK VALUATION FOR ENERGY-COMMODITIES SWAPS

## Impact of volatilities and correlation

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### *Abstract*

It is commonly accepted that Commodities futures and forward prices, in principle, agree under some simplifying assumptions. One of the most relevant assumptions is the absence of counterparty risk. Indeed, due to margining, futures have practically no counterparty risk. Forwards, instead, may bear the full risk of default for the counterparty when traded with brokers or outside clearing houses, or when embedded in other contracts such as swaps. In this paper we focus on energy commodities and on Oil in particular. We use a hybrid commodities-credit model to assess impact of counterparty risk in pricing formulas, both in the gross effect of default probabilities and on the subtler effects of credit spread volatility, commodities volatility and credit-commodities correlation. We illustrate our general approach with a case study based on an oil swap, showing that an accurate valuation of counterparty risk depends on volatilities and correlation and cannot be accounted for precisely through a pre-defined multiplier.

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## 1 Introduction

In this paper we consider counterparty risk for commodities payoffs in presence of correlation between the default event and the underlying commodity, while taking into account volatilities both for credit and commodities. We focus on Oil but much of our reasoning can be adapted to other commodities with similar characteristics (storability, liquidity, and similar seasonality).

Past work on *pricing* counterparty risk for different asset classes is in Sorensen and Bollier (1994), Brigo and Masetti (2006) and Brigo and Pallavicini (2007, 2008) for interest rate swaps and exotics underlyings. Leung and Kwok (2005) and Brigo and Chourdakis (2008) worked on counterparty risk for credit (CDS) underlyings.

Here we analyze in detail counterparty-risky (or default-risky) Oil forward and swaps contracts.

In general the reason to introduce counterparty risk when evaluating a contract is linked to the fact that many financial contracts are traded over the counter, so that the credit quality of the counterparty can be relevant. This is particularly appropriated when thinking of the different defaults experienced by some important companies during the last years, especially in the energy sector.

Earlier works in counterparty risk for commodities include for example Cannabaro, Picoult and Wilde (2005), who analyze this notion more from a capital adequacy/ risk management point of view. In particular, their approach is not dynamical and does not consider explicitly credit spread volatility and especially correlation between the underlying commodity and credit spread. In our approach wrong way risk is modeled through said correlation. Mostly, however, the difference is in the purpose. We are valuing counterparty risk more from a pricing than a risk management perspective, resorting to a fully arbitrage free and fine-tuned risk neutral approach. This is why all our processes are calibrated to liquid market information both on forward curves and volatilities. Correlations are harder to estimate but we analyze their impact by letting them range across a set of possible values. A limitation of our approach is that we refer to a single counterparty.

In general we are looking at the problem from the viewpoint of a safe (default-free) institution entering a financial contract with another counterparty having a positive probability of defaulting before the final maturity. We formalize the general and reasonable fact that the value of a generic claim subject to counterparty risk is always smaller than the value of a similar claim having a null default probability, expressing the discrepancy in precise quantitative terms.

We consider Credit Default Swaps for the counterparty as liquid sources of market default probabilities. Different models can be used to calibrate CDS data and obtain default probabilities: here we resort to Brigo and Alfonsi (2005) stochastic intensity model, whose jump extension with analytical formulas for CDS options is illustrated in Brigo and El-Bachir (2008).

As a model for oil we adopt a two factor model shaping both the short term deviation in prices and the equilibrium price level, as in Smith and Schwartz (2000). This model can be shown to be equivalent to a more classical convenience yield model like in Gibson and Schwartz (1990), and a stochastic volatility extension of a similar approach is considered in Geman (2000). What is modeled is the oil spot price, under the implicit assumption that such a spot price process exists. This is not true for electricity, for example, and even for markets like crude oil where spot prices are quoted daily, the exact meaning of the spot is difficult to single out. Nonetheless, we assume, along with most of the industry and with Carmona and Ludkowski (2004), that there is a traded spot asset.

In the paper we find that counterparty risk has a relevant impact on the products prices and that, in turn, correlation between oil and credit spreads of the counterparty has a relevant

impact on the adjustment due to counterparty risk. Similarly, oil and credit spread volatilities have reasonable impacts on the adjustment. The impact patterns do not involve the peculiar behaviour one observes in the case of credit underlyings, observed in Brigo and Chourdakis (2008). Nonetheless, the impact is quantitatively relevant, and we illustrate this with a case study based on an oil swap.

The paper is organized as follows: Section 2 lays down the general framework for the valuation of counterparty risk. In section 3 we present the CIR++ specification which serves as the credit model, and in section 4 we outline the two-factor Smith and Schwartz commodity model. Sections 5 and 6 illustrate the counterparty adjustments for forwards and swaps respectively. An example, based on a swap contract with between a bank and an airline company is presented in section 7.

## 2 General valuation of counterparty risk

We denote by  $\tau$  the default time of the counterparty and we assume the investor who is considering a transaction with the counterparty to be default-free. We place ourselves in a probability space  $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ . The filtration  $(\mathcal{G}_t)_t$  models the flow of information of the whole market, including credit and defaults.  $\mathbb{Q}$  is the risk neutral measure. This space is endowed also with a right-continuous and complete sub-filtration  $\mathcal{F}_t$  representing all the observable market quantities but the default event (hence  $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$  where  $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$  is the right-continuous filtration generated by the default event). We set  $\mathbb{E}_t(\cdot) := \mathbb{E}(\cdot | \mathcal{G}_t)$ , the risk neutral expectation leading to prices.

Let us call  $T$  the final maturity of the payoff we need to evaluate. If  $\tau > T$  there is no default of the counterparty during the life of the product and the counterparty has no problems in repaying the investors. On the contrary, if  $\tau \leq T$  the counterparty cannot fulfill its obligations and the following happens. At  $\tau$  the Net Present Value (NPV) of the residual payoff until maturity is computed: If this NPV is negative (respectively positive) for the investor (defaulted counterparty), it is completely paid (received) by the investor (counterparty) itself. If the NPV is positive (negative) for the investor (counterparty), only a recovery fraction  $R_{EC}$  of the NPV is exchanged.

Let us call  $\Pi^D(t, T)$  (sometimes abbreviated into  $\Pi^D(t)$ ) the discounted payoff of a generic claim at  $t$  under counterparty risk. This is the sum of all cash flows from  $t$  to  $T$ , each discounted back at  $t$ , and under counterparty risk. This is a stochastic payoff, whose price would be given by risk neutral expectation. We denote by  $\Pi(t, T)$  the analogous quantity when counterparty risk is absent, or when the counterparty is default free. All payoffs are seen from the point of view of the “investor” (i.e. the company facing counterparty risk). Then we have  $\text{NPV}(\tau) = \mathbb{E}_\tau\{\Pi(\tau, T)\}$  and

$$\begin{aligned} \Pi^D(t) &= \mathbf{1}_{\{\tau > T\}} \Pi(t, T) + \\ &\quad \mathbf{1}_{\{t < \tau \leq T\}} \left[ \Pi(t, \tau) + D(t, \tau) \left( R_{EC} (\text{NPV}(\tau))^+ - (-\text{NPV}(\tau))^+ \right) \right] \end{aligned} \quad (2.1)$$

being  $D(u, v)$  the stochastic discount factor at time  $u$  for maturity  $v$ . This last expression is the general price of the payoff under counterparty risk. Indeed, if there is no early counterparty default this expression reduces to risk neutral valuation of the payoff (first term in the right hand side); in case of early default, the payments due before default occurs are received (second term), and then if the residual net present value is positive only a recovery of it is received (third term), whereas if it is negative it is paid in full (fourth term).

Calling  $\Pi(t)$  the discounted payoff for an equivalent claim with a default-free counterparty, i.e.  $\Pi(t) = \Pi(t, T)$ , it is possible to prove the following

**Proposition 2.1. (General counterparty-risk credit-valuation adjustment (CR-CVA) formula).** *At valuation time  $t$ , and on  $\{\tau > t\}$ , the price of our payoff under counterparty risk is*

$$\mathbb{E}_t\{\Pi^D(t)\} = \mathbb{E}_t\{\Pi(t)\} - \underbrace{L_{GD} \mathbb{E}_t\{\mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) (NPV(\tau))^+\}}_{\text{Positive CR-CVA}} \quad (2.2)$$

where  $L_{GD} = 1 - R_{EC}$  is the Loss Given Default and the recovery fraction  $R_{EC}$  is assumed to be deterministic. It is clear that the value of a defaultable claim is the value of the corresponding default-free claim minus an option part, in the specific a call option (with zero strike) on the residual NPV giving nonzero contribution only in scenarios where  $\tau \leq T$ . Counterparty risk adds an optionality level to the original payoff.

For a proof see for example Brigo and Masetti (2006).

Notice finally that the previous formula can be approximated as follows. Take  $t = 0$  for simplicity and write, on a discretization time grid  $T_0, T_1, \dots, T_b = T$ ,

$$\begin{aligned} \mathbb{E}[\Pi^D(0, T_b)] &= \mathbb{E}[\Pi(0, T_b)] - L_{GD} \sum_{j=1}^b \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau \leq T_j\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+] \\ &\approx \mathbb{E}[\Pi(0, T_b)] - \underbrace{L_{GD} \sum_{j=1}^b \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau \leq T_j\}} D(0, T_j) (\mathbb{E}_{T_j} \Pi(T_j, T_b))^+] }_{\text{approximated (positive) adjustment}} \end{aligned} \quad (2.3)$$

where the approximation consists in postponing the default time to the first  $T_i$  following  $\tau$ . From this last expression, under independence between  $\Pi$  and  $\tau$ , one can factor the outer expectation inside the summation in products of default probabilities times option prices. This way we would not need a default model for the counterparty but only survival probabilities and an option model for the underlying market of  $\Pi$ . This is what led to earlier results on swaps with counterparty risk in interest rate payoffs in Brigo and Masetti (2006). In this paper we do not assume zero correlation, so that in general we need to compute the counterparty risk without factoring the expectations.

### 3 Default modeling assumptions

In this section we consider a reduced form model that is stochastic in the default intensity for the counterparty. We will later correlate the credit spread of this model with the underlying commodity model.

More in detail, we assume that the counterparty default intensity is  $\lambda$ , and we denote the cumulated intensity by  $\Lambda(t) = \int_0^t \lambda(s) ds$ . We assume intensities to be strictly positive, so that  $t \mapsto \Lambda$  are invertible functions.

We assume deterministic default-free instantaneous interest rate  $r$  (and hence deterministic discount factors  $D(s, t, \dots)$ ), although our analysis would work well even with stochastic rates independent of oil and credit spreads.

We set ourselves in a Cox process setting, where

$$\tau = \Lambda^{-1}(\xi),$$

with  $\xi$  standard (unit-mean) exponential random variable.

### 3.1 CIR++ stochastic intensity models

For the stochastic intensity model we set

$$\lambda(t) = y(t) + \psi(t; \beta), \quad t \geq 0, \quad (3.1)$$

where  $\psi$  is a deterministic function, depending on the parameter vector  $\beta$  (which includes  $y_0$ ), that is integrable on closed intervals. The initial condition  $y_0$  is one more parameter at our disposal: We are free to select its value as long as

$$\psi(0; \beta) = \lambda_0 - y_0.$$

We take  $y$  to be a Cox Ingersoll Ross process (see for example Brigo and Mercurio (2001) or (2006)):

$$dy(t) = \kappa(\mu - y(t))dt + \nu\sqrt{y(t)}dZ_y(t),$$

where the parameter vector is  $\beta = (\kappa, \mu, \nu, y_0)$ , with  $\kappa, \mu, \nu, y_0$  positive deterministic constants. As usual,  $Z_y$  is a standard Brownian motion processes under the risk neutral measure, representing the stochastic shock in our dynamics. We assume the origin to be inaccessible, i.e.

$$2\kappa\mu > \nu^2.$$

We will often use the integrated quantities

$$\Lambda(t) = \int_0^t \lambda_s ds, \quad Y(t) = \int_0^t y_s ds, \quad \text{and} \quad \Psi(t, \beta) = \int_0^t \psi(s, \beta) ds.$$

### 3.2 CIR++ model: CDS calibration

Since we are assuming deterministic rates, the default time  $\tau$  and interest rate quantities  $r, D(s, t), \dots$  are trivially independent. It follows that the (receiver) CDS valuation at time 0 becomes model independent and is given by the formula

$$\text{CDS}_{a,b}(0, S_{a,b}, L_{\text{GD}}; \mathbb{Q}(\tau > \cdot)) = S_{a,b} \left[ - \int_{T_a}^{T_b} P(0, t)(t - T_{\gamma(t)-1}) d_t \mathbb{Q}(\tau \geq t) \right] \quad (3.2)$$

$$+ \sum_{i=a+1}^b P(0, T_i) \alpha_i \mathbb{Q}(\tau \geq T_i) \quad (3.3)$$

$$+ L_{\text{GD}} \left[ \int_{T_a}^{T_b} P(0, t) d_t \mathbb{Q}(\tau \geq t) \right] \quad (3.4)$$

This means that if we strip survival probabilities from CDS in a model independent way at time 0, to calibrate the market CDS quotes we just need to make sure that the survival probabilities we strip from CDS are correctly reproduced by the CIR++ model. Since the survival probabilities in the CIR++ model are given by

$$\mathbb{Q}(\tau > t)_{\text{model}} = \mathbb{E}(e^{-\lambda(t)}) = \mathbb{E} \exp(-\Psi(t, \beta) - Y(t)) \quad (3.5)$$

we just need to make sure

$$\mathbb{E} \exp(-\Psi(t, \beta) - Y(t)) = \mathbb{Q}(\tau > t)_{market}$$

from which

$$\Psi(t, \beta) = \ln \left( \frac{\mathbb{E}(e^{-Y(t)})}{\mathbb{Q}(\tau > t)_{market}} \right) = \ln \left( \frac{P^{CIR}(0, t, y(0); \beta)}{\mathbb{Q}(\tau > t)_{market}} \right) \quad (3.6)$$

where we choose the parameters  $\beta$  in order to have a positive function  $\psi$  (i.e. an increasing  $\Psi$ ) and  $P^{CIR}$  is the closed form expression for bond prices in the time homogeneous CIR model with initial condition  $y_0$  and parameters  $\beta$  (see for example Brigo and Mercurio (2001, 2006)). Thus, if  $\psi$  is selected according to this last formula, as we will assume from now on, the model is easily and automatically calibrated to the market survival probabilities for the counterparty (possibly stripped from CDS data).

Once we have done this and calibrated CDS data through  $\psi(\cdot, \beta)$ , we are left with the parameters  $\beta$ , which can be used to calibrate further products. However, this will be interesting when single name option data on the credit derivatives market will become more liquid. Currently the bid-ask spreads for single name CDS options are large and suggest to either consider these quotes with caution, or to try and deduce volatility parameters from more liquid index options. At the moment we content ourselves of calibrating only CDS's. To help specifying  $\beta$  without further data we set some values of the parameters implying possibly reasonable values for the implied volatility of hypothetical CDS options on the counterparty.

#### 4 Commodity model

We consider crude oil as a first important case.

Suppose we have a airline company that buys a forward contract on oil from a bank with a very high credit quality, so that we assume the bank to be default-free. The bank wants to charge counterparty risk to the airline in defining the forward price, as there is no collateral posted and no margining is occurring.

As a model for oil we adopt a two factor model shaping both the short term deviation in prices and the equilibrium price level, as in Smith and Schwartz (2000). This model can be shown to be equivalent to a more classical convenience yield model like in Gibson and Schwartz (1990), and a stochastic volatility extension of a similar approach is considered in Geman (2000). What is modeled is the oil spot price, under the implicit assumption that such a spot price process exists. This is not true for electricity, for example, and even for markets like crude oil where spot prices are quoted daily, the exact meaning of the spot is difficult to single out. Nonetheless, we assume, along with most of the industry, that there is a traded spot asset.

If we denote by  $S_t$  the oil spot price at time  $t$ , the log-price process is written as

$$\ln(S_t) = x(t) + L(t) + \varphi(t),$$

where, under the risk neutral measure,

$$dx(t) = -k_x x(t)dt + \sigma_x dZ_x, \quad (4.1)$$

$$dL(t) = \mu_L dt + \sigma_L dZ_L, \quad dZ_x dZ_L = \rho_{x,L} dt, \quad (4.2)$$

and  $\varphi$  is a deterministic shift we will use to calibrate quoted futures prices. The process  $x$  represents the short term deviation, whereas  $L$  represents the backbone of the equilibrium price level in the long run.

For applications it can be important to derive the transition density of the spot commodity in this model. For the two factors we have a joint Gaussian transition,

$$\begin{aligned} \begin{bmatrix} x(t) \\ L(t) \end{bmatrix} \Big|_{x(s), L(s)} &\sim \mathcal{N} \left( \begin{bmatrix} x(s) \exp(-k_x(t-s)) \\ L(s) + \mu_L(t-s) \end{bmatrix}, \begin{bmatrix} \frac{\sigma_x^2}{2k_x}(1 - \exp(-2k_x(t-s))) & \text{Cov}_{x,L}(s,t) \\ & \sigma_L^2(t-s) \end{bmatrix} \right) \\ \text{Cov}_{x,L}(s,t) &= \rho_{x,L} \frac{\sigma_x \sigma_L}{k_x} (1 - \exp(-k_x(t-s))). \end{aligned}$$

This can be used for exact simulation between times  $s$  and  $u$ . As we know that the sum of two jointly Gaussian random variables is Gaussian, we have

$$\begin{aligned} \ln(S(t)) &= x(t) + L(t) + \varphi(t) \Big|_{x(s), L(s)} \sim \mathcal{N}(m(t,s), V(s,t)) \\ m(t,s) &= x(s) \exp(-k_x(t-s)) + L(s) + \mu_L(t-s) + \varphi(t), \\ V(t,s) &= \frac{\sigma_x^2}{2k_x}(1 - \exp(-2k_x(t-s))) + \sigma_L^2(t-s) + 2\text{Cov}_{x,L}(s,t) \end{aligned}$$

from which, in particular, we see that

$$\mathbb{E}[S(t)|x(s), L(s)] = \exp(x(s) \exp(-k_x(t-s)) + L(s) + \mu_L(t-s) + \varphi(t) + V(s,t)/2)$$

Hence we can compute the forward price  $\mathbb{E}[S(T)|x(t), L(t)]$  at time  $t$  of the commodity at maturity  $T$  when counterparty risk is negligible and under deterministic interest rates, as

$$F(t, T) = \exp(x(t) \exp(-k_x(T-t)) + L(t) + \mu_L(T-t) + \varphi(T) + V(T,t)/2) \quad (4.3)$$

In particular, given the forward curve  $T \mapsto F^M(0, T)$  from the market, the expression for the shift  $\varphi^M(T)$  that makes the model consistent with said curve is

$$\varphi^M(T) = \ln(F^M(0, T)) - x_0 \exp(-k_x T) - L_0 - \mu_L T - V(T, 0)/2.$$

The short term/equilibrium price model  $(x, L)$ , when  $\varphi = 0$ , is equivalent to the more classical Gibson and Schwartz (1990) model, formulated as

$$\begin{aligned} d \ln(S_t) &= (r(t) - q(t) - \sigma_S^2/2)dt + \sigma_S dZ_S, \\ dq(t) &= k_q(\alpha - q(t))dt + \sigma_q dZ_q, \quad dZ_S dZ_q = \rho_{q,S} dt, \end{aligned} \quad (4.4)$$

the relationships being

$$\begin{aligned}
 x(t) &= \frac{1}{k_q}(q(t) - \alpha) \\
 L(t) &= \ln(S_t) - \frac{1}{k_q}(q(t) - \alpha) \\
 k_x &= k_q \\
 \mu_L &= r - \alpha - \sigma_S^2/2 \\
 \sigma_x &= \sigma_q/k_q \\
 \sigma_L^2 &= \sigma_S^2 + \sigma_q^2/k_q^2 - 2\rho_{q,S}\sigma_S\sigma_q/k_q \\
 dZ_x &= dZ_q \\
 dZ_L &= \left( \sigma_S dZ_S - \frac{\sigma_q}{k_q} dZ_q \right) / \sqrt{\sigma_S^2 + \sigma_q^2/k_q^2 - 2\rho_{q,S}\sigma_S\sigma_q/k_q} \\
 \rho_{x,L} &= \left( \sigma_S \rho_{q,s} - \frac{\sigma_q}{k_q} \right) / \sqrt{\sigma_S^2 + \sigma_q^2/k_q^2 - 2\rho_{q,S}\sigma_S\sigma_q/k_q}
 \end{aligned}$$

## 5 Forward vs Future prices and counterparty risk

Consider now a forward contract. The prototypical forward contract agrees on the following.

Let  $t$  be the valuation time. At the future time  $T$  a party agrees to buy from a second party a commodity at the price  $K$  fixed today. This is expressed by saying that the first party has entered a payer forward rate agreement. The second party has agreed to enter a receiver forward rate agreement. The value of this contract to the first and second party respectively, at maturity, will be

$$S_T - K, \quad K - S_T$$

i.e. the actual price of the commodity at maturity minus the pre-agreed price in the payer case, and the opposite of this in the receiver case. Let us focus on the payer case. When this is discounted back at  $t$  with deterministic interest rates, and risk neutral expectation is taken, this leads to the price being given by

$$\mathbb{E}_t[D(t, T)(S_T - K)] = D(t, T)(\mathbb{E}_t[S_T] - K) = D(t, T)(F(t, T) - K). \quad (5.1)$$

Note that the forward price is exactly the value of the pre-agreed rate  $K$  that sets the contract price to zero, i.e.  $K = F(t, T)$ . Let us maintain a general  $K$  in the forward contract under examination.

In the oil model above, the forward contract price is given by plugging Formula (4.3) into (5.1). Let us denote by  $\text{Fwdp}(t, T; K)$  such price (“p” is for payer),

$$\text{Fwdp}(t, T; K) = D(t, T) (\exp(x(t)) \exp(-k_x(T - t)) + L(t) + \mu_L(T - t) + \varphi(T) + V(T, t)/2) - K$$

whereas the opposite of this quantity is denoted by  $\text{Fwdr}(t, T; K)$ .

We may apply our counterparty risk framework to the forward contract, where now  $\Pi(t, T) = D(t, T)(S_T - K)$ , and  $\text{NPV}(t) = \text{Fwdp}(t, T; K)$ . We obtain as price of the payer forward under



counterparty risk from Equation (2.2). We obtain

$$\text{Fwdp}^D(t, T; K) = \text{Fwdp}(t, T; K) - \underbrace{\text{L}_{\text{GD}} \mathbb{E}_t\{\mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) (\text{Fwdp}(\tau, T; K))^+\}}_{\text{Positive counterparty-risk adjustment}} \quad (5.2)$$

Under the bucketing approximation given by Equation (2.3), we obtain

$$\text{Fwdp}^D(t, T; K) = \text{Fwdp}(t, T; K) - \text{L}_{\text{GD}} \sum_{j=1}^b D(t, T_j) \mathbb{E}_t[\mathbf{1}_{\{T_{j-1} < \tau \leq T_j\}} (\text{Fwdp}(T_j, T; K))^+].$$

If one assumes independence between the underlying commodity and the counterparty default, one may factor the above expectation obtaining

$$\text{Fwdp}^D(t, T; K) = \text{Fwdp}(t, T; K) - \text{L}_{\text{GD}} \sum_{j=1}^b \mathbb{Q}(\{T_{j-1} < \tau \leq T_j\}) \mathbb{E}_t[D(t, T_j) (\text{Fwdp}(T_j, T; K))^+].$$

The last term is the price of a option on a forward price, that is known in closed form in the Schwartz and Smith model, although we have to incorporate the shift in our formulation. We have

$$\begin{aligned} & \mathbb{E}_t[D(t, T_j) (\text{Fwdp}(T_j, T; K))^+] = \\ &= D(t, T) \exp(M(t, T, T_j; x_t, L_t) + \bar{V}(t, T, T_j)/2) \Phi\left(\frac{M(t, T, T_j; x_t, L_t) + \bar{V}(t, T, T_j) - \ln K}{\sqrt{V}(t, T, T_j)}\right) \\ & \quad - D(t, T) K \Phi\left(\frac{M(t, T, T_j; x_t, L_t) - \ln K}{\sqrt{V}(t, T, T_j)}\right) \\ & M(t, T, T_j; x_t, L_t) = x_t \exp(-k_x(T-t)) + L_t + \varphi(T) + V(T, T_j)/2 \\ & \bar{V}(t, T, T_j) = \exp(-2k_x(T-T_j)) \frac{\sigma_x^2}{2k_x} (1 - \exp(-2k_x(T_j-t))) \\ & + \sigma_L^2(T_j-t) + \exp(-k_x(T-T_j)) 2\rho_{x,L} \frac{\sigma_x \sigma_L}{k_x} (1 - \exp(-k_x(T_j-t))) \end{aligned}$$

where  $\Phi$  is the cumulative distribution function of the standard Gaussian.

So we have the adjustment as a stream of options on forwards weighted by default probabilities.

If we do not assume independence then we need to substitute for the intensity model. Through iterated conditioning we obtain easily

$$\begin{aligned} & \text{Fwdp}^D(t, T; K) = \text{Fwdp}(t, T; K) \\ & - \text{L}_{\text{GD}} \sum_{j=1}^b D(t, T_j) \mathbb{E}_t[(\exp(-\Lambda(T_{j-1})) - \exp(-\Lambda(T_j))) (\text{Fwdp}(T_j, T; K; x(T_j), L(T_j)))^+]. \end{aligned}$$

If in particular we select  $K = F(t, T)$  then  $\text{Fwdp}(t, T; K)$  will be zero.

This price can be computed by joint simulation of  $\lambda$ ,  $x$  and  $L$ . We may correlate the credit spread to the commodity by correlating the shock  $Z_y$  in the default intensity to the shocks  $Z_x, Z_L$  in the commodity. If we assume

$$dZ_x dZ_y = \rho_{x,y} dt, \quad dZ_L dZ_y = \rho_{y,L} dt$$

then the instantaneous correlation of interest is

$$\text{corr}(d\lambda_t, dS_t) = \frac{\sigma_x \rho_{x,y} + \sigma_L \rho_{L,y}}{\sqrt{\sigma_x^2 + \sigma_L^2 + 2\rho_{x,L} \sigma_x \sigma_L}}$$

This is the correlation one may try to infer from the market, through historical estimation or implying it from liquid market quotes. In general the only parameters that have not been calibrated previously are  $\rho_{x,y}$  and  $\rho_{L,y}$ . If we make for example the assumption that the two are the same,

$$\rho_{x,y} = \rho_{L,y} =: \rho_1,$$

then we get the model correlation parameters as a function of the already calibrated parameters and of the market correlation as

$$\rho_1 = \text{corr}(d\lambda_t, dS_t) \frac{\sqrt{\sigma_x^2 + \sigma_L^2 + 2\rho_{x,L} \sigma_x \sigma_L}}{\sigma_x + \sigma_L}$$

## 6 Swaps and counterparty risk

Consider now a swap contract. The prototypical swap contract is actually a portfolio of forward contracts with different maturities, and agrees on the following.

Let  $t$  be the valuation time. At the future times  $T_i$  in  $T_{a+1}, T_{a+2}, \dots, T_b$ , a party agrees to buy from a second party a commodity at the price  $K$  fixed today, on a notional  $\alpha_i$ . This is expressed by saying that the first party has entered a payer swap agreement. The second party has agreed to enter a receiver swap. The value of the payer commodity swap (CS) contract to the first party, at time  $t$ , will be

$$\text{PCS}_{a,b}(t, K) = \mathbb{E}_t \sum_{i=a+1}^b D(t, T_i) \alpha_i (S_{T_i} - K) = \sum_{i=a+1}^b \alpha_i D(t, T_i) (F(t, T_i) - K) = \sum_{i=a+1}^b \alpha_i \text{Fwdp}(t, T_i; K).$$

Since the last formula is known in our oil model, in terms of the processes  $x(t)$  and  $L(t)$ , we easily obtain a formula for the commodity swap by summation.

If we look for the value of  $K$  that sets the contract price to zero, i.e. the so called forward swap commodity price  $S_{a,b}(t)$ , we have

$$S_{a,b}(t) = \frac{\sum_{i=a}^b \alpha_i D(t, T_i) F(t, T_i)}{\sum_{i=a}^b \alpha_i D(t, T_i)}.$$

Using this rate we can also express the payer commodity swap price at a general strike  $K$  as

$$\text{PCS}_{a,b}(t, K) = (S_{a,b}(t) - K) \sum_{i=a+1}^b \alpha_i D(t, T_i)$$

whereas the receiver commodity swap would be

$$RCS_{a,b}(t, K) = (K - S_{a,b}(t)) \sum_{i=a+1}^b \alpha_i D(t, T_i).$$

These formulas provide the value of these contracts when a clearing house or margining agreements are in place. However, swaps are often traded outside such contexts and as such they embed counterparty risk.

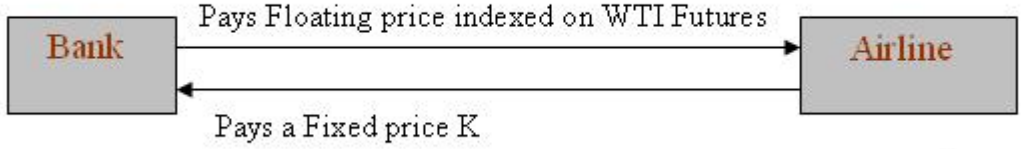
Our general formula (2.2), for a payer CS, when including counterparty risk, would read in the swap case:

$$\begin{aligned} PCS_{a,b}^D(t; K) &= PCS_{a,b}(t, K) - \underbrace{LGD \mathbb{E}_t \{ \mathbf{1}_{\{t < \tau \leq T_b\}} D(t, \tau) (PCS_{a,b}(\tau, K))^+ \}}_{\text{Positive counterparty-risk adjustment}} \quad (6.1) \\ &= PCS_{a,b}(t, K) - LGD \mathbb{E}_t \left\{ \mathbf{1}_{\{t < \tau \leq T_b\}} D(t, \tau) \left( \sum_{i=a+1}^b \alpha_i Fwdp(\tau, T_i; K) \right)^+ \right\} \end{aligned}$$

Since the forward formula is known in our model, we can proceed similarly to the forward case to value the counterparty risk adjustment for the swap case through simulation. The receiver case is completely analogous.

## 7 A case study and conclusions

### ***FIXED-FLOATING (for hedge purposes)***



As a case study we consider an oil swap. An airline needs to buy oil in the future and is concerned about possible changes in the oil price. To hedge this price movement the airline asks a bank to enter a swap where the bank pays periodically to the airline a (floating) amount indexed at a relevant oil futures price at the coupon date. In exchange for this, the airline pays periodically an amount  $K$  that is fixed in the beginning.

In the following, we are taking an example of a bank with currently high credit spreads as receiver, and one international airline as the payer of the swap. We will look at the counterparty risk adjustment from the point of view of each of the two parties separately, by calibrating the credit model adequately in each case.

The Oil model has been calibrated to the At The Money Futures options implied volatility shown in Figure 1

The shift  $\varphi$  has been calibrated in order to fit the forward curve extracted from West Texas Intermediate Futures in Figure 2

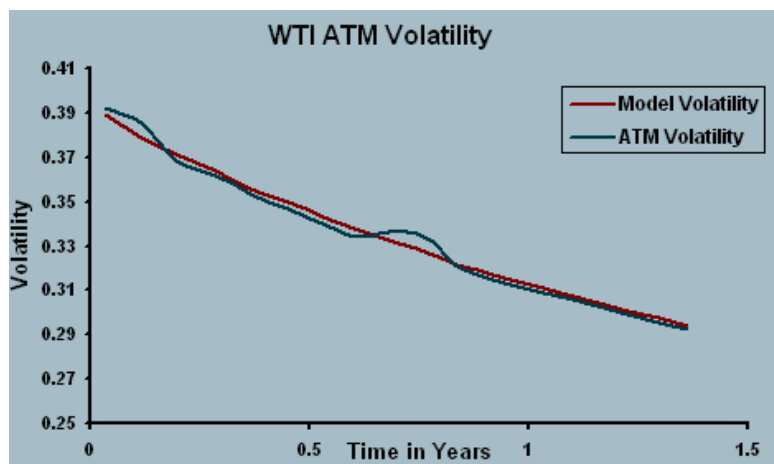


Fig. 1: Calibration: ATM Volatility Curve.

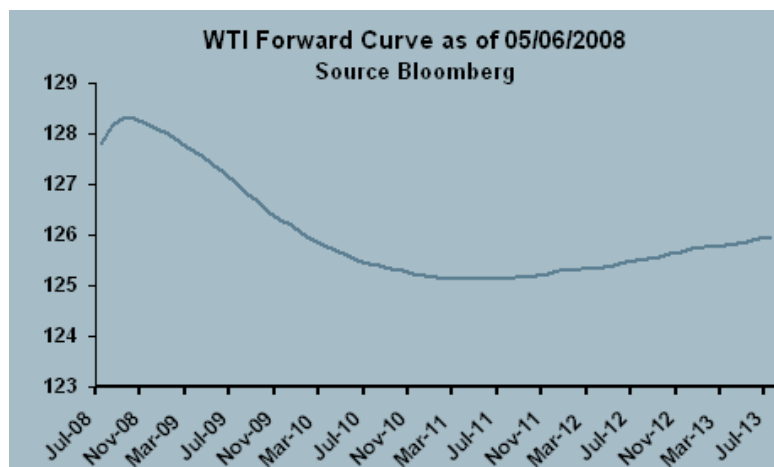


Fig. 2: Calibration: Forward Curve.

We reformulated the commodity model by setting the parameter  $\mu_L$  in the long term equilibrium price to zero, as it can be included implicitly in the deterministic shift  $\varphi$ . The resulting Oil model parameters are shown in Table 1.

The oil swap we consider has a final maturity of 5 years, monthly payments and strike  $K$  given by  $K = 126$  USD, that is the strike setting to zero the value of the 5 years default free oil swap.  $\alpha_i$  is equal to one (barrel).

$k_x$	$\sigma_x$	$\sigma_L$	$\rho_{x,L}$
0.7170	0.3522	0.19	-0.0392

Tab. 1: Calibration Parameters

### 7.1 Counterparty Risk from the Payer Perspective (the Airline computes counterparty risk)

First, we use the CDS spreads for the bank, which are given in Table 2

maturity (years)	0.5	1	2	3	4	5
spread (bps)	345	332	287	256	232	2.17

Tab. 2: CDS spreads term structure for the bank

The yield curve is given in Table 3

maturity (years)	3/12	6/12	2	5	10	30
yield (percent)	2.68	2.92	3.40	4.27	4.87	5.376

Tab. 3: Zero coupon continuously compound spot interest rates

In the following we assume the bank credit quality to be characterized by a CIR++ stochastic intensity model that, as spread levels, is consistent with Table 2 through the shift  $\psi$ , while allowing for credit spread volatility through the CIR dynamics. We use the base CIR parameter set given in Table 4. Later, we change the spread volatility parameter  $\nu$  by reducing it through multiplicative factors smaller than one, and recalibrate the model shift to maintain consistency with Table 2. This way we investigate the impact of the spread volatility on the counterparty adjustment.

$y_0$	$\kappa$	$\mu$	$\nu$
0.0560	0.6331	0.0293	0.5945

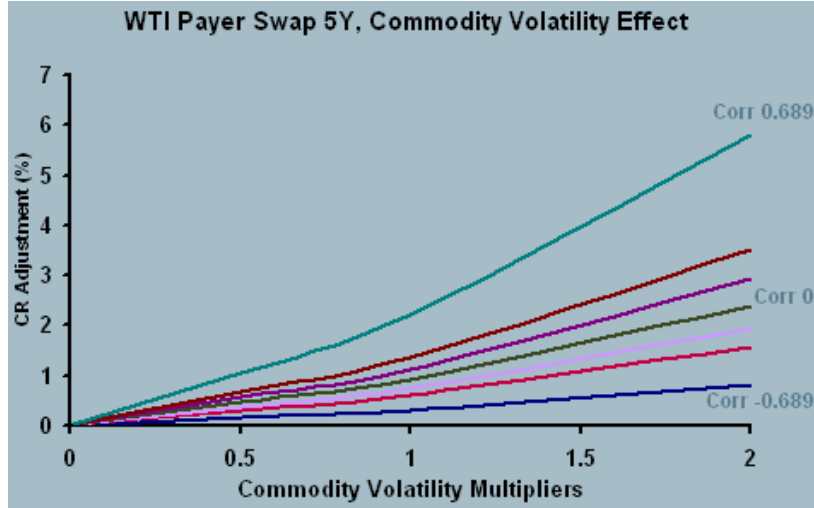
Tab. 4: CIR parameters for the base case for the bank credit spread volatility

The graphs in Fig. 3 and Fig. 5 illustrate some of our results for the CR-CVA. The counterparty risk is expressed as a percentage of a 5Y maturing swap fixed leg value, which is 6852.35 USD. First we observe the effect of varying the commodity volatility while keeping the credit intensity volatility fixed at  $\nu_{Bank} = 59\%$ <sup>1</sup> The commodity volatility was varied by applying multiplicative factors to the two factors instantaneous volatilities  $\sigma_x$  and  $\sigma_L$ .

As an indication of implied volatility levels, the term structure of the commodity implied volatility when we apply the multiplicative factor 2 is given in Fig. 4

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<sup>1</sup> The CDS implied volatility associated to these parameters is 26%. Brigo (2005, 2006), under the CDS market model, shows that implied volatilities for CDS options can easily exceed 50%



Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, CR-CVA as a (%) of the Fixed leg price

Fig. 3: Commodity Swap CR-CVA Results Overview : Commodity Volatility Effect

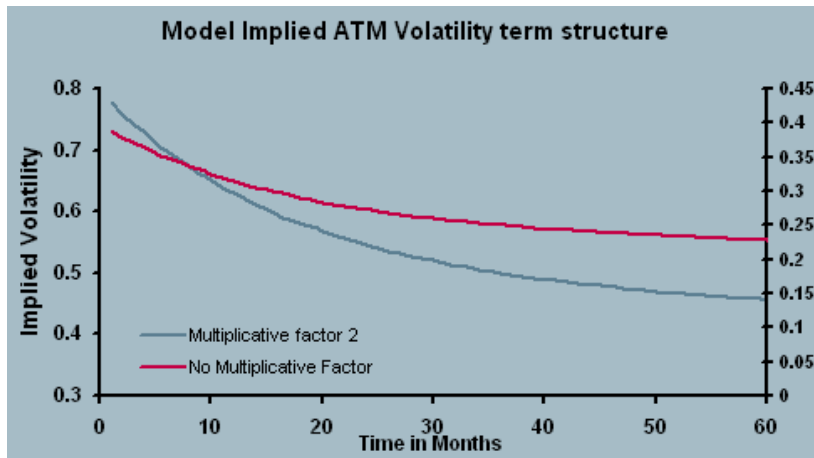
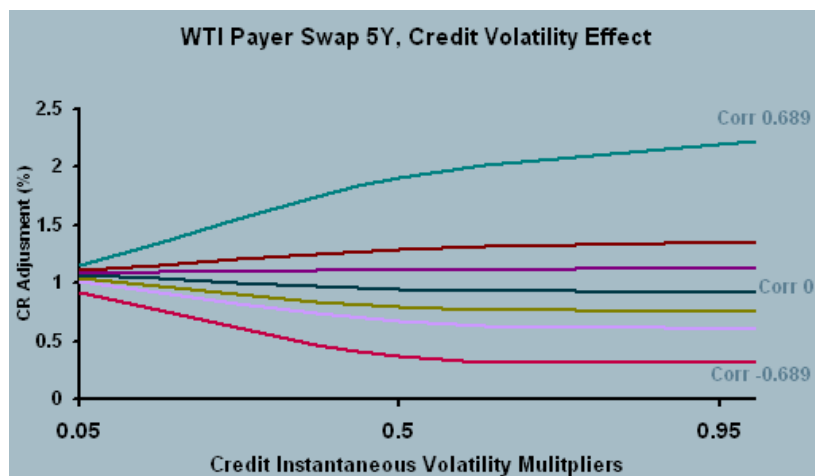


Fig. 4: Model Implied Volatility Without (right scale) and with (left scale) multiplicative factors



Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, CR-CVA as a (%) of the Fixed leg price

Fig. 5: Commodity Swap CR-CVA Results Overview : Credit volatility effect

Secondly, we observe the effect of varying the intensity volatility while keeping the commodity spot volatility fixed at  $\sigma_S = 32.82\%$  as implied by Table 1.

The same results are presented in a different way in Tables 5 and 6. In these tables, we give the absolute value of the adjustment in USD. We also express it as an adjusted Strike price that the payer might choose to pay to its counterparty by taking into account the estimated adjustment:

$\bar{\rho}$	intensity volatility $\nu_R$	0.0295	0.295	0.59
-68.9	Adjustment in USD	63.49	25.17	21.58
	Adjusted Strike	124.84	125.54	125.60
-27.6	CR-CVA (USD)	69.99	45.89	41.5
	Adjusted Strike	124.71	125.16	125.24
-13.8	CR-CVA (USD)	71.83	55.02	51.48
	Adjusted Strike	124.68	124.99	125.05
0	CR-CVA (USD)	73.3	65.23	63.42
	Adjusted Strike	124.66	124.80	124.84
+13.8	CR-CVA (USD)	74.62	76.63	77.36
	Adjusted Strike	124.63	124.59	124.58
+27.6	CR-CVA (USD)	75.88	88.93	93.08
	Adjusted Strike	124.61	124.37	124.29
+68.9	CR-CVA (USD)	79.32	130.39	152.05
	Adjusted Strike	124.54	123.61	123.21

Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, Fair Strike without Counterparty Risk 126 USD.

Tab. 5: Effect of credit spread volatility on the CR-CVA

$\bar{\rho}$	Comdty spot vol $\sigma_S$	0.033	0.1642	0.3285	0.657
-68.9	CR-CVA (USD)	1.17	11.05	21.58	57.11
	Adjusted Strike	125.98	125.79	125.60	124.95
-27.6	CR-CVA (USD)	1.63	21.75	41.5	107.48
	Adjusted Strike	125.97	125.60	125.24	124.03
-13.8	CR-CVA (USD)	1.8	26.71	51.48	133.49
	Adjusted Strike	125.96	125.51	125.05	123.55
0	CR-CVA (USD)	1.98	32.4	63.42	164.27
	Adjusted Strike	125.96	125.41	124.84	122.98
+13.8	CR-CVA (USD)	2.15	38.85	77.36	200.08
	Adjusted Strike	125.96	125.28	124.58	122.33
+27.6	CR-CVA (USD)	2.34	46.05	93.08	240.41
	Adjusted Strike	125.96	125.15	124.29	121.59
+68.9	CR-CVA (USD)	2.92	72.47	152.05	397.87
	Adjusted Strike	125.95	124.67	123.21	118.70

Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, Fair Strike without Counterparty Risk 126 USD.

Tab. 6: Effect of oil volatility on the CR-CVA



7.2 Counterparty Risk from the Receiver Perspective (the Bank computes counterparty risk)

Now we place ourselves from the point of view of the bank, and we use the CDS spreads for the airline, which are given in Table 7

maturity (years)	0.5	1	2	3	4	5
spread (bps)	76	82	104	122	139	154

Tab. 7: CDS spreads term structure for the airline

We use the same discount curve as in Table 3.

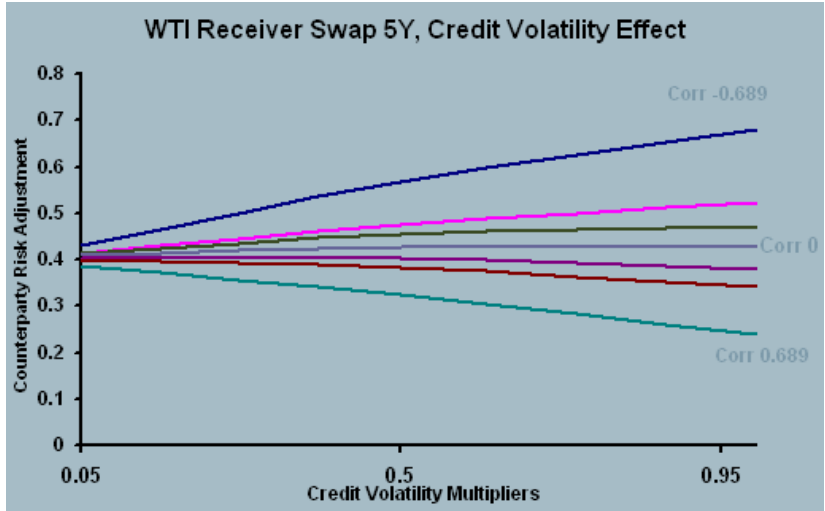
Here, the airline credit quality is represented by a CIR++ stochastic intensity model that, as spreads levels, is consistent with Table 7 through the shift  $\psi$ , while allowing for credit spread volatility through the CIR dynamics. We use the base CIR parameter set given in Table 8. Later, we reduce the spread volatility parameter  $\nu$  via multiplicative factors smaller than one, and recalibrate the shift to maintain each time the model consistent with Table 7. This way we investigate again the impact of the spread volatility on the counterparty adjustment.

$y_0$	$\kappa$	$\mu$	$\nu$
0.0000	0.5341	0.0328	0.2105

Tab. 8: CIR parameters for the base case for the airline credit spread volatility

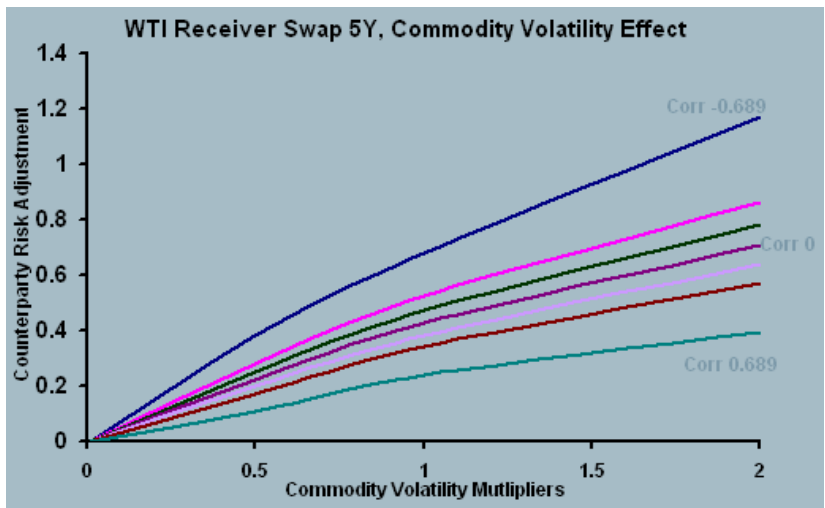
As before, we observe the effect of varying the commodity volatility and of the airline credit intensity volatility, starting from  $\sigma_S = 32.82\%$  as from Table 1 and  $\nu_{Airline} = 21\%$ . We apply the same multiplicative factors as before and the results are summarized in the graphs in Fig. 7 and Fig. 6.

The same results are presented more in detail in Tables 9 and 10.



Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, CR-CVA as a (%) of the Fixed leg price

Fig. 6: Commodity Swap CR-CVA Results Overview : Credit volatility effect



Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, CR-CVA as a (%) of the Fixed leg price

Fig. 7: Commodity Swap CR-CVA Results Overview : Commodity volatility effect

$\bar{\rho}$	intensity volatility $\nu_R$	0.0295	0.295	0.59
-68.9	CR-CVA (USD)	29.62	38.95	46.62
	Adjusted Strike	126.54	126.71	126.85
-27.6	CR-CVA (USD)	28.41	32.58	35.82
	Adjusted Strike	126.52	126.59	126.66
-13.8	CR-CVA (USD)	28.21	31.02	32.4
	Adjusted Strike	126.52	126.57	126.59
0	CR-CVA (USD)	27.99	29.37	29.16
	Adjusted Strike	126.51	126.54	126.53
+13.8	CR-CVA (USD)	27.78	27.72	26.09
	Adjusted Strike	126.51	126.51	126.48
hline +27.6	CR-CVA (USD)	27.49	26.15	23.42
	Adjusted Strike	126.50	126.48	126.43
+68.9	CR-CVA (USD)	26.48	22.23	16.31
	Adjusted Strike	126.48	126.41	126.30

Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, Fair Strike without Counterparty Risk 126 USD.

Tab. 9: Effect of credit spread volatility on the CR-CVA

$\bar{\rho}$	Comdty spot vol $\sigma_S$	0.033	0.1642	0.3285	0.657
-68.9	CR-CVA (USD)	0.12	26.33	46.62	80.26
	Adjusted Strike	126.00	126.48	126.85	127.47
-27.6	CR-CVA (USD)	0.09	19.33	35.82	59.23
	Adjusted Strike	126.00	126.35	126.65	127.08
-13.8	CR-CVA (USD)	0.08	17.35	32.4	53.64
	Adjusted Strike	126.00	126.32	126.59	126.98
0	CR-CVA (USD)	0.07	15.42	29.16	48.59
	Adjusted Strike	126.00	126.28	126.53	126.89
+13.8	CR-CVA (USD)	0.06	13.58	26.09	43.88
	Adjusted Strike	126.00	126.25	126.48	126.80
+27.6	CR-CVA (USD)	0.05	11.86	23.42	39.09
	Adjusted Strike	126.00	126.22	126.43	126.72
+68.9	CR-CVA (USD)	0.03	7.4	16.31	27.16
	Adjusted Strike	126.00	126.13	126.30	126.50

Fixed Leg Price maturity 5Y: 6852.35 USD for a notional of 1 Barrel per Month, Fair Strike without Counterparty Risk 126 USD.

Tab. 10: Effect of oil volatility on the CR-CVA

### 7.3 Conclusions

The patterns we observe in the counterparty-risk credit valuation adjustment (CR-CVA) are natural. Starting with the receiver case, for a fixed credit spread volatility, the receiver CR-CVA increases in oil volatility and decreases in correlation. Given the embedded oil option, the increase with respect to oil volatility is natural (as is in the payer case). As concerns correlation, as this increases, the oil tends to move in line with credit spreads. This means that higher credit spreads will lead to higher oil values, and the option will end up less in the money as the oil spot goes up. The opposite appears in the payer case. Patterns in credit spread volatility are similarly explained.

The size of the CVA hence depends on the precise value of the volatility and correlation dynamic parameters that cannot be explained via rough multipliers.

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