Constant Maturity Credit Default Swap Pricing with Market Models *

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Abstract

In this work we derive an approximated no-arbitrage market valuation formula for Constant Maturity Credit Default Swaps (CMCDS). We move from the CDS options market model in Brigo (2004), and derive a formula for CMCDS that is the analogous of the formula for constant maturity swaps in the default free swap market under the LIBOR market model. A "convexity adjustment"-like correction is present in the related formula. Without such correction, or with zero correlations, the formula returns an obvious deterministic-credit-spread expression for the CMCDS price. To obtain the result we derive a joint dynamics of forward CDS rates under a single pricing measure, as in Brigo (2004). Numerical examples of the "convexity adjustment" impact complete the paper.

Keywords: CDS Options, CDS Options Market Model, Constant Maturity CDS, Convexity Adjustment, Participation Rate, CDS rates volatility, CDS rates correlation.

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Contents

1 Introduction

Constant Maturity Credit Default Swaps (CMCDS) are receiving increasing attention in the financial community. In this work, we aim at deriving an approximated no-arbitrage market valuation formula for CMCDS. We move from the CDS options market model in Brigo (2004), and derive a formula for CMCDS that is the analogous of the formula for constant maturity swaps in the default free swap market under the LIBOR market model.

In Brigo (2004) the focus is somehow different, since in that paper we derive CDS option prices for CDS payoffs given in the market and for some postponed approximated CDS payoffs. We do so by means of a rigorous change of numeraire technique following Jamshidian (2002) and based on a result of Jeanblanc and Rutkowski (2000). We also establish a link with callable defaultable floaters.

This paper starts by recalling again one alternative expression for CDS payoffs, stemming from different conventions on the premium flows and on the protection leg. We briefly introduce CDS forward rates, postponing the detailed definitions. We define CMCDS and give immediately the main result of the paper, the approximated pricing formula, in terms of CDS forward rates and of their volatilities and correlations. We point out some analogies with constant maturity swaps, showing that a "convexity adjustment"-like correction is present. Without such correction, or with zero correlations, the formula returns an obvious deterministic-credit-spread expression for the CMCDS price.

Once the main result has been described, we move to introduce the formal apparatus that allows to prove it. We introduce rigorously CDS forward rates. This leads to an investigation on the possibility to express such rates in terms of some basic one-period rates and to a discussion on a possible analogy with the LIBOR and swap default-free models. We discuss the change of numeraire approach to deriving a joint dynamics of forward CDS rates under a single pricing measure, derivation that is only hinted at in Brigo (2004) for deriving the Black-like formula for CDS options. Through a drift-freezing approximation we then prove the formula for CMCDS pricing and give some numerical examples highlighting the role of the participation rate and of the convexity adjustment.

Remark 1.1. (How to use this paper) *The CDS Options market model is the same as in Brigo (2004) and is based on a complex change of numeraire involving two families of forward CDS rates. This apparatus is reported in Section 4 but there is no need to go through it if one does not plan to test the formula against Monte Carlo simulation. As far as the formula derivation is concerned, the reader may skip Section 4 and go directly to the much simpler approximated model in Section 5.*

2 Credit Default Swaps (CDS) and Constant Maturity CDS

We recall briefly some basic definitions for CDS's.

Definition 2.1. (Credit Default Swap). *A CDS is a contract where the "protection buyer" "A" pays rates R ("R" stands for Rates) at times $T_{a+1}, \ldots, T_b$ (the "premium leg") in exchange for a single protection payment* LGD *(loss given default, the protection leg). "A" receives the protection leg by the "protection seller" "B" at the default time τ of a reference entity "C", provided that $T_a < \tau \leq T_b$. The rates R paid by "A" stop in case of default. We thus have*

Protection Seller	$\rightarrow$	*protection* $\mathrm{L{\scriptstyle GD}}$ *at default* τ_C *of "C" if in* $[T_a, T_b]$	$\rightarrow$	*Protection Buyer*
"B"	$\leftarrow$	*rate* R *at* $T_{a+1}, \ldots, T_b$ *or until default* τ_C	$\leftarrow$	*"A"*

This is called a "running CDS" (RCDS) discounted payoff.

We explicitly point out that we are assuming the offered protection amount $\mathrm{L{\scriptstyle GD}}$ to be deterministic. Typically $\mathrm{L{\scriptstyle GD}} = 1 - \mathrm{R{\scriptstyle EC}}$, where the recovery rate $\mathrm{R{\scriptstyle EC}}$ is assumed to be deterministic and the notional is set to one.

Sometimes a slightly different payoff is considered for RCDS contracts. Instead of considering the exact default time τ, the protection payment $\mathrm{L{\scriptstyle GD}}$ is postponed to the first time T_i following default. If the grid is three or six months spaced, this postponement consists in a few months at worst.

We term "Postponed payments Running CDS" (PRCDS) the CDS payoff under the postponed formulation. The advantage of the postponed protection payment is that no accrued-interest term is necessary, and also that all payments occur at the canonical grid of the T_i's. The postponed payout is better for deriving market models of CDS rates dynamics, as we shall see shortly. It is also fundamental in establishing the "defaultable floater analogy" as we have seen in Brigo (2004).

We denote by $\mathrm{CDS}_{a,b}(t, R, \mathrm{L{\scriptstyle GD}})$ the price at time t of the above standard running CDS flows to the protection seller "B". We add the prefix "PR" to denote the analogous price for the postponed payoff. The pricing formulas for these payoffs depend on the assumptions on interest-rate dynamics and on the default time τ. In general the CDS forward rate $R_{a,b}(0)$ for protection in T_a, T_b at initial time 0 is obtained (in the postponed case) by solving the equation $\mathrm{PRCDS}_{a,b}(0, R_{a,b}(0), \mathrm{L{\scriptstyle GD}}) = 0$. We will give details on this later: now we only say that the market provides quotes for $R_{0,b}(0)$'s for increasing maturities T_b (notice that $T_0 = 0$, so that only "spot" CDS rates are quoted in the market). The $R_{a,b}(0)$ rate makes the CDS contract fair at the valuation time. A special role in our work will be covered by one-period rates $R_i(0) = R_{i-1,i}(0)$ (protection in $[T_{i-1}, T_i]$). These rates may seem artificial but they can be easily computed from quoted spot rates. An important role is also assigned to the corporate zero-coupon bond price $\bar{P}(0, T)$, which is the price at time 0 of one unit of currency made available by name "C" at maturity T if no default occurs, and with no recovery in case of early default. The corresponding default-free zero coupon bond is denoted by $P(0,T)$.

We are now ready to introduce CMCDS's.

In a CMCDS with first reset in T_a and with final maturity T_b, protection $\mathrm{L{\scriptsize GD}}$ on a reference credit "C" against default in $[T_a, T_b]$ is given from a protection seller "B" to a protection buyer "A". However, in exchange for this protection, a "constant maturity" CDS rate is paid.

We know by definition that the fair rate to be paid at T_i for protection against default in $[T_{i-1}, T_i]$ would be R_i. This leads us to the following

Remark 2.2. (A "floating-rate" CDS). *A contract that protects in T_a, T_b can be in principle decomposed into a stream of contracts, each single contract protecting in $[T_{j-1}, T_j]$, for $j = a+1, \ldots, b$, say with protection payment $\mathrm{L{\scriptsize GD}}$ postponed to T_j if default occurs in $[T_{j-1}, T_j]$. In each single period, the rate $R_j(T_{j-1})$ paid at T_j makes the exchange fair, so that in total a contract offering protection $\mathrm{L{\scriptsize GD}}$ on a reference credit "C" in $[T_a, T_b]$ in exchange for payment of rates $R_{a+1}(T_a), \ldots, R_j(T_{j-1}), \ldots, R_b(T_{b-1})$ at times $T_{a+1}, \ldots, T_j, \ldots, T_b$ is fair, i.e. has zero initial present value. This product can be seen as a sort of floating rate CDS.*

However, in CMCDS's the rate that is paid at each period for protection is not the related one-period CDS rate, as would be natural from the above remark, but a longer period CDS rate. Consider indeed the following

Definition 2.3. (Constant Maturity CDS). *Consider a contract protecting in $[T_a, T_b]$ against default of a reference credit "C". If default occurs in $[T_a, T_b]$, a protection payment $\mathrm{L{\scriptsize GD}}$ is made from the protection seller "B" to the protection buyer "A" at the first T_j following the default time. This is called "protection leg". In exchange for this protection "A" pays to "B" at each T_j before default a "$c+1$–long" (constant maturity) CDS rate $R_{j-1,j+c}(T_{j-1})$ (times a year fraction $\alpha_j = T_j - T_{j-1}$), with "c" an integer larger than zero. Notice that for $c = 0$ we would obtain the fair "floating rate" CDS above, whose initial value would be zero.*

Given that $c > 0$ in our definition, the value of the contract will be nonzero in general, so that we have to find this value at the initial time 0 if we are to price this kind of transaction. We face this task by resorting to the market model derived below.

The value of the CMCDS to "B" is the value of the premium leg minus the value of the protection leg. The protection leg valuation is trivial, since this is the same leg as in a standard forward start $[T_a, T_b]$ CDS. As such, it is for example equal to

$$R_{a,b}(0) \sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j) = \sum_{j=a+1}^{b} \alpha_j R_j(0) \bar{P}(0,T_j).$$

This value has to be subtracted to the premium leg. The non-trivial part is indeed computing the premium leg value at initial time 0. Notice that the final formula can be implemented easily on a spreadsheet, requiring no numerical apparatus.

Proposition 2.4. (Main Result: An approximated formula for CMCDS) *Consider the Constant Maturity CDS defined in Definition 2.3. The present value at initial time 0 of the CMCDS to the protection seller "B" is*

$$\mathit{CDS}_{\mathit{CM}a,b,c}(0, L_{GD}) = \sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j) \left\{ \sum_{i=j}^{j+c} \frac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0,T_h)} \cdot \right. \tag{1}$$
$$\left. \widetilde{R}_i(0) \exp\left[T_{j-1}\sigma_i \cdot \left(\sum_{k=j+1}^{i} \rho_{j,k} \frac{\sigma_k \widetilde{R}_k(0)}{\widetilde{R}_k(0) + L_{GD}/\alpha_k} \right) \right] - R_j(0) \right\}$$

where $R_k(0)$ are the one-period CDS forward rates for protection in $[T_{k-1}, T_k]$. These CDS forward rates can be computed from quoted spot CDS rates $R_{0,k}(0)$ and corporate zero coupon bonds $\bar{P}(0,T_k)$ via

$$R_k(0) = \frac{R_{0,k}(0) \sum_{h=1}^{k} \alpha_h \bar{P}(0,T_h) - R_{0,k-1}(0) \sum_{h=1}^{k-1} \alpha_h \bar{P}(0,T_h)}{\alpha_k \bar{P}(0,T_k)}$$

while $\widetilde{R}_k(0)$ are approximations of the $R_k(0)$ (equal in case of independence of interest rates and credit spreads) in terms of corporate $\bar{P}$ and default free P zero coupon bonds given by

$$R_k(0) \approx \widetilde{R}_k(0) = L_{GD} \frac{\bar{P}(0,T_{k-1}) P(0,T_k)/P(0,T_{k-1}) - \bar{P}(0,T_k)}{\alpha_k \bar{P}(0,T_k)}$$

and where: σ_k is the volatility of $R_k(t)$, assumed constant (we deal with the time-varying volatility in the proof below);

$\rho_{i,j}$ is the instantaneous correlation between R_i and R_j;

One-period forward CDS rates volatilities σ_k can in principle be stripped from longer period CDS volatilities, similarly to how forward LIBOR rates volatilities can be stripped from swaptions volatilities in the LIBOR model. This stripping is made possible from an approximated volatility formula based on drift freezing (formula (6.58) for the LIBOR case in Brigo and Mercurio (2001)). Cascade methods are also available for this (as in Brigo and Morini (2004)), although for the time being the only available CDS options all have short maturities and the lack of a liquid market discourages this kind of approach. For the time being the above formula can be employed with stylized values of volatilities to have an idea of the impact of the "convexity adjustments". Finally, one may consider using historical volatilities and correlations in the formula as first guesses.

As a further remark we notice that, if not for the exponential term (which vanishes for example when ρ's are set to zero) this expression would be, not surprisingly,

$$\mathit{CDS}_{\mathit{CM}a,b,c}(0, L_{GD}; \rho = 0) = \sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j)(R_{j-1,j+c}(0) \; - R_{j-1,j}(0)) \tag{2}$$

Notation	Description
$\tau = \tau^C$	Default time of the reference entity "C"
$T_a, (T_{a+1}, \ldots, T_{b-1}), T_b$	initial and final dates in the protection schedule of the CDS and CMCDS
$T_{\beta(\tau)}, T_{\beta(t)}$	First of the T_i's following τ and t, respectively
α_i	year fraction between T_{i-1} and T_i
$L(S,T)$	LIBOR rate at time S for maturity T
$R_{a,b}$	Rate in the premium leg of a CDS, paid by "A", the protection buyer
R_i	CDS Rate to be paid by "A", the protection buyer at T_i for protection in $[T_{i-1}, T_i]$
$R_{i-2,i}$	CDS Rate to be paid by "A" at T_{i-1} and T_i for protection in $[T_{i-2}, T_i]$
Rec	Recovery fraction on a unit notional
$\text{Lgd} = 1 - \text{Rec}$	Protection payment against a Loss (given default of "C" in $[T_a, T_b]$)
$\Pi\text{PRCDS}_{a,b}(t)$	Discounted payoff of a postponed running CDS to "B", the protection seller
$\text{PRCDS}_{a,b}(t, R, \text{Lgd})$	Price of a running postponed CDS to "B", protecting against default of "C" in $[T_a, T_b]$
$\text{CDScm}_{a,b,c}(t, \text{Lgd})$	Price at time t of a CMCDS to "B", protecting against default of "C" in $[T_a, T_b]$ in exchange for a periodic constant maturity "$c+1$"-long CDS rate
$1_{\{\tau > T\}}$	Survival indicator, is one if default occurs after T and zero otherwise
$1_{\{\tau \le T\}}$	Default indicator, is one if default occurs before or at T, and zero otherwise
$B(t)$	Bank account numeraire of the risk neutral measure at time t
$D(t,T) = B(t)/B(T)$	Stochastic discount factor at time t for maturity T
$P(t,T)$	Zero coupon bond at time t for maturity T
$1_{\{\tau > t\}} \bar{P}(t,T)$	Defaultable Zero coupon bond at time t for maturity T
$\widehat{C}_{a,b}(t),\ \widehat{\mathbb{Q}}^{a,b}$	Defaultable "Present value per basis point" numeraire and associated measure
$\mathcal{F}_t$	Default free market information up to time t
$\mathcal{G}_t$	Default free market information plus explicit monitoring of default up to time t
$DC(\cdot)$	$DC(X_t)$ is the row vector $\mathbf{v}$ in $dX_t = (\ldots)dt + \mathbf{v}\, dW_t$ for diffusion processes X with W vector Brownian motion common to all relevant diffusion processes

Table 1: Main notation in the paper.

The exponential term can be considered indeed to be a sort of "convexity adjustment" similar in spirit to the convexity adjustment needed to value constant maturity swaps with the LIBOR model in the default-free market.

Finally, this formula should be tested against prices obtained via Monte Carlo simulation of the dynamics (11) before being employed massively. One should make sure that for the order of magnitude of volatilities, correlations and initial CDS rates present in the market at a given time the freezing approximation works well. We plan to analyze this approximation against Monte Carlo simulation in further work. This future work is the reason why we derive the exact rates dynamics below.

3 CDS Option Market Model Dynamics

In the remaining part of the paper we build the apparatus allowing us to prove the main result rigorously. We specify the probabilistic framework in the following remark.

Remark 3.1. (Probabilistic Framework: τ as first jump of a Cox process) *Here we place ourselves in a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ where the default time random variable τ will be defined. The probability measure $\mathbb{Q}$ is the risk neutral probability. This space is endowed with a filtration $(\mathcal{F}_t)_t$, typically representing the basic filtration without default, i.e. the "information flow" of interest rates, intensities and possibly other default-free market quantities. Obviously $\mathcal{F}_t \subseteq \mathcal{G}$ for all t.*

We consider a non-negative, $(\mathcal{F}_t)_t$ progressively measurable process λ with integrable sample paths in $(\Omega, \mathcal{G}, \mathbb{Q})$.

The space $(\Omega, \mathcal{G}, \mathbb{Q})$ is assumed to be sufficiently rich to support a random variable U uniformly distributed on $[0, 1]$ and independent of $(\mathcal{F}_t)_t$. The random default time τ can then be defined as

$$\tau := \inf \left\{ t \geq 0 : \exp\left(-\int_0^t \lambda_s ds \right) \leq U \right\}$$

With this definition λ is indeed the $\mathcal{F}_t$ stochastic intensity of the default time τ, in that $\mathbb{Q}(\tau > t | \mathcal{F}_t) = \exp(-\int_0^t \lambda_s ds)$.

We consider also the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < s\}, s \leq t)$ and assume $\mathcal{G}_t \subseteq \mathcal{G}$. The second sigma field $\sigma(\{\tau < s\}, s \leq t)$ contributing to $\mathcal{G}_t$ represents the information on whether default occurred before t, and if so when exactly. Since this information is available to us when we price, we need to condition on $\mathcal{G}_t$ rather than on $\mathcal{F}_t$ alone.

For more details on the canonical construction of a default time with a given hazard rate see e.g. Bielecki and Rutkowski (2001), p. 226.

The remark above amounts to saying that default τ is modeled as the first jump time of a Cox process with the given intensity process. We will not model the intensity directly but rather some market quantities embedding the impact of the relevant intensity model that is consistent with them. An explicit tractable stochastic intensity/ interest-rate model with automatic analytical and separable calibration to interest rate derivatives and CDS's is given for example in Brigo and Alfonsi (2003), where an analytical formula for CDS options based on Jamshidian's decomposition is also presented.

Formally, we may write the RCDS discounted value at time t as

$$\Pi_{\mathrm{RCDS}_{a,b}}(t) := \text{DiscountedPremiumLeg} - \text{DiscountedProtectionLeg} \quad (3)$$

$$= D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^{b} D(t, T_i)\alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(t,\tau) \,\mathrm{L_{GD}}$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t, and where α_i is the year fraction between T_{i-1} and T_i. The stochastic discount factor at time t for maturity T is denoted by $D(t, T) = B(t)/B(T)$, where $B(t)$ denotes the risk-neutral measure bank-account numeraire.

Under the postponed formulation, where the protection payment is moved from τ to $T_{\beta(\tau)}$, the CDS discounted payoff can be written as

$$\Pi_{\text{PRCDS}_{a,b}}(t) := \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau \leq T_i\}} D(t,T_i) \text{ L}_{\text{GD}}, \tag{4}$$

which we term "Postponed payments Running CDS" (PRCDS) discounted payoff. Compare with the earlier discounted payout (3) where the protection payment occurs exactly at τ: The advantage of the postponed protection payment is that no accrued-interest term in $(\tau - T_{\beta(\tau)-1})$ is necessary, and also that all payments occur at the canonical grid of the T_i's.

In general, we can compute the CDS price according to risk-neutral valuation (see for example Bielecki and Rutkowski (2001) for the most general result of this kind):

$$\text{CDS}_{a,b}(t,R,\text{L}_{\text{GD}}) = \mathbb{E}\left\{\Pi_{\text{RCDS}_{a,b}}(t)|\mathcal{G}_t\right\} \tag{5}$$

where we recall that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$. In the Cox process setting default is unpredictable and this is why observation of $\mathcal{F}_t$ alone does not imply observation of the default time, contrary to standard structural (Merton, Black and Cox, etc) models where instead $\mathcal{F}_t = \mathcal{G}_t$. At times we denote by $\mathbb{E}_t$ and $\mathbb{Q}_t$ the expectation and probability conditional on the default-free sigma field $\mathcal{F}_t$. The above expected value can also be written as

$$\text{CDS}_{a,b}(t,R,\text{L}_{\text{GD}}) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\left\{\Pi_{\text{RCDS}_{a,b}}(t)|\mathcal{F}_t\right\} \tag{6}$$

(see again Bielecki and Rutkowski (2001), or more in particular Jeanblanc and Rutkowski (2000), where the most general form of this result is reported). This second expression, and especially the analogous definitions with postponed payoffs, is fundamental for introducing the market model for CDS options in a rigorous way. We explicit the postponed expression by substituting the payoff:

$$\text{PRCDS}_{a,b}(t,R) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau>t)}\Big\{ - \text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{E}_t[\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} D(t,T_i)] + R \sum_{i=a+1}^{b} \mathbb{E}_t[D(t,T_i)\alpha_i \mathbf{1}_{\{\tau \geq T_i\}}]\Big\} \tag{7}$$

Let us deal with the definition of postponed (running) CDS forward rate $R_{a,b}(t)$. This can be defined as that R that makes the PRCDS value equal to zero at time t, so that

$$\text{PRCDS}_{a,b}(t, R_{a,b}(t), \text{L}_{\text{GD}}) = 0$$

(notice that this $R_{a,b}$ is the $R^{PR}_{a,b}$ of Brigo (2004)). The idea is then solving this equation in $R_{a,b}(t)$. In doing this one has to be careful. It is best to start moving from expression (6) rather than (5). Equate this expression to zero and derive R correspondingly. Strictly speaking, the resulting R would be defined on $\{\tau > t\}$ only, since elsewhere the equation is satisfied automatically thanks to the indicator in front of the expression, regardless of R. Since the value of R does not matter when $\{\tau < t\}$, the equation being satisfied automatically, we need not worry about $\{\tau < t\}$ and may define, in general, R =ProtectionLegValue/PremiumLegValue,

$$R_{a,b}(t) = \frac{\text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)}. \quad (8)$$

where

$$\mathbf{1}_{\{\tau>t\}} \boxed{\bar{P}(t,T)} := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}} \boxed{\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t)}$$

is the price at time t of a defaultable zero-coupon bond maturing at time T. The corresponding default free zero coupon bond is denoted by $P(t,T)$. This approach amounts to equating to zero only the expected value part in (6), and in a sense is a way of privileging "partial information" $\mathcal{F}_t$ expected values to "complete information" $\mathcal{G}_t$ ones. Our R above is defined everywhere and not only conditional on $\tau > t$. The technical tool allowing us to do this is the above-mentioned Jeanblanc Rutkowski (2000) result, and this is the spirit of part of the work in Jamshidian (2002).

A remark on how the market quotes running CDS prices is in order at this point. First we notice that typically the T's are three- months spaced. Usually at time $t = 0$, provided default has not yet occurred, the market sets R to a value $R_{a,b}^{\text{MID}}(0)$ that makes the CDS fair at time 0, i.e. such that $\text{CDS}_{a,b}(0, R_{a,b}^{\text{MID}}(0), \text{L}_{\text{GD}}) = 0$. Actually, bid and ask levels are quoted for R. Typically in quoted CDS we have $T_a = 0$ and T_b spanning a set of increasing maturities, even though in recent times the quoting mechanism has changed in some respects. Indeed, the quoting mechanism has become more similar to the mechanism of the futures markets. Let 0 be the current time. Maturities $T_a, \ldots, T_b$ are fixed at the original time 0 to some values such as 1y, 2y, 3y etc and then, as time moves for example to $t = 1day$, the CDS maturities are not shifted correspondingly of 1 day as before but remain 1y,2y etc from the original time 0. This means that the times to maturity of the quoted CDS's decrease as time passes. When the quoting time approaches maturity, a new set of maturities are fixed and so on. A detail concerning the "constant maturities" paradigm is that when the first maturity T_a is less than one month away from the quoting time (say 0), the payoff two terms

$$T_a D(0,T_a) R \mathbf{1}_{\{\tau>T_a\}} + (T_{a+1} - T_a) D(0,T_{a+1}) R \mathbf{1}_{\{\tau>T_{a+1}\}}$$

are replaced by

$$T_{a+1} D(0,T_{a+1}) R \mathbf{1}_{\{\tau>T_{a+1}\}}$$

in determining the "fair" R. If we neglect this last convention, once we fix the quoting time (say to 0) the method to strip implied hazard functions is the same under the two quoting paradigms. The same happens when not neglecting the convention if we are exactly at one of the "0 dates", so that for example $T_1 - t = 1y$. Brigo and Alfonsi (2003) present a more detailed section on the "constant time-to-maturity" earlier paradigm, and illustrate the notion of implied deterministic intensity (hazard function).

We also set

$$\widehat{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i),$$

the denominator in (8), so that $R_{a,b}$ is a tradable asset (upfront CDS) divided by $\widehat{C}_{a,b}(t)$. It follows that under the measure $\widehat{\mathbb{Q}}^{a,b}$ having $\widehat{C}_{a,b}(t)$ as numeraire, $R_{a,b}$ follows a martingale. For more details see Brigo (2004). We will be interested in the particular cases $a = i-1, b = i$ and $a = i-2, b = i$.

In the following it will be useful to consider a running postponed CDS on a one-period interval, with $T_a = T_{j-1}$ and $T_b = T_j$. We obtain, for the related CDS forward rate:

$$R_j(t) := \mathrm{L}_{\mathrm{GD}} \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} = \mathrm{L}_{\mathrm{GD}} \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}|\mathcal{F}_t]}{\widehat{C}_{j-1,j}(t)}$$

where we have set $R_j := R_{j-1,j}$.

A last remark concerns an analogy with the default-free swap market model, where we have a formula linking swap rates to forward rates through a weighted average. This is useful since it leads to an approximated formula for swaptions in the LIBOR model, see for example Brigo and Mercurio (2001), Chapter 6. A similar approach can be obtained for CDS forward rates. It is easy to check that

$$R_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \alpha_i R_i(t) \bar{P}(t,T_i)}{\sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i)} = \sum_{i=a+1}^{b} \bar{w}_i^{a,b}(t) R_i(t) \approx \sum_{i=a+1}^{b} \bar{w}_i^{a,b}(0) R_i(t). \quad (9)$$

A possible lack of analogy with the swap rates is that the $\bar{w}$'s

$$\bar{w}_i^{a,b}(t) := \frac{\alpha_i \bar{P}(t,T_i)}{\sum_{h=a+1}^{b} \alpha_h \bar{P}(t,T_h)}$$

cannot be expressed as functions of the R_i's only, unless we make some particular assumptions on the correlation between default intensities and interest rates. However, if we freeze the $\bar{w}$'s to time 0, which we have seen to work in the default-free LIBOR model, we obtain easily a useful approximated expression for $R_{a,b}$ and its volatility in terms of R_i's and their volatilities/correlations. **A similar approach is pursued in Section 5 below, and the reader who is interested only in the derivation of the approximated formula given in the beginning may go there directly. Here instead we hint at deriving the real dynamics without compromises, which will be useful in future work for Monte Carlo tests of the numerical approximation.**

In general, when not freezing, the presence of stochastic intensities besides stochastic interest rates adds degrees of freedom. Now the $\bar{P}$'s (and thus the $\bar{w}$'s) can be determined as functions for example of one- and two-period rates. Indeed, it is easy to show that

$$\bar{P}(t,T_i) = \bar{P}(t,T_{i-1}) \frac{\alpha_{i-1}(R_{i-1}(t) - R_{i-2,i}(t))}{\alpha_i(R_{i-2,i}(t) - R_i(t))}, \quad \frac{\bar{P}(t,T_j)}{\bar{P}(t,T_i)} = \frac{\alpha_i}{\alpha_j} \prod_{k=i+1}^{j} \frac{R_{k-1} - R_{k-2,k}}{R_{k-2,k} - R_k} \quad (10)$$

To have the formula working we need to assume $R_{i-2,i}(t) \neq R_i(t)$. We will therefore assume in the paper that at the initial time $R_{i-2,i}(0) \neq R_i(0)$. Under the approximated frozen dynamics for these two quantities we will derive below, we can see that the probability of them to be equal at any future time t is generally 0, since this can be computed as the probability of the difference of two correlated continuous random variables to be zero.

We show below how this formula helps us in obtaining a market model for CDS rates. For the time being let us keep in mind that the exact weights $\bar{w}(t)$ in (9) are completely specified in terms of $R_i(t)$'s and $R_{i-2,i}(t)$'s, so that if we include these two rates in our dynamics the "system" is closed in that we also know all the relevant $\bar{P}$'s. The difference with the LIBOR/Swap model is that here to close the system we need also two-period rates.

We close this section by summarizing our notation in Table (1).

4 One- and Two- Period CDS Rates joint Dynamics under a single pricing measure

Let us postulate the following dynamics for one- and two- period CDS forward rates. Recall that $R_j = R_{j-1,j}$.

$$dR_j(t) = \sigma_j(t) R_j(t) dZ_j^j(t)$$
$$dR_{j-2,j}(t) = \nu_j(t; R) R_{j-2,j}(t) dV_j^{j-2,j}(t)$$

In the Brownian shocks Z and V the upper index denotes the measure (i.e. the measure associated with the numeraires $\widehat{C}_{j-1,j}, \widehat{C}_{j-2,j}$ in the above case) and the lower index denotes to which component of the one- and two- period rate vectors the shock refers. **The volatilities σ are deterministic, whereas the ν's depend on the one-period R's.** We assume correlations

$$dZ_i dZ_j = \rho_{i,j} dt, \quad dV_i dV_j = \eta_{i,j} dt, \quad dZ_i dV_j = \theta_{i,j} dt$$

and $R_{i-2,i}(t) \in (\min(R_{i-1}(t), [R_{i-1}(t) + R_i(t)]/2), \max(R_{i-1}(t), [R_{i-1}(t) + R_i(t)]/2))$**. This latter condition ensures that the resulting $\bar{P}$ from formula (10) be positive and decreasing with respect to the maturity, i.e.** $0 < \bar{P}(t, T_i)/\bar{P}(t, T_{i-1}) < 1$**. The specific definition of ν ensuring this property is currently under investigation.**

We aim at finding the drift of a generic R_j under the measure associated with $\widehat{C}_{i-1,i}$, let us say for $j \geq i$.

The change of numeraire toolkit provides the formula relating shocks under $\widehat{C}_{i-1,i}$ to shocks under $\widehat{C}_{j-2,j}$, see for example Formula (2.13) in Brigo and Mercurio (2001), Chapter 2. We can write

$$d\begin{bmatrix} Z^{j-2,j} \\ V^{j-2,j} \end{bmatrix} = d\begin{bmatrix} Z^i \\ V^i \end{bmatrix} - \text{CorrMatrix} \times \text{VectorDiffusionCoefficient}\left(\ln\left(\frac{\widehat{C}_{j-2,j}}{\widehat{C}_{i-1,i}}\right)\right)' dt$$

Let us abbreviate "Vector Diffusion Coefficient" by "DC".

This is actually a sort of operator for diffusion processes that works as follows. $DC(X_t)$ is the row vector $\mathbf{v}$ in

$$dX_t = (...)dt + \mathbf{v}\ d \begin{bmatrix} Z_t \\ V_t \end{bmatrix}$$

for diffusion processes X with Z and V column vectors Brownian motions common to all relevant diffusion processes. This is to say that if for example $dR_1 = \sigma_1 R_1 dZ_1^1$, then

$$DC(R_1) = [\sigma_1 R_1,\ 0,\ 0, \ldots,\ 0].$$

Let us call Q the total correlation matrix including ρ, η and θ. We have

$$d \begin{bmatrix} Z^{j-2,j} \\ V^{j-2,j} \end{bmatrix} = d \begin{bmatrix} Z^{i} \\ V^{i} \end{bmatrix} - Q\ \mathrm{DC}\left(\ln \left(\frac{\widehat{C}_{j-2,j}}{\widehat{C}_{i-1,i}} \right) \right) dt$$

Now we need to compute

$$\mathrm{DC}\left(\ln \left(\frac{\widehat{C}_{j-2,j}}{\widehat{C}_{i-1,i}} \right) \right) = \mathrm{DC}\left(\ln \left(\frac{\alpha_{j-1}\bar{P}(t,T_{j-1}) + \alpha_j \bar{P}(t,T_j)}{\alpha_i \bar{P}(t,T_i)} \right) \right) =$$

$$= \mathrm{DC}\left(\ln \left(\frac{\alpha_{j-1}}{\alpha_i}\frac{\alpha_i}{\alpha_{j-1}} \prod_{k=i+1}^{j-1} \frac{R_{k-1} - R_{k-2,k}}{R_{k-2,k} - R_k} + \frac{\alpha_j}{\alpha_i}\frac{\alpha_i}{\alpha_j} \prod_{k=i+1}^{j} \frac{R_{k-1} - R_{k-2,k}}{R_{k-2,k} - R_k} \right) \right)$$

$$= \mathrm{DC}\left(\ln \left(\left[\prod_{k=i+1}^{j-1} \frac{R_{k-1} - R_{k-2,k}}{R_{k-2,k} - R_k} \right] \left[1 + \frac{R_{j-1} - R_{j-2,j}}{R_{j-2,j} - R_j} \right] \right) \right)$$

$$= \mathrm{DC}\left(\sum_{k=i+1}^{j-1} \ln \left(\frac{R_{k-1} - R_{k-2,k}}{R_{k-2,k} - R_k} \right) \right) + \mathrm{DC}\left(\ln \left(\frac{R_{j-1} - R_j}{R_{j-2,j} - R_j} \right) \right)$$

$$= \sum_{k=i+1}^{j-1} \mathrm{DC}\left(\ln \left(\frac{R_{k-1} - R_{k-2,k}}{R_{k-2,k} - R_k} \right) \right) + \mathrm{DC}\left(\ln \left(\frac{R_{j-1} - R_j}{R_{j-2,j} - R_j} \right) \right) =$$

$$= \sum_{k=i+1}^{j-1} [\mathrm{DC}(\ln(R_{k-1} - R_{k-2,k})) - \mathrm{DC}(\ln(R_{k-2,k} - R_k))] +$$

$$+\mathrm{DC}(\ln(R_{j-1} - R_j)) - \mathrm{DC}(\ln(R_{j-2,j} - R_j))$$

$$= \sum_{k=i+1}^{j-1} \frac{\mathrm{DC}(\ R_{k-1} - R_{k-2,k})}{R_{k-1} - R_{k-2,k}} - \sum_{k=i+1}^{j-1} \frac{\mathrm{DC}(\ R_{k-2,k} - R_k)}{R_{k-2,k} - R_k} +$$

$$+\frac{\mathrm{DC}(\ R_{j-1} - R_j)}{R_{j-1} - R_j} - \frac{\mathrm{DC}(\ R_{j-2,j} - R_j)}{R_{j-2,j} - R_j} =$$

$$= \sum_{k=i+1}^{j-1} \frac{(\ \mathrm{DC}(R_{k-1}) - \mathrm{DC}(R_{k-2,k}))}{R_{k-1} - R_{k-2,k}} - \sum_{k=i+1}^{j-1} \frac{(\ \mathrm{DC}(R_{k-2,k}) - \mathrm{DC}(R_k))}{R_{k-2,k} - R_k}$$

$$+\frac{\mathrm{DC}(R_{j-1}) - \mathrm{DC}(R_j)}{R_{j-1} - R_j} - \frac{\mathrm{DC}(R_{j-2,j}) - \mathrm{DC}(R_j)}{R_{j-2,j} - R_j}$$

It follows that

$$dZ_m^{j-2,j} - dZ_m^i = -\sum_{k=i+1}^{j-1} \frac{(\rho_{k-1,m}\sigma_{k-1}R_{k-1} - \theta_{m,k}\nu_k R_{k-2,k})}{R_{k-1} - R_{k-2,k}} dt + \sum_{k=i+1}^{j-1} \frac{(\theta_{m,k}\nu_k R_{k-2,k} - \rho_{k,m}\sigma_k R_k)}{R_{k-2,k} - R_k} dt$$
$$- \frac{\rho_{j-1,m}\sigma_{j-1}R_{j-1} - \rho_{j,m}\sigma_j R_j}{R_{j-1} - R_j} dt + \frac{\theta_{m,j}\nu_j R_{j-2,j} - \rho_{j,m}\sigma_j R_j}{R_{j-2,j} - R_j}\ dt$$

and

$$dV_m^{j-2,j} - dV_m^i = -\sum_{k=i+1}^{j-1} \frac{(\theta_{k-1,m}\sigma_{k-1}R_{k-1} - \eta_{m,k}\nu_k R_{k-2,k})}{R_{k-1} - R_{k-2,k}} dt + \sum_{k=i+1}^{j-1} \frac{(\eta_{m,k}\nu_k R_{k-2,k} - \theta_{k,m}\sigma_k R_k)}{R_{k-2,k} - R_k} dt +$$
$$- \frac{\theta_{j-1,m}\sigma_{j-1}R_{j-1} - \theta_{j,m}\sigma_j R_j}{R_{j-1} - R_j} dt + \frac{\eta_{j,m}\nu_j R_{j-2,j} - \theta_{j,m}\sigma_j R_j}{R_{j-2,j} - R_j} dt =: \bar{\phi}_m^{i,j}\ dt$$

Therefore, by subtracting from the first equation, taking $h > i$:

$$dZ_m^h - dZ_m^i = dZ_m^{j-2,j} - dZ_m^i - (dZ_m^{j-2,j} - dZ_m^h) =$$
$$= -\sum_{k=i+1}^{h} \frac{(\rho_{k-1,m}\sigma_{k-1}R_{k-1} - \theta_{m,k}\nu_k R_{k-2,k})}{R_{k-1} - R_{k-2,k}} dt + \sum_{k=i+1}^{h} \frac{(\theta_{m,k}\nu_k R_{k-2,k} - \rho_{k,m}\sigma_k R_k)}{R_{k-2,k} - R_k} dt =: \bar{\mu}_m^{i,h} dt$$

so that we finally obtain (taking $h = j$)

$$dR_j(t) = \sigma_j R_j(t)(\bar{\mu}_j^{i,j} dt + dZ_j^i(t))$$
$$dR_{j-2,j}(t) = \nu_j R_{j-2,j}(t))(\bar{\phi}_j^{i,j} dt + dV_j^i(t)),$$

or, by setting

$$\mu_j^i := \bar{\mu}_j^{i,j}\ \sigma_j, \quad \phi_j^i := \bar{\phi}_j^{i,j}\ \nu_j,$$

we have

$$dR_j(t) = R_j(t)(\mu_j^i dt + \sigma_j dZ_j^i(t)), \quad dR_{j-2,j}(t) = R_{j-2,j}(t)(\phi_j^i dt + \nu_j dV_j^i(t)),$$

and since μ and ϕ are completely determined by one- and two- period rates vectors $R = [R_{i-1,i}]_i$ and $R^{(2)} = [R_{i-2,i}]_i$, **the system is closed.** We can write a vector SDE which is a vector diffusion for all the one- and two- period rates under any of the $\widehat{C}_{i-1,i}$ measures:

$$d\begin{bmatrix} R \\ R^{(2)} \end{bmatrix} = \mathrm{diag}(\mu(R, R^{(2)}), \phi(R, R^{(2)})) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} dt + \mathrm{diag}(\sigma, \nu) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix}\ d\begin{bmatrix} Z^i \\ V^i \end{bmatrix}$$

At this point a Monte Carlo simulation of the process, based on a discretization scheme for the above vector SDE is possible. One only needs to know the initial CDS rates $R(0), R^{(2)}(0)$, which if not directly available one can build by suitably stripping spot CDS rates. Given the volatilities and correlations, one can easily simulate the scheme by means of standard Gaussian shocks.

If C is the Cholesky decomposition of the correlation Q ($Q = CC'$ with "C" lower triangular matrix) and W is a standard Brownian motion under $\widehat{C}_{i-1,i}$, we can write

$$d \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} = \mathrm{diag}(\mu(R, R^{(2)}), \phi(R, R^{(2)})) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} dt + \mathrm{diag}(\sigma, \nu) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} C\, dW \quad (11)$$

The log process can be easily simulated with a Milstein scheme.

5 An approximated model with a single family of rates and proof of the main result

Consider now the approximation

$$R_j(t) := \mathrm{L_{GD}} \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} = \mathrm{L_{GD}} \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} =$$

$$= \mathrm{L_{GD}} \frac{\mathbb{E}[D(t,T_{j-1})\mathbf{1}_{\{\tau>T_{j-1}\}} \boxed{D(t,T_j)/D(t,T_{j-1})} |\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} = \dots$$

At this point we approximate the boxed ratio of stochastic discount factors with the related zero-coupon bonds, obtaining

$$\approx \mathrm{L_{GD}} \frac{\mathbb{E}[D(t,T_{j-1})\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] \boxed{P(t,T_j)/P(t,T_{j-1})} - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

$$= \mathrm{L_{GD}} \frac{\bar{P}(t,T_{j-1})P(t,T_j)/P(t,T_{j-1}) - \bar{P}(t,T_j)}{\alpha_j \bar{P}(t,T_j)} = \frac{\mathrm{L_{GD}}}{\alpha_j} \left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(t))\bar{P}(t,T_j)} - 1 \right)$$

$$\approx \frac{\mathrm{L_{GD}}}{\alpha_j} \left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(0))\bar{P}(t,T_j)} - 1 \right) = \widetilde{R}_j(t)$$

where F is the forward LIBOR rate between T_{j-1} and T_j. This last definition can be inverted so as to have

$$\frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)} = \left(\frac{\alpha_j}{\mathrm{L_{GD}}} \widetilde{R}_j + 1 \right) (1 + \alpha_j F_j(0)) > 1 \quad (12)$$

as long as $\widetilde{R} > 0$, provided that $F_j(0) > 0$ as should be. This means that we are free to select any martingale dynamics for $\widetilde{R}_j$ under $\widehat{\mathbb{Q}}^{j-1,j}$, as long as $\widetilde{R}_j$ remains positive. Choose than such a family of $\widetilde{R}$ as building blocks

$$d\widetilde{R}_i(t) = \sigma_i(t)\widetilde{R}_i(t)dZ_i^i(t), \quad \text{for all} \quad i$$

and define the $\bar{P}$ by using (12) to obtain inductively $\bar{P}(t,T_j)$ from $\bar{P}(t,T_{j-1})$ and from $\widetilde{R}_j$. **This way, the numeraires $\bar{P}$ become functions only of the $\widetilde{R}$'s, so that now the system is closed and all one has to model is the one-period rates $\widetilde{R}$ vector. No need to model two-period rates in this framework.**

In this context the change of numeraire becomes

$$dZ^j = dZ^i - \rho \mathrm{DC}\left(\ln\left(\frac{\widehat{C}_{j-1,j}}{\widehat{C}_{i-1,i}}\right)\right)' dt = dZ^i - \rho \mathrm{DC}\left(\ln\left(\frac{\bar{P}(t,T_j)}{\bar{P}(t,T_i)}\right)\right)' dt =$$

$$= dZ^i - \rho \mathrm{DC} \ln\left[\left(\prod_{h=j+1}^{i}\left(\frac{\alpha_h}{\mathrm{L_{GD}}}\widetilde{R}_h + 1\right)(1+\alpha_h F_h(0))\right)\right]' dt$$

$$= dZ^i - \rho \sum_{h=j+1}^{i} \mathrm{DC}\ln\left(\left(\frac{\alpha_h}{\mathrm{L_{GD}}}\widetilde{R}_h + 1\right)(1+\alpha_h F_h(0))\right)' dt =$$

$$= dZ^i - \rho \sum_{h=j+1}^{i} \mathrm{DC}\ln\left(\left(\frac{\alpha_h}{\mathrm{L_{GD}}}\widetilde{R}_h + 1\right)\right)' dt = dZ^i - \rho \sum_{h=j+1}^{i} \frac{1}{\widetilde{R}_h + \frac{\mathrm{L_{GD}}}{\alpha_h}} \mathrm{DC}(\widetilde{R}_h)' dt$$

so that we can write

$$dZ^j_k = dZ^i_k - \sum_{h=j+1}^{i} \rho_{k,h} \frac{\sigma_h(t)\widetilde{R}_h}{\widetilde{R}_h + \frac{\mathrm{L_{GD}}}{\alpha_h}} dt$$

from which we have the dynamics of $\widetilde{R}_i$ under $\mathbb{Q}^j$:

$$d\widetilde{R}_i = \sigma_i \widetilde{R}_i dZ^i_i = \sigma_i \widetilde{R}_i \left(dZ^j_i + \sum_{h=j+1}^{i} \rho_{j,h} \frac{\sigma_h \widetilde{R}_h}{\widetilde{R}_h + \frac{\mathrm{L_{GD}}}{\alpha_h}} dt\right) =: \widetilde{R}_i(\widetilde{\mu}^j_i(\widetilde{R})dt + \sigma_i dZ^j_i) \quad (13)$$

Consider the drift term in the last formula. If we compute $E^{j-1,j}[\widetilde{R}_i(T_{j-1})]$ we obtain

$$\begin{aligned} E^{j-1,j}[\widetilde{R}_i(T_{j-1})] &\approx \widetilde{R}_i(0)\exp\left\{\int_0^{T_{j-1}} \widetilde{\mu}^j_i(\widetilde{R}(0))du\right\} \\ &= \widetilde{R}_i(0)\exp\left\{\sum_{k=j+1}^{i} \frac{\widetilde{R}_k(0)}{\widetilde{R}_k(0) + \mathrm{L_{GD}}/\alpha_k}\rho_{j,k}\int_0^{T_{j-1}} \sigma_i(u)\sigma_k(u)du\right\} \quad (14) \end{aligned}$$

and, if we take volatilities σ to be constant, we have

$$\approx \widetilde{R}_i(0)\exp\left\{T_{j-1}\sigma_i \cdot \left(\sum_{k=j+1}^{i} \rho_{j,k} \frac{\sigma_k \widetilde{R}_k(0)}{\widetilde{R}_k(0) + \mathrm{L_{GD}}/\alpha_k}\right)\right\}$$

Under independence between intensities and interest rates (and in particular under deterministic intensities, which are a common assumption when stripping one-period

CDS rates from multi-period ones), by definition of R_j it is easy to show that at time 0, both the original $R_j(0)$ and the approximated $\widetilde{R}_j(0)$ are given in terms of the survival probabilities as

$$R_j(0) = \widetilde{R}_j(0) = \mathrm{L}_{\mathrm{GD}}/\alpha_j \left(\frac{\mathbb{Q}(\tau > T_{j-1})}{\mathbb{Q}(\tau > T_j)} - 1 \right) \tag{15}$$

and hence (14) is reduced to

$$E^{j-1,j}[\widetilde{R}_i(T_{j-1})] = \frac{\mathrm{L}_{\mathrm{GD}}}{\alpha_i} \left(\frac{\mathbb{Q}(\tau > T_{i-1})}{\mathbb{Q}(\tau > T_i)} - 1 \right) \exp \left\{ T_{j-1}\sigma_i \sum_{k=j+1}^{i} \rho_{j,k}\sigma_k \left(1 - \frac{\mathbb{Q}(\tau > T_k)}{\mathbb{Q}(\tau > T_{k-1})} \right) \right\} \tag{16}$$

It follows that the convexity effect vanishes if the ratio $\frac{\mathbb{Q}(\tau>T_j)}{\mathbb{Q}(\tau>T_{j-1})}$ is close to one.

Now, based on the approximated dynamics (13) and the related expectation above, we prove the main result of the paper, i.e. Proposition 2.4.

Proof. To prove the proposition, we compute the price of the premium leg as

$$\sum_{j=a+1}^{b} \alpha_j \mathbb{E}_0[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}} R_{j-1,j+c}(T_{j-1})] = \dots$$

The first approximation we consider is (9) applied to $R_{j-1,j+c}(T_{j-1})$, so that

$$R_{j-1,j+c}(T_{j-1}) \approx \sum_{i=j}^{j+c} \bar{w}_i^j(0) R_i(T_{j-1}), \quad \bar{w}_i^j(0) = \frac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0,T_h)}$$

Then by substituting this in the premium leg expression we have

$$\dots \approx \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{w}_i^j(0) \mathbb{E}_0[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}} R_i(T_{j-1})] =$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{w}_i^j(0) \mathbb{E}_0[D(0,T_j) R_i(T_{j-1}) \mathbb{E}(\mathbf{1}_{\{\tau>T_j\}} | \mathcal{F}_{T_j})]$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \bar{w}_i^j(0) \mathbb{E}_0 \left[\frac{B(0)}{B(T_j)} (R_i(T_{j-1})\ \widehat{C}_{j-1,j}(T_j)) \right]$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \bar{w}_i^j(0) \widehat{C}_{j-1,j}(0) \widehat{\mathbb{E}}_0^{j-1,j}[R_i(T_{j-1})] = \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{w}_i^j(0) \bar{P}(0,T_j) \widehat{\mathbb{E}}_0^{j-1,j}[R_i(T_{j-1})] = \dots$$

where we have applied the change of numeraire, moving from the risk neutral numeraire B to the numeraires $\widehat{C}_{j-1,j}$'s. The last expected value can be computed based on (16). By substituting the expected value expression, we obtain the final formula.

□

6 A few numerical examples

We report input data and outputs for a name with relatively large CDS forward rates. We consider the FIAT car company CDS market quotes as of December 20, 2004. Since in Brigo and Alfonsi (2003) we have some evidence on the fact that CDS prices depend very little on the correlation between interest rates and credit spreads, when stripping credit spreads from CDS data we may assume independence between interest rates and credit spreads. This leads to a model where it is easy to strip default probabilities (hazard rates) from CDS prices, as hinted at again in Brigo and Alfonsi (2003). Using this independence assumption, we strip default (or survival) probabilities from CDS quotes with increasing maturities.

6.1 Inputs

We take as inputs the following Fiat CDS rates and use mid quotes

T_b	$R_{0,b}^{BID}(bps)$	$R_{0,b}^{ASK}$
1Y	99.9	175.57
2Y	172.5	231.38
3Y	243.73	286.13
5Y	348.85	366.54
7Y	380	410
10Y	395.16	412.73

We take $\text{R}_{\text{EC}} = 0.4$ (so that $\text{L}_{\text{GD}} = 0.6$). The input zero coupon curve, and the survival risk-neutral probabilities stripped from the above CDS quotes are reported in Appendix 1 below.

6.2 Outputs

We start by giving a table for

$$\text{Conv}(\sigma, \rho) := \text{CDS}_{\text{CM}_{a,b,c}}(0, \text{L}_{\text{GD}}, \sigma, \rho) - \text{CDS}_{\text{CM}_{a,b,c}}(0, \text{L}_{\text{GD}}; \rho = 0).$$

The first term is computed by assuming the volatilities σ_i of forward one-period CDS rates R_i to have a common value σ and the pairwise correlations $\rho_{i,j}$ to have a common value ρ. This first term is then given by formula (1). The second term is the simpler value (2) where no correction due to CDS forward rate dynamics is accounted for. This difference then gives us the impact of volatilities and correlations of CDS rates on the CMCDS price. The difference is always positive, similarly to what happens to analogous constant maturity swaps in default free markets under similar conditions on volatilities and correlation. It is the impact of "convexity" on the CMCDS valuation. We take

$a = 0$, $b = 20$ (5y final maturity) and $c = 20$ (which means we are considering non-standard 5y3m CDS rates in the CMCDS premium leg, $c = 19$ would amount to a 5y CDS rate).

We obtain

$\text{Conv}(\sigma, \rho)$	ρ: 0.7	0.8	0.9	0.99
σ: 0.1	0.000659	0.000754	0.000848	0.000933
0.2	0.002662	0.003047	0.003435	0.003784
0.4	0.011066	0.012742	0.014442	0.015995
0.6	0.026619	0.030964	0.035464	0.039652

The "convexity difference" increases with respect both to correlation and volatility, as expected.

The next table reports the so called "participation rate" $\phi_{a,b,c}(\sigma, \rho)$ for a CMCDS with final $T_b = 5y$ ($a = 0, b = 20$, recalling that resets occur quarterly), with $5y3m$ constant maturity CDS rates ($c = 20$),

$$\phi_{0,20,20}(\sigma, \rho) = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS”}} = \frac{\sum_{j=1}^{20} \alpha_j \bar{P}(0, T_j) R_{0,20}(0)}{\sum_{j=1}^{20} \alpha_j \mathbb{E}_0^{j-1,j}[D(0, T_j)\mathbf{1}_{\{\tau > T_j\}} R_{j-1,j+20}(T_{j-1})]},$$

The CMCDS premium leg is computed with our approximated market model based on one-period rates $\widetilde{R}$. As we see from the outputs, the participation rate increases with volatility and correlation, as is expected from the "convexity adjustment" effect.

$\phi_{0,20,20}(\sigma, \rho)$	ρ: 0.7	0.8	0.9	0.99
σ: 0.1	0.71358	0.71325	0.71292	0.71262
0.2	0.70664	0.70532	0.704	0.70281
0.4	0.67894	0.67368	0.66842	0.66368
0.6	0.63302	0.62128	0.60957	0.59907

Finally, we fix volatilities and correlations and check how the patterns change when changing final maturity $T_b = T_i$. We consider the following quantities at time 0 and with $T_a = 0$:

$$x_i = \frac{\text{“Constant maturity rate”}}{\text{“standard rate”}} = \frac{R_{i-1,i+c}(0)}{R_{0,b}(0)}, \quad i = 1, \dots, b$$

$$y_i = \frac{\mathbb{E}_0^{i-1,i}[D(0, T_i)\mathbf{1}_{\{\tau > T_i\}} R_{i-1,i+c}(T_{i-1})]}{\bar{P}(0, T_i) R_{0,b}(0)}, \quad i = 1, \dots, b$$

$$z_i = \frac{\mathbb{E}_0^{i-1,i}[D(0, T_i)\mathbf{1}_{\{\tau > T_i\}} R_{i-1,i+c}(T_{i-1})]}{\bar{P}(0, T_i) R_{i-1,i+c}(0)}, \quad i = 1, \dots, b$$

$$\psi_i = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS”}} = \frac{\sum_{j=1}^{i} \alpha_j \bar{P}(0, T_j) R_{0,i}(0)}{\sum_{j=1}^{i} \alpha_j \bar{P}(0, T_j) R_{j-1,j+c}(0)}, \quad i = 1, \dots, b$$

$$\phi_i = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS with convexity”}} = \frac{\sum_{j=1}^{i} \alpha_j \bar{P}(0,T_j) R_{0,i}(0)}{\sum_{j=1}^{i} \alpha_j \mathbb{E}_0^{j-1,j}[D(0,T_j)\mathbf{1}_{\{\tau > T_j\}} R_{j-1,j+c}(T_{j-1})]}.$$

x_i	y_i	z_i	ψ_i	ϕ_i	
					$\sigma = 0.4$;
					$\rho = 0.9$;
1.0668	1.0668	1	0.37773	0.37773	
1.1288	1.1359	1.0063	0.36281	0.36162	Rec = 0.4;
1.1914	1.2075	1.0135	0.35281	0.35039	a=0;
1.2525	1.2792	1.0214	0.34359	0.33993	c = 20;
1.3107	1.3495	1.0297	0.33512	0.33024	b = 20;
1.3673	1.4193	1.038	0.34187	0.33548	
1.4171	1.4826	1.0462	0.36905	0.36064	
1.4515	1.53	1.0541	0.40755	0.39664	
1.4716	1.5622	1.0616	0.45262	0.43881	
1.4798	1.5818	1.0689	0.49477	0.47785	
1.4837	1.5979	1.0769	0.52661	0.50671	
1.4905	1.6175	1.0852	0.55072	0.52799	
1.4999	1.6403	1.0936	0.56931	0.54384	
1.5122	1.666	1.1018	0.58674	0.55846	
1.5236	1.69	1.1092	0.60704	0.57574	
1.5275	1.706	1.1168	0.62715	0.5928	
1.5274	1.7174	1.1244	0.64681	0.60938	
1.5249	1.7236	1.1303	0.67017	0.62939	
1.5106	1.7173	1.1368	0.69254	0.64843	
1.4924	1.7047	1.1422	0.71589	0.66842	

The x_i's measure how the constant maturity CDS rate differs multiplicatively from the standard CDS rate, so they are a measure of how the constant maturity CDS differs from a standard CDS in the premium rate paid at each period. We find an increasing pattern in T_i as partly expected from the fact that the input CDS rates $R_{0,b}^{BID,ASK}$ are increasing with respect to maturity T_b.

The y_i's measure the same effect while taking into account “convexity”, i.e. future randomness of the payoff and correlation. The y_i's would reduce to the x_i's if correlations ρ were taken equal to 0. The y maintain the increasing pattern with respect to T_i.

The z_i's measure the multiplicative impact of “convexity”, in that they are due to contributions stemming from volatilities σ and correlations ρ of CDS rates. The impact is increasing with maturity T_i, as expected from the sign in the exponent of the convexity adjustments and from the positive signs of correlations (and volatilities).

Finally, as seen above, the ψ_i's are the so called “participation rates” for different terminal maturities T_i. They give the ratio between the premium leg in a standard CDS protecting in $[0, T_i]$ and the premium leg in CMCDS for the same protection interval when ignoring the convexity adjustment due to correlation and volatilities. The ϕ_i's are the participation rates computed when taking into account convexity due to volatilities and correlations. We have seen a particular participation rate ϕ earlier. In the table above for ϕ_i we obtain an initially decreasing pattern followed by a longer increasing pattern for both ψ and ϕ. Notice that, on the longest participation rate, in the last row of the related table, convexity has an impact moving from a 71.59% participation rate when not including "convexity" (ignoring correlations and volatilities) to a 66.84% participation rate when including convexity. There is a 4.8% difference in the participation rate of this FIAT 5y-5y3m CMCDS with correlations set at 0.9 and volatilities at 40%.

7 Further work

In further research we need to propose a realistic dynamics for one- and two- period rates that completely specifies the market model, along the guidelines given in this paper and in Brigo (2004). Then we may test the market formula proposed here against Monte Carlo simulation of the exact dynamics. Moreover, examining the formula outputs for ρ matrices with more realistic decorrelation patterns and for different names can be appropriated.

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Appendix 1. Input data and survival probabilities

The input zero coupon curve, and the survival risk-neutral probabilities stripped from FIAT CDS quotes are:

α_i	T_i	$P(0,T_i)$	$\mathbb{Q}(\tau > T_i)$
0	0	0.99994	0.99994
0.24444	0.24444	0.99459	0.99429
0.25556	0.5	0.989	0.98856
0.25556	0.75556	0.98309	0.98279
0.25278	1.0083	0.97709	0.97712
0.25	1.2583	0.97098	0.97155
0.25556	1.5139	0.96458	0.96433
0.25556	1.7694	0.958	0.95409
0.25278	2.0222	0.95133	0.94108
0.25	2.2722	0.94448	0.92552
0.25556	2.5278	0.9373	0.9086
0.25556	2.7833	0.92995	0.89227
0.25278	3.0361	0.92251	0.87669
0.25278	3.2889	0.91502	0.86165
0.25556	3.5444	0.90731	0.84618
0.25556	3.8	0.89929	0.82931
0.25278	4.0528	0.89139	0.81203
0.25	4.3028	0.88373	0.79449
0.25556	4.5583	0.87544	0.77495
0.25556	4.8139	0.8673	0.75531
0.25278	5.0667	0.85906	0.73503
0.25	5.3167	0.85085	0.7142
0.25556	5.5722	0.84255	0.69403
0.25556	5.8278	0.83417	0.67559

(continues in the next page)

α_i	T_i	$P(0,T_i)$	$\mathbb{Q}(\tau > T_i)$
0.25278	6.0806	0.82572	0.65879
0.25	6.3306	0.81735	0.64351
0.25556	6.5861	0.80892	0.62968
0.25556	6.8417	0.80035	0.61709
0.25278	7.0944	0.79182	0.60594
0.25278	7.3472	0.78344	0.59601
0.25556	7.6028	0.77494	0.58651
0.25556	7.8583	0.76641	0.57685
0.25278	8.1111	0.75794	0.56715
0.25	8.3611	0.74977	0.55744
0.25556	8.6167	0.74141	0.5474
0.25556	8.8722	0.73303	0.53726
0.25278	9.125	0.72474	0.52713
0.25	9.375	0.7168	0.51705
0.25556	9.6306	0.70869	0.50667
0.25556	9.8861	0.70041	0.49601
0.25278	10.139	0.69241	0.48565
0.25	10.389	0.6849	0.4756