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Credit Default Swaps Calibration and Option Pricing with the SSRD Stochastic Intensity and Interest-Rate Model

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Abstract

In the present paper we introduce a two-dimensional shifted square-root diffusion (SSRD) model for interest rate derivatives and single-name credit derivatives, in a stochastic intensity framework. The SSRD is the unique model, to the best of our knowledge, allowing for an automatic calibration of the term structure of interest rates and of credit default swaps (CDS's). Moreover, the model retains free dynamics parameters that can be used to calibrate option data, such as caps for the interest rate market and options on CDS's in the credit market. The calibrations to the interest-rate market and to the credit market can be kept separate, thus realizing a superposition that is of practical value. We discuss the impact of interest-rate and default-intensity correlation on calibration and pricing, and test it by means of Monte Carlo simulation. We use a variant of Jamshidian's decomposition to derive an analytical formula for CDS options under CIR++ stochastic intensity. Finally, we develop an analytical approximation based on a Gaussian dependence mapping for some basic credit derivatives terms involving correlated CIR processes.

JEL classification code: G13.
AMS classification codes: 60H10, 60J60, 60J75, 91B70

1 Credit Default Swaps

A credit default swap is a contract ensuring protection against default. This contract is specified by a number of parameters. Let us start by assigning a maturity T.

Consider two companies "A" and "B" who agree on the following:

If a third reference company "C" defaults at time $\tau < T$, "B" pays to "A" a certain cash amount Z, supposed to be deterministic in the present paper, either at maturity T or at the default time τ itself. This cash amount is a *protection* for "A" in case "C" defaults. A typical case occurs when "A" has bought a corporate bond issued from "C" and is waiting for the coupons and final notional payment from this bond: If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this danger, asking "B" a payment that roughly amounts to the bond notional in case "C" defaults.

In case the protection payment occurs at T we talk about "protection at maturity", whereas in the second case, with a payment occurring at τ, we talk about "protection at default".

Typically Z is equal to a notional amount, or to a notional amount minus a recovery rate.

In exchange for this protection, company "A" agrees to pay periodically to "B" a fixed amount R_f. Payments occur at times $\mathcal{T} = \{T_1, \dots, T_n\}$, $\alpha_i = T_i - T_{i-1}$, $T_0 = 0$, fixed in advance at time 0 up to default time τ if this occurs before maturity T, or until maturity T if no default occurs. We assume $T_n \leq T$, typically $T_n = T$.

Assume we are dealing with "protection at default", as is more frequent in the market. Formally we may write the CDS discounted value to "B" at time t as

$$\mathbf{1}_{\{\tau>t\}} \left[D(t,\tau)(\tau - T_{\beta(\tau)-1}) R_f \mathbf{1}_{\{\tau<T_n\}} + \sum_{i=\beta(t)}^{n} D(t,T_i)\alpha_i R_f \mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{\tau<T\}} D(t,\tau)\ Z \right] \tag{1}$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date of $T_1, \dots, T_n$ following t. The stochastic discount factor at time t for maturity T is denoted by $D(t,T) = B(t)/B(T)$, where $B(t) = \exp(\int_0^t r_u du)$ denotes the bank-account numeraire, r being the instantaneous short interest rate.

We denote by $\mathrm{CDS}(t, \mathcal{T}, T, R_f, Z)$ the price at time t of the above CDS. The pricing formula for this product depends on the assumptions on interest-rate dynamics and on the default time τ.

In general, we can compute the CDS price according to risk-neutral valuation (see for example Bielecki and Rutkowski (2002)):

$$\begin{aligned} \mathrm{CDS}(t, \mathcal{T}, T, R_f, Z) = \mathbf{1}_{\{\tau>t\}} \mathbb{E} \Big\{ D(t,\tau)(\tau - T_{\beta(\tau)-1}) R_f \mathbf{1}_{\{\tau<T_n\}} \\ + \sum_{i=\beta(t)}^{n} D(t,T_i)\alpha_i R_f \mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{\tau<T\}} D(t,\tau)\ Z \Big| \mathcal{F}_t \vee \sigma(\{\tau<u\}, u \leq t) \Big\} \end{aligned} \tag{2}$$

where $\mathcal{F}_t$ is the basic filtration without default, typically representing the information flow of interest rates, intensities and possibly other default-free market quantities (see Bielecki and Rutkowski (2001)), and $\mathbb{E}$ denotes the risk-neutral expectation in the enlarged probability space supporting τ. Finally, we explain shortly how the market quotes CDS prices. Usually at time t, provided default has not yet occurred, the market sets R_f to a value $R_f^{\text{MID}}(t,T)$ that makes the CDS fair at time t, i.e. such that $\text{CDS}(t,\mathcal{T},T,R_f^{\text{MID}}(t,T),Z) = 0$. In fact, in the market CDS's are quoted at a time t through a bid and an ask value for this "fair" $R_f^{\text{MID}}(t,T)$, for a set of canonical maturities $T = t + 1y$ up to $T = t + 10y$.

2 A deterministic-intensity model

We consider the following model for default times. We denote by τ the default time and assume it to be the first jump-time of a time-inhomogeneous Poisson process with strictly increasing, continuous (and thus invertible) hazard function Γ and hazard rate (deterministic intensity) γ, with $\int_0^T \gamma(t)dt = \Gamma(T)$. We place ourselves under the risk-neutral measure $\mathbb{Q}$, so that all expected values and probabilities in the following concern the risk neutral world.

In general intensity can be stochastic, as we will see later on. In such a case it is denoted by λ and the related hazard process is denoted by $\Lambda(T) = \int_0^T \lambda_t dt$.

In this section we consider the time-inhomogeneous Poisson process with deterministic intensity γ. Such a process N_t has the following well known properties: the related process $M_t = N_{\Gamma^{-1}(t)}$ is a time-homogeneous Poisson process with constant intensity equal to $\bar{\gamma} = 1$. This means that M is a unit-jump increasing, right continuous process with stationary independent increments and $M_0 = 0$. Moreover we know that

$$M_t - M_s \sim \mathcal{P}(\bar{\gamma}(t-s)),$$

with $\mathcal{P}(a)$ denoting the Poisson law with parameter a.

Notice that we can also write $N_t = M_{\Gamma(t)}$. It follows that if N jumps the first time at τ, then M jumps the first time at time $\Gamma(\tau)$. But since M is Poisson with intensity one, its first jump time $\Gamma(\tau)$ is distributed as an exponential random variable with parameter 1, so that

$$\mathbb{Q}\{\Gamma(\tau) < s\} = 1 - \exp(-s).$$

In particular, notice that since Γ is strictly increasing,

$$\mathbb{Q}\{s < \tau \le t\} = \mathbb{Q}\{\Gamma(s) < \Gamma(\tau) \le \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)).$$

Finally, if we assume for example interest rates to come from a diffusion process for the short-rate,

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t,$$

with W a Brownian motion under the risk-neutral measure $\mathbb{Q}$, we have the following. Since a Poisson process and a Brownian motion defined on a common probability space are independent (see for example Bielecki and Rutkowski (2001), p. 188), this means that the processes N and r are independent. We can thus assume the stochastic discount factor for rates, $D(s,t) = \exp(-\int_s^t r_u du)$, and the default time τ to be independent whenever intensities are deterministic. We will be able to introduce dependence between interest rates and default by means of a stochastic intensity that will be correlated with the short rate.

2.1 Pricing and calibrating CDS with deterministic intensity models

Consider the CDS payoff (1) and price (2) in the context of deterministic intensities. Since interest rates are independent of τ, we can set $\tau = \Gamma^{-1}(\xi)$, with ξ an exponential random variable of parameter 1 independent of interest rates.

Consider first

$$\begin{aligned}
\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left\{ D(t,\tau)(\tau - T_{\beta(\tau)-1})R_f \mathbf{1}_{\{\tau<T_n\}} \middle| \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \le t)\right\} = \\
\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left\{ \mathbb{E}\left\{ D(t,\tau)(\tau - T_{\beta(\tau)-1})R_f \mathbf{1}_{\{\tau<T_n\}} \middle| \mathcal{F}_t \vee \tau\right\} \middle| \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \le t)\right\} = \\
\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left\{ P(t,\tau)(\tau - T_{\beta(\tau)-1})R_f \mathbf{1}_{\{\tau<T_n\}} \middle| \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \le t)\right\} = \\
\mathbf{1}_{\{t<\tau\}} R_f \int_t^{T_n} P(t,u)(u - T_{\beta(u)-1}) d\mathbb{Q}\{\tau \le u | \sigma(\{\tau < s\}, s \le t)\} = \\
\mathbf{1}_{\{t<\tau\}} R_f \int_t^{T_n} P(t,u)(T_{\beta(u)-1} - u) d_u(e^{-(\Gamma(u)-\Gamma(t))}).
\end{aligned}$$

Also, by similar arguments,

$$\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left\{ Z D(t,\tau)\mathbf{1}_{\{\tau<T\}} \middle| \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \le t)\right\} = -\mathbf{1}_{\{t<\tau\}} Z \int_t^T P(t,u) d_u(e^{-(\Gamma(u)-\Gamma(t))}),$$

and, finally,

$$\begin{aligned}
\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left\{ D(t,T_i)\mathbf{1}_{\{\tau>T_i\}} \middle| \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \le t)\right\} = \\
\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left\{ D(t,T_i) \middle| \mathcal{F}_t\right\} \mathbb{E}\left\{ \mathbf{1}_{\{\tau>T_i\}} \middle| \sigma(\{\tau < u\}, u \le t)\right\} = \\
\mathbf{1}_{\{t<\tau\}} P(t,T_i) e^{\Gamma(t)-\Gamma(T_i)},
\end{aligned}$$

so that the CDS price (2) is in this case

$$\begin{aligned}
\text{CDS}(t,\mathcal{T},T,R_f,Z;\Gamma(\cdot)) = \mathbf{1}_{\{t<\tau\}} \Bigg[R_f \int_t^{T_n} P(t,u)(T_{\beta(u)-1} - u) d_u(e^{-(\Gamma(u)-\Gamma(t))}) + \\
\sum_{i=\beta(t)}^{n} P(t,T_i) R_f \alpha_i e^{\Gamma(t)-\Gamma(T_i)} + Z \int_t^T P(t,u) d_u(e^{-(\Gamma(u)-\Gamma(t))}) \Bigg].
\end{aligned} \tag{3}$$

One may wish to calibrate the determinisic-intensity model to CDS market quotes $R_f^{\text{MID}}(0,T)$ in order to value different payoffs. To do so, one has to invert the model

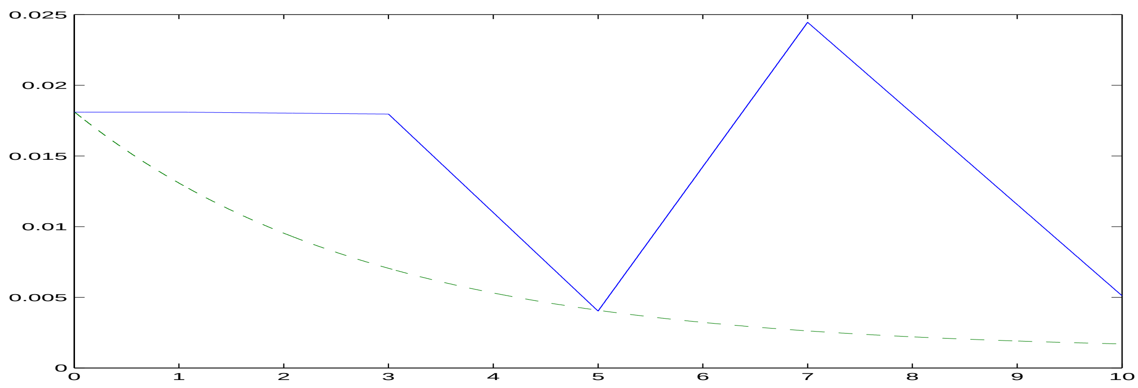

Figure 1: Graph of the implied deterministic intensity $t \mapsto \gamma^{\text{mkt}}(t)$ for Merrill-Lynch CDS's of several maturities on October 25, 2002 (continuous line) and the best approximating hazard rate coming from a time-homogeneous CIR model (dashed line) that we will extend to CIR++ to recover exactly γ^{mkt}

formula and find the Γ's that match the given CDS market quotes, by solving in Γ a set of equations for increasing T: Solve

$$\text{CDS}(0, \mathcal{T}, T, R_f^{\text{MID}}(0,T), Z; \Gamma(\cdot)) = 0$$

in Γ for different T's.

We can assume a piecewise constant intensity γ, constant among different maturities T, and invert prices in an iterative way as T increases, deriving each time the new part of γ that is consistent with the CDS quote R_f for the new maturity. Other possibilities include a piecewise *linear* γ (Prampolini (2002)) or some parametric forms for γ such as Nelson and Siegel or extensions thereof. In all such cases CDS prices in γ with the quoted R_f^{MID} have to be set to zero and such equations or error minimizations in γ have to be solved. In the following we denote by γ^{mkt} and Γ^{mkt} respectively the hazard rate and hazard function that are obtained in a deterministic model when calibrating CDS market data as above. We close this section by giving an example in Figure 1 of a piecewise linear hazard rate $\gamma^{\text{mkt}}(t)$ obtained by calibrating the 1y, 3y, 5y, 7y and 10y CDS's on Merrill-Lynch on October 2002. In Figure 2 the related risk-neutral default probabilities are given. These are equal, first order in the hazard function, to the hazard function $\Gamma(t)$ itself, since $\mathbb{Q}\{\tau < t\} = 1 - \exp(-\Gamma(t)) \approx \Gamma(t)$ for small Γ.

3 A two-factor shifted square-root diffusion model for intensity and interest rates

In this section we consider a model with stochastic intensity and interest rates.

In this kind of models λ is a stochastic process but, conditional on the filtration generated by λ itself, N remains a time-inhomogeneous Poisson process with intensity

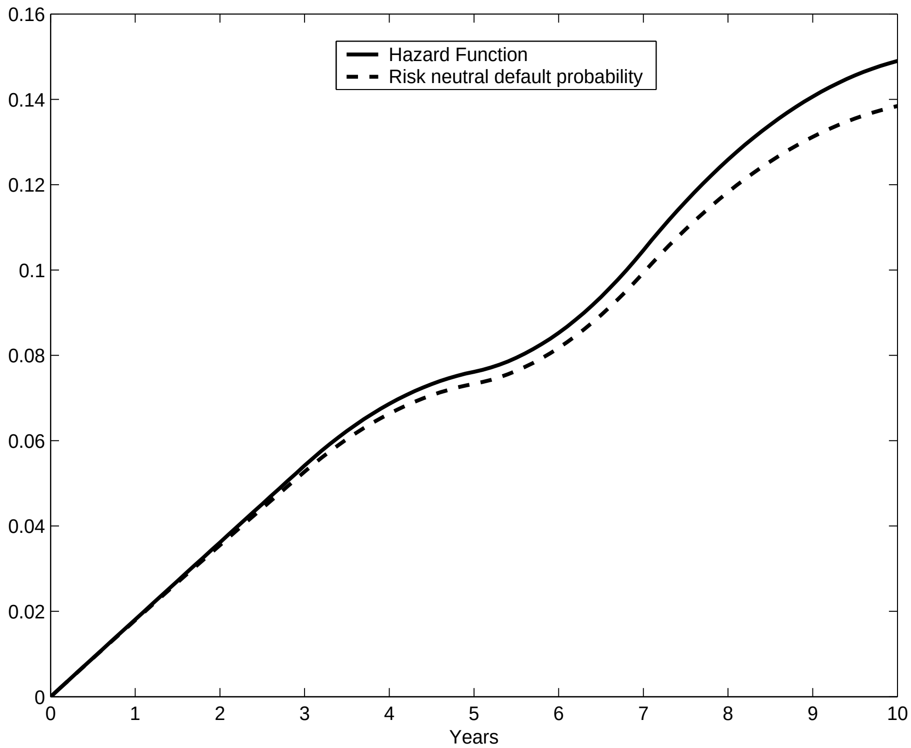

Figure 2: Graph of the implied hazard function $t \mapsto \Gamma^{\mathrm{mkt}}(t)$ and implied risk-neutral default probability for Merrill-Lynch CDS's of several maturities on October 25, 2002

λ, and conditional on this filtration all results seen at the beginning of Section 2 on survival and default probabilities are still valid. N is called a Cox process.

We now describe our assumptions on the short-rate process r and on the intensity dynamics. For more details on the use of the shifted dynamics, on a default-free interest rate context, see for example Avellaneda and Newman (1998), or Brigo and Mercurio (2001, 2001b).

3.1 CIR++ short-rate model

We write the short-rate r_t as the sum of a deterministic function φ and of a Markovian process x_t^α:

$$r_t = x_t^\alpha + \varphi(t;\alpha)\ ,\quad t \geq 0, \tag{4}$$

where φ depends on the parameter vector α (which includes x_0^α) and is integrable on closed intervals. Notice that x_0^α is indeed one more parameter at our disposal: we are free to select its value as long as

$$\varphi(0;\alpha) = r_0 - x_0\ .$$

We take as reference model for x the Cox-Ingersoll-Ross (1985) process:

$$dx_t^\alpha = k(\theta - x_t^\alpha)dt + \sigma\sqrt{x_t^\alpha}dW_t,$$

where the parameter vector is $\alpha = (k, \theta, \sigma, x_0^\alpha)$, with k, θ, σ, x_0^α positive deterministic constants. The condition

$$2k\theta > \sigma^2$$

ensures that the origin is inaccessible to the reference model, so that the process x^α remains positive. As is well known, this process x^α features a noncentral *chi-square* distribution, and yields an affine term-structure of interest rates. Accordingly, analytical formulae for prices of zero-coupon bond options, caps and floors, and, through Jamshidian's decomposition, coupon-bearing bond options and swaptions, can be derived. We can therefore consider the CIR++ model, consisting of our extension (4), and calculate the analytical formulae implied by such a model, by simply adapting the analogous explicit expressions for the reference CIR model as given in Cox et al. (1985). Denote by f instantaneous forward rates, i.e. $f(t,T) = -\partial \ln P(t,T)/\partial T$.

The initial market zero-coupon interest-rate curve $T \mapsto P^M(0,T)$ is automatically calibrated by our model if we set $\varphi(t;\alpha) = \varphi^{\text{CIR}}(t;\alpha)$ where

$$\begin{aligned} \varphi^{\text{CIR}}(t;\alpha) &= f^M(0,t) - f^{\text{CIR}}(0,t;\alpha), \\ f^{\text{CIR}}(0,t;\alpha) &= 2k\theta \frac{(\exp\{th\}-1)}{2h+(k+h)(\exp\{th\}-1)} \\ &+ x_0 \frac{4h^2 \exp\{th\}}{[2h+(k+h)(\exp\{th\}-1)]^2} \end{aligned}$$

with

$$h = \sqrt{k^2 + 2\sigma^2}.$$

For restrictions on the α's that keep r positive see Brigo and Mercurio (2001, 2001b).

Moreover, the price at time t of a zero-coupon bond maturing at time T is

$$P(t,T) = \frac{P^M(0,T)A(0,t;\alpha)\exp\{-B(0,t;\alpha)x_0\}}{P^M(0,t)A(0,T;\alpha)\exp\{-B(0,T;\alpha)x_0\}} P^{\text{CIR}}(t,T,r_t - \varphi^{\text{CIR}}(t;\alpha);\alpha) \quad (5)$$

where

$$P^{\text{CIR}}(t,T,x_t;\alpha) = \mathbb{E}_t(e^{-\int_t^T x^\alpha(u)du}) = A(t,T;\alpha)\exp\{-B(t,T;\alpha)x_t\}$$

is the bond price formula for the basic CIR model, with

$$\begin{aligned} A(t,T;\alpha) &= \left[\frac{2h\ \exp\{(k+h)(T-t)/2\}}{2h+(k+h)(\exp\{(T-t)h\}-1)}\right]^{2k\theta/\sigma^2}, \\ B(t,T;\alpha) &= \frac{2(\exp\{(T-t)h\}-1)}{2h+(k+h)(\exp\{(T-t)h\}-1)}, \end{aligned}$$

from which the continuously compound spot rate $R(t,T)$ (still affine in r_t), the spot LIBOR rate $L(t,T)$, forward LIBOR rates $F(t,T,S)$ and all other kind of rates can be easily computed as explicit functions of r_t. We omit the argument α when clear from the context.

The cap option price formula for the CIR++ model can be derived easily in closed form from the corresponding formula for the basic CIR model. This formula is a function of the parameters α. In our application we will calibrate the parameters α to cap prices, by inverting the analytical CIR++ formula, so that our interest rate model is calibrated to the initial zero coupon curve through ϕ and to the cap market through α. For more details, see Brigo and Mercurio (2001, 2001b).

3.2 CIR++ intensity model

For the intensity model we adopt a similar approach, in that we set

$$\lambda_t = y_t^\beta + \psi(t;\beta) \ , \quad t \geq 0, \tag{6}$$

where ψ is a deterministic function, depending on the parameter vector β (which includes y_0^β), that is integrable on closed intervals. As before, y_0^β is indeed one more parameter at our disposal: We are free to select its value as long as

$$\psi(0;\beta) = \lambda_0 - y_0 \ .$$

We take y again of the form:

$$dy_t^\beta = \kappa(\mu - y_t^\beta)dt + \nu\sqrt{y_t^\beta}dZ_t,$$

where the parameter vector is $\beta = (\kappa, \mu, \nu, y_0^\beta)$, with κ, μ, ν, y_0^β positive deterministic constants. Again we assume the origin to be inaccessible, i.e.

$$2\kappa\mu > \nu^2.$$

For restrictions on the β's that keep λ positive, as is required in intensity models, see Brigo and Mercurio (2001, 2001b). We will often use the integrated process, that is $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y^\beta(t) = \int_0^t y_s^\beta ds$ and $\Psi(t,\beta) = \int_0^t \psi(s,\beta)ds$.

We take the short interest-rate and the intensity processes to be correlated, by assuming the driving Brownian motions W and Z to be instantaneously correlated according to

$$dW_t \ dZ_t = \rho \ dt.$$

This way to model the intensity and the short interest rate can be viewed as a generalization of a particular case of the Lando's (1998) approach, and can also be seen as a generalization of a particular case of the Duffie and Singleton (1997, 1999) square-root diffusion model (see for example Bielecki and Rutkowski (2001), pp 253-258). In both cases we add a non homogeneous term to recover exactly fundamental market data in the spirit of Brigo and Mercurio (2001, 2001b).

3.3 Calibrating the joint stochastic model to CDS: Separability

With the above choice for λ, in the credit derivatives world we have formulae that are analogous to the ones for interest-rate derivatives products. Consider for example the risk-neutral survival probability. We have easily

$$\mathbb{E}(\mathbf{1}_{\tau>t}) = \mathbb{E}[\mathbb{E}(\mathbf{1}_{\tau>t}|\mathcal{F}^\lambda)] = \mathbb{E}[\mathbb{E}(\mathbf{1}_{\Lambda(\tau)>\Lambda(t)}|\mathcal{F}^\lambda)] = \mathbb{E}e^{-\Lambda(t)} = \mathbb{E}(e^{-\int_0^t \lambda(u)du}),$$

since, conditional on λ, $\Lambda(\tau)$ is an exponential random variable with parameter one. Notice that, if λ were a short-rate process, the last expectation of the "stochastic discount factor" would simply be the zero-coupon bond price in our interest-rate model. So we see that survival probabilities for the λ model are the analogous of zero-coupon bond prices P in the r model. Thus if we choose for λ a CIR++ process, survival probabilities will be given by the CIR++ model bond price formula.

In particular, by expressing credit default swaps data through the implied hazard function Γ^{mkt}, according to the method described in Section 2.1, we see that in order to reproduce such data with our λ model we need have, in case $\rho = 0$ (independence between interest-rates r and default intensities λ),

$$\mathbb{Q}(\tau > t)_{model} = \mathbb{E}(e^{-\Lambda(t)}) = e^{-\Gamma^{\text{mkt}}(t)} = \mathbb{Q}(\tau > t)_{market}.$$

Taking into account our particular specification (6) of λ, the central equality reads

$$\exp(-\Gamma^{\text{mkt}}(t)) = \mathbb{E}\exp\left(-\Psi(t,\beta) - Y^\beta(t)\right)$$

from which

$$\Psi(t,\beta) = \Gamma^{\text{mkt}}(t) + \ln(\mathbb{E}(e^{-Y^\beta(t)})) = \Gamma^{\text{mkt}}(t) + \ln(P^{\text{CIR}}(0,t,y_0;\beta)), \tag{7}$$

where we choose the parameters β in order to have a positive function ψ (i.e. an increasing Ψ). Thus, if ψ is selected according to this last formula, as we will assume from now on, the model is calibrated to the market implied hazard function Γ^{mkt}, i.e. to CDS data.

Recall that in the above calibration procedure we assumed $\rho = 0$. Indeed, it is easy to show via iterated conditioning that in such a case calibrating the implied hazard function to the model survival probabilities is equivalent to directly calibrate the (r, λ)-model by setting to zero CDS prices corresponding to the market quoted R_f's. More precisely, one can show by straightforward calculations that if $\rho = 0$ and $\psi(\cdot;\beta)$ is selected according to (7), then the price of the CDS under the stochastic intensity model λ is the same price obtained under deterministic intensity γ^{mkt} and is given by (3). So in a sense when $\rho = 0$ the CDS price does not depend on the dynamics of (λ, r), and in particular it does not depend on $k, \theta, \sigma, \kappa, \nu$ and μ. We will verify this also numerically in Table 6: by amplifying instensity randomness through an increase of κ, ν and μ we do not substantially affect the CDS price in case $\rho = 0$.

However, if $\rho \neq 0$, the CDS becomes in principle dependent on the dynamics, and the two procedures are not equivalent, and the correct one would be to equate to zero the model CDS prices (now depending on ρ, given the nonlinear nature of some terms in the payoff) corresponding to market quoted R_f's.

This is rather annoying, since the attractive feature of the model is the separate and semi-automatic calibration of the interest-rate part to interest-rate data and of the intensity part to credit market data. Indeed, in the separable case the credit derivatives desk might ask for the α parameters and the $\phi(\cdot;\alpha)$ curve to the interest-rate derivatives desk, and then proceed with finding β and $\psi(\cdot;\beta)$ from CDS data. This ensures also a consistency of the interest rate model that is used in credit derivatives evaluation with the interest rate model that is used for default-free derivatives. This separate automatic calibration no longer holds if we introduce ρ, since now the dynamics of interest rates is also affecting the CDS price.

However, we will see below in table 6 that the impact of ρ is typically negligible on CDSs, even in case intensity randomness is increased by a factor from 3 to 5.

We can thus calibrate CDS data with $\rho = 0$, using the separate calibration procedure outlined above, and then set ρ to a desired value.

After calibrating CDS data through $\psi(\cdot,\beta)$, we are left with the parameters β, which can be used to calibrate further products, similarly to the way the α parameters of the r model are used to calibrate cap prices after calibration of the zero-coupon curve in the interest rate market. However, this will be interesting when option data on the credit derivatives market will become more liquid. Even as we write, the first proposals for CDS options have reached our bank through Bloomberg, but the bid-ask spreads are very large and suggest to consider these first quotes with caution (Prampolini (2002)). At the moment we content ourselves of calibrating only CDS's for the credit part. To help specifying β without further data we impose a constraint on the calibration of CDS's. We require the β's to be found that keep Ψ positive and increasing and that minimize $\int_0^T \psi(s,\beta)^2 ds$. This minimization amounts to contain the departure of λ from its time-homogeneous component y^β as much as possible. Indeed, if we take as criterion the integrated squared difference between "instantaneous forward rates" γ^{mkt} in the market and $f^{\text{CIR}}(\cdot;\beta)$ in our homogeneous CIR model with β parameters, constraining these differences to be positive at all points, the related minimization gives us the time-homogeneous CIR model β that is closest to market data under the given constraints.

We calibrated the same CDS data as at the end of Section 2.1 up to a ten years maturity and obtained the following results

$$\beta: \quad \kappa = 0.354201,\ \mu = 0.00121853,\ \nu = 0.0238186;\ y_0 = 0.0181,$$

with the ψ function plotted in Fig 3. The interest-rate model part has been calibrated to the initial zero curve and to cap prices, along the lines of Brigo and Mercurio (2001, 2001b), which we do not repeat here. The parameters are

$$\alpha: \quad k = 0.528905,\ \theta = 0.0319904, \quad \sigma = 0.130035,\ x_0 = 8.32349 \times 10^{-5}.$$

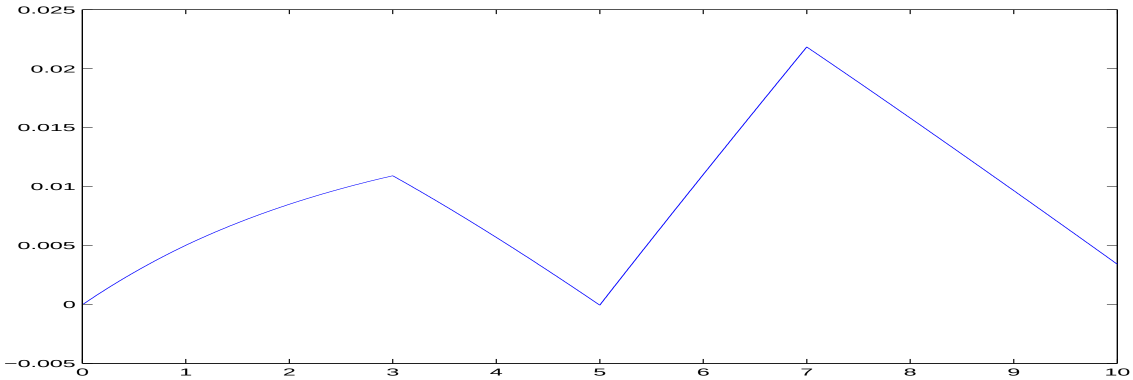

Figure 3: ψ function for the CIR++ model for λ calibrated to Merrill-Lynch CDS's of maturities up to 10y on October 25, 2002

To check that, as anticipated above, the impact of the correlation ρ is negligible on CDS's we reprice the 5y CDS we used in the above calibration with $\rho = 0$, ceteris paribus, by setting first $\rho = -1$ and then $\rho = 1$. As usual, the amount R_f renders the CDS fair at time 0, thus giving $\text{CDS}(0, \mathcal{T}, T, R_f, Z) = 0$ with the deterministic model or with the stochastic model when $\rho = 0$. In our case (market data of October 25, 2002) the MID value R_f corresponding to $R_f^{\text{BID}} = 0.009$ and $R_f^{\text{ASK}} = 0.0098$ is $R_f = 0.0094$, while $Z = 0.593$, corresponding to a recovery rate of 0.407. With this R_f and the above (r, λ) model calibrated with $\rho = 0$ we now set ρ to different values and, by the "Gaussian mapping" approximation technique described below to model (r, λ), we obtain the results given in Table 5. It is evident that the impact of rates/intensities correlation is almost negligible on CDS's, and typically well within a small fraction of the bid-ask spread (Prampolini (2002)). Indeed, with the above market quotes, in the case $\rho = 0$, we have

$$\text{CDS}(0, \mathcal{T}, T, R_f^{\text{BID}}, Z) = -17.14E-4, \quad \text{CDS}(0, \mathcal{T}, T, R_f^{\text{ASK}}, Z) = 17.16E-4. \quad (8)$$

So we see that the possible excursion of the CDS value due to correlation as from Table 5 is less than one tenth of the CDS excursion corresponding to the market bid-ask spread, and is thus negligible. This is further confirmed when Monte Carlo valuation replaces the Gaussian dependence mapping approximation, as one can see from Table 6.

3.4 Euler and Milstein explicit schemes for simulating (λ, r)

The SSRD model allows for known non-central chi-squared transition densities in the case with 0 correlation. However, when ρ is not zero we need to resort to numerical methods to obtain the joint distribution of r and λ and of their functionals needed for discounting and evaluating payoffs. The typical technique consists in adopting a discretization scheme for the relevant SDEs and then to simulate the Gaussian shocks corresponding to the joint Brownian motions increments in the discretized dynamics.

The easiest choice is given by the Euler Scheme. Let $t_0 = 0 < t_1 < ... < t_n = T$ be a discretization of the interval $[0, T]$. We write Z as $Z_t = \rho W_t + \sqrt{1-\rho^2} W'_t$ (Cholesky decomposition), where W'_t is a Brownian motion independent of W, and we obtain the increments of (W, Z) between t_i and t_{i+1} through simulation of the increments of W and W' (independent, centered Gaussian variables with variance $t_{i+1} - t_i$). We thus obtain:

$$\begin{aligned}
\tilde{x}^{\alpha}_{t_{i+1}} &= \tilde{x}^{\alpha}_{t_i} + k(\theta - \tilde{x}^{\alpha}_{t_i})(t_{i+1} - t_i) + \sigma\sqrt{\tilde{x}^{\alpha}_{t_i}}(W_{t_{i+1}} - W_{t_i}) \\
\tilde{y}^{\beta}_{t_{i+1}} &= \tilde{y}^{\beta}_{t_i} + \kappa(\mu - \tilde{y}^{\beta}_{t_i})(t_{i+1} - t_i) + \nu\sqrt{\tilde{y}^{\beta}_{t_i}}(Z_{t_{i+1}} - Z_{t_i}).
\end{aligned}$$

Although the regularity conditions that ensure a better convergence for the Milstein scheme are not satisfied here (the diffusion coefficient is not Lipschitz), one may try to apply it anyway. The related equations for $(\tilde{x}^{\alpha}_{t_i}, \tilde{y}^{\beta}_{t_i})$ are as follows:

$$\begin{aligned}
\tilde{x}^{\alpha}_{t_{i+1}} &= \tilde{x}^{\alpha}_{t_i} + k(\theta - \tilde{x}^{\alpha}_{t_i})(t_{i+1} - t_i) + \sigma\sqrt{\tilde{x}^{\alpha}_{t_i}}(W_{t_{i+1}} - W_{t_i}) + \frac{1}{4}\sigma^2[(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)] \\
\tilde{y}^{\beta}_{t_{i+1}} &= \tilde{y}^{\beta}_{t_i} + \kappa(\mu - \tilde{y}^{\beta}_{t_i})(t_{i+1} - t_i) + \nu\sqrt{\tilde{y}^{\beta}_{t_i}}(Z_{t_{i+1}} - Z_{t_i}) + \frac{1}{4}\nu^2[(Z_{t_{i+1}} - Z_{t_i})^2 - (t_{i+1} - t_i)]
\end{aligned}$$

However, for the SSRD model and for CIR processes in general we may obtain a more effective ad-hoc scheme as follows.

3.5 The Euler Implicit Positivity-Preserving Scheme

The previous explicit schemes present us with two major drawbacks. The first one is that such schemes do not ensure positivity of $\tilde{x}^{\alpha}_{t_i}$ (resp. $\tilde{y}^{\beta}_{t_i}$). It is possible to correct the above problem as follows: when we obtain a negative value, we can simulate a Brownian bridge on $[t_i, t_{i+1}]$, with a time step small enough to retrieve the positivity which is ensured in the continuous case when $2k\theta > \sigma^2$. The related second drawback is that the above basic explicit schemes do not preserve the following property of positivity. Let $\bar{\alpha} = (k, \theta, \sigma, \bar{x}_0)$, corresponding to a different initial condition $\bar{x}_0$ for x. "For a given path $(W_{t_i}(\omega))_i$, $x_0 \leq \bar{x}_0$ implies $\tilde{x}^{\alpha}_{t_i}(\omega) \leq \tilde{x}^{\bar{\alpha}}_{t_i}(\omega)$ for all t_i's". This property is important, since by taking a positive initial condition we would be sure that the simulation keeps the process positive. This positivity preserving property holds for the original process in continuous time[1]. We then set to find a scheme satisfying this property.

Let us remark that, for a sufficiently regular partition of $[0, T]$, when $\max\{t_{i+1} -$

[1]Indeed, if we set $\delta_t = x^{\bar{\alpha}}_t - x^{\alpha}_t$ with $\bar{x}_0 > x_0$, we have $d\delta_t = \delta_t(-kdt + \sigma/(\sqrt{x^{\alpha}_t} + \sqrt{x^{\bar{\alpha}}_t})dW_t)$. Thus, δ_t appears as a Doleans exponential process and remains positive for all t.

$t_i, 0 \leq i \leq n\} \to 0$ we have

$$\begin{aligned} x_t^\alpha &= x_0^\alpha + \int_0^t k(\theta - x_s)ds + \sigma \int_0^t \sqrt{x_s^\alpha} dW_s \\ &= x_0^\alpha + \sum_{i;t_i<t} k(\theta - x_{t_i})(t_{i+1} - t_i) + \sigma \sum_{i;t_i<t} \sqrt{x_{t_i}^\alpha}(W_{t_{i+1}} - W_{t_i}) + O((\max_i(t_{i+1} - t_i))^{1/2}) \\ &= x_0^\alpha + \sum_{i;t_i<t} k(\theta - x_{t_{i+1}})(t_{i+1} - t_i) + \sigma \sum_{i;t_i<t} \sqrt{x_{t_{i+1}}^\alpha}(W_{t_{i+1}} - W_{t_i}) \\ &\quad -\sigma \sum_{i;t_i<t} (\sqrt{x_{t_{i+1}}^\alpha} - \sqrt{x_{t_i}^\alpha})(W_{t_{i+1}} - W_{t_i}) + O((\max_i(t_{i+1} - t_i))^{1/2}) \\ &= x_0^\alpha + \sum_{i;t_i<t} (k\theta - \frac{\sigma^2}{2} - kx_{t_{i+1}})(t_{i+1} - t_i) + \sigma \sum_{i;t_i<t} \sqrt{x_{t_{i+1}}^\alpha}(W_{t_{i+1}} - W_{t_i}) \\ &\quad +O((\max_i(t_{i+1} - t_i))^{1/2}), \end{aligned}$$

in L^2, since $d\langle\sqrt{x_t^\alpha}, W_t\rangle = \sigma\ dt/2$. We will then introduce the following implicit scheme:

$$\begin{aligned} \tilde{x}_{t_{i+1}}^\alpha &= \tilde{x}_{t_i}^\alpha + (k\theta - \frac{\sigma^2}{2} - k\tilde{x}_{t_{i+1}}^\alpha)(t_{i+1} - t_i) + \sigma\sqrt{\tilde{x}_{t_{i+1}}^\alpha}(W_{t_{i+1}} - W_{t_i}) \\ \tilde{y}_{t_{i+1}}^\beta &= \tilde{y}_{t_i}^\beta + (\kappa\mu - \frac{\nu^2}{2} - \kappa\tilde{y}_{t_{i+1}}^\beta)(t_{i+1} - t_i) + \nu\sqrt{\tilde{y}_{t_{i+1}}^\beta}(Z_{t_{i+1}} - Z_{t_i}). \end{aligned}$$

It follows that $\sqrt{\tilde{x}_{t_{i+1}}^\alpha}$ is the unique positive root (when $2k\theta > \sigma^2$) of the second-degree polynomial $P(X) = (1+k(t_{i+1}-t_i))X^2 - \sigma(W_{t_{i+1}} - W_{t_i})X - (\tilde{x}_{t_i}^\alpha + (k\theta - \frac{\sigma^2}{2})(t_{i+1} - t_i))$, and we get $\tilde{x}_{t_{i+1}}^\alpha :=$

$$\left(\frac{\sigma(W_{t_{i+1}} - W_{t_i}) + \sqrt{\sigma^2(W_{t_{i+1}} - W_{t_i})^2 + 4(\tilde{x}_{t_i}^\alpha + (k\theta - \frac{\sigma^2}{2})(t_{i+1} - t_i))(1 + k(t_{i+1} - t_i))}}{2(1 + k(t_{i+1} - t_i))}\right)^2, \tag{9}$$

with a similar formula for $\tilde{y}_{t_{i+1}}^\beta$. Since this expression is clearly increasing in $\tilde{x}_{t_i}^\alpha$, we obtain the positivity preserving property above, and the positivity of $\tilde{x}_{t_{i+1}}^\alpha$ is guaranteed by construction. Thus, this Euler implicit positivity preserving scheme may be preferred to the explicit ones. We note here that it is also possible to construct other implicit schemes with a convex combination of the Euler explicit scheme and the implicit one described above. Finally, we briefly mention that control variate variance reduction techniques may be used to reduce the number of paths. As control variables one may use exponentials of integrals of λ and r, whose expectations are known in closed form.

3.6 Gaussian dependence mapping: A tractable approximated SSRD

To obtain an acceptable precision with a Monte-Carlo algorithm, it is unfortunately necessary to simulate a quite large number of scenarios. Indeed, the variance of the CDS is quite large in relative terms, due essentially to the indicator term in $\mathbf{1}_{\{\tau<T\}} ZD(0,\tau)$. A quick example can help us to clarify this important point. Compute the variance

$$\mathrm{Var}(\mathbf{1}_{\{\tau<T\}}) = E\mathbf{1}^2_{\{\tau<T\}} - (E\mathbf{1}_{\{\tau<T\}})^2 = E\mathbf{1}_{\{\tau<T\}} - (E\mathbf{1}_{\{\tau<T\}})^2.$$

Consider for example the ML data given in Fig 2 and take $T = 5y$. Notice that $E\mathbf{1}_{\{\tau<T\}}$ is the risk neutral probability to default in 5y for ML. From the graph we see that this is about 0.07. Then the above variance is about $0.07 - 0.07^2 = 0.0651$, and the standard deviation is $\sqrt{0.0651} = 0.2551$. We know that the standard error in the Monte Carlo method is given by the standard deviation of the object we are simulating divided by the square root of the number of paths. So we have that the standard error is about $0.2551/\sqrt{\text{npaths}}$. Now, we are estimating a quantity that is about 0.07 and we would like to have a standard error below one basis point. But if we wish our standard error to be below one basis point (i.e. 1/10000) we need set npaths$> (10000 * 0.2551)^2 = 6507601$.

We may slightly improve the situation by using a threshold barrier $\bar{B}$ such that $\mathbb{Q}(\Lambda(T) < \bar{B}) \simeq 1$. We thus assume that default may occur only when $\Lambda(\tau) < \bar{B}$. The idea is then to simulate default times conditional on $\xi := \Lambda(\tau) < \bar{B}$. Indeed, we see that if "DCDS" is the CDS discounted payoff, recalling that $\Lambda(\tau)$ is exponential with parameter 1 independent of $\mathcal{F}$, we have that

$$\mathbb{E}\ \mathrm{DCDS} = \mathbb{E}[\mathrm{DCDS}|\Lambda(\tau) < \bar{B}](1 - e^{-\bar{B}}) + \mathbb{E}[\mathrm{DCDS}|\Lambda(\tau) \geq \bar{B}]e^{-\bar{B}}.$$

The CDS value is known in case $\xi > \bar{B}$, since in this case default has not occurred and the price is $R_f \sum_{i=1}^n P(0,T_i)\alpha_i$. Our simulations then need concern only the first term, so if ξ is an exponential random variable with parameter one we just simulate $\xi|\xi < \bar{B}$, whose density is easily seen to be

$$p_{\xi|\xi<\bar{B}}(u) = 1_{\{u<\bar{B}\}} e^{-u}/(1 - e^{-\bar{B}}).$$

From the exponential distribution we see that simulating N scenarios for ξ amounts to simulate $N(1 - e^{-\bar{B}})$ scenarios with $\xi < \bar{B}$ and $Ne^{-\bar{B}}$ with $\xi \geq \bar{B}$. So in turn simulating $M = N(1 - e^{-\bar{B}})$ scenarios for $\xi < \bar{B}$, as we will do, amounts to simulate in total $N = M/(1 - e^{-\bar{B}})$ scenarios, the extra scenarios corresponding to the known value $R_f \sum_{i=1}^n P(0,T_i)\alpha_i$ of the CDS in case of default. Dividing by $1 - e^{-\bar{B}}$ may help us increase efficiency (in our examples typically it increases the number of scenarios by a factor 10), but a large amount of scenarios remains to be generated, and the time needed for Monte Carlo simulation remains large.

With the SSRD, using the independence of $\xi = \Lambda(\tau)$ from $\mathcal{F}$ (and thus λ and r), the value of the CDS at time 0 can be written, by simple passages, as:

$$\begin{aligned}
&\mathbb{E}\left\{D(0,\tau)(\tau - T_{\beta(\tau)-1})R_f \mathbf{1}_{\{\tau<T_n\}} + \sum_{i=1}^n D(0,T_i)\alpha_i R_f \mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{\tau<T\}} D(0,\tau)\ Z\right\}\\
&= \mathbb{E}\int_0^T \Big\{D(0,u)(u - T_{\beta(u)-1})R_f \mathbf{1}_{\{u<T_n\}} + \sum_{i=1}^n D(0,T_i)\alpha_i R_f \mathbf{1}_{\{u>T_i\}}\\
&\qquad - \mathbf{1}_{\{u<T\}} D(0,u)\ Z\Big\}\, d\mathbf{1}_{\{\tau\le u\}}\\
&= \mathbb{E}\Big\{R_f \int_0^{T_n} \exp\left(-\int_0^u (r_s+\lambda_s)ds\right)\lambda_u (u - T_{\beta(u)-1})du\\
&\qquad + \sum_{i=1}^n \alpha_i R_f \exp\left(-\int_0^{T_i}(r_s+\lambda_s)ds\right)\\
&\qquad - Z\int_0^T \exp\left(-\int_0^u (r_s+\lambda_s)ds\right)\lambda_u du\Big\}\\
&= R_f \int_0^{T_n} \mathbb{E}\left[\exp\left(-\int_0^u (r_s+\lambda_s)ds\right)\lambda_u\right](u - T_{\beta(u)-1})du\\
&+ \sum_{i=1}^n \alpha_i R_f \mathbb{E}\left[\exp\left(-\int_0^{T_i}(r_s+\lambda_s)ds\right)\right] - Z\int_0^T \mathbb{E}\left[\exp\left(-\int_0^u (r_s+\lambda_s)ds\right)\lambda_u\right]du,
\end{aligned}$$

where we have used iterated conditioning with respect to $\mathcal{F}_T$. The terms in λ and r appearing in the above formula are quite common in credit derivatives evaluation and it would be a good idea to have an approximated formula to compute them when $\rho \neq 0$.

Our idea is to "map" the two-dimensional CIR dynamics in an analogous tractable two-dimensional Gaussian dynamics that preserves as much as possible of the original CIR structure, and then do calculations with the Gaussian model. Recall that the CIR process and the Vasicek process for interest rates give both affine models. The first one is more convenient because it ensures positive values while the second one is more analytically tractable. Indeed, in the SSRD we have no formula for $\mathbb{E}[\exp(-\int_0^T(x_s^\alpha + y_s^\beta)ds)]$ and $\mathbb{E}[\exp(-\int_0^T(x_s^\alpha + y_s^\beta)ds)y_T^\beta]$ when $\rho \neq 0$, while in the Vasicek case, we can easily derive such formulae from the following

Lemma 3.1. *Let $A = m_A + \sigma_A N_A$ and $B = m_B + \sigma_B N_B$ be two random variables such that N_A and N_B are two correlated standard Gaussian random variables with $[N_A, N_B]$ jointly Gaussian vector with correlation $\bar\rho$. Then,*

$$\mathbb{E}(e^{-A}B) = m_B e^{-m_A + \frac{1}{2}\sigma_A^2} - \bar\rho\sigma_A\sigma_B e^{-m_A + \frac{1-\bar\rho^2}{2}\sigma_A^2} \qquad (10)$$

Lemma 3.2. *Let $x_t^{\alpha,V}$ and $y_t^{\beta,V}$ be two Vasicek processes as follows:*

$$\begin{aligned}
dy_t^{\beta,V} &= \kappa(\mu - y_t^{\beta,V})dt + \nu dZ_t,\\
dx_t^{\alpha,V} &= k(\theta - x_t^{\alpha,V})dt + \sigma dW_t
\end{aligned} \qquad (11)$$

with $dW_t\ dZ_t = \rho\ dt$. *Then* $A = \int_0^T (x_t^{\alpha,V} + y_t^{\beta,V})dt$ *and* $B = y_T^{\beta,V}$ *are Gaussian random variables with respective means:*

$$\begin{aligned} m_A &= (\mu+\theta)T - [(\theta - x_0)g(k,T) + (\mu - y_0)g(\kappa,T)] \\ m_B &= \mu - (\mu - y_0)e^{-\kappa T} \end{aligned}$$

respective variances:

$$\begin{aligned} \sigma_A^2 &= \left(\frac{\nu}{\kappa}\right)^2 (T - 2g(\kappa,T) + g(2\kappa,T)) + \frac{2\rho\nu\sigma}{k\kappa}(T - g(\kappa,T) - g(k,T) + g(\kappa+k,T)) \\ &\quad + \left(\frac{\sigma}{k}\right)^2 (T - 2g(k,T) + g(2k,T)) \\ \sigma_B^2 &= \nu^2 g(2\kappa,T) \end{aligned}$$

and correlation:

$$\bar{\rho} = \frac{1}{\sigma_A\sigma_B}\left[\frac{\nu^2}{\kappa}(g(\kappa,T) - g(2\kappa,T)) + \frac{\rho\sigma\nu}{k}(g(\kappa,T) - g(\kappa+k,T))\right]$$

where $g(k,T) = (1 - e^{-kT})/k$.

Thus, we are able to calculate $\mathbb{E}[\exp(-\int_0^T (x_t^{\alpha,V} + y_t^{\beta,V})dt)y_T^{\beta,V}]$ and $\mathbb{E}[\exp(-\int_0^T (x_t^{\alpha,V} + y_t^{\beta,V})dt)]$ (taking $m_B = 1$ and $\sigma_B = 0$); and taking for y^V a degenerated case ($\mu = \kappa = y_0 = 1, \nu = 0$), we obtain the well known formula for the bond price in the Vasicek model, which in our notation reads

$$\mathbb{E}\left[\exp(-\int_0^T x_s^{\alpha,V} ds)\right] = A^V(0,T;\alpha)\exp(-B^V(0,T;\alpha)x_0) \tag{12}$$
$$= \exp\left(-\theta t + (\theta - x_0)g(k,t) + \frac{1}{2}\left(\frac{\sigma}{k}\right)^2 (t - 2g(k,t) + g(2k,t))\right).$$

The idea is then to approximate the expectation by these formulae. More precisely, on $[0,T]$ we consider a particular Vasicek volatility in the dynamics (11), corresponding to taking $\alpha_T := (x_0, k, \theta, \sigma^{V,T})$ (resp. $\beta_T = (y_0, \kappa, \mu, \nu^{V,T})$) such that

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha_T,V} ds\right)\right] &= \mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha} ds\right)\right] \\ \text{(resp. } \mathbb{E}\left[\exp\left(-\int_0^T y_s^{\beta_T,V} ds\right)\right] &= \mathbb{E}\left[\exp\left(-\int_0^T y_s^{\beta} ds\right)\right] \text{)} \end{aligned}$$

where on the right hand sides we have the CIR processes. In the above equations expectations on both sides are analytically known, being bond price formulae for the Vasicek and CIR models respectively, and the inversions needed to retrieve $\sigma^{V,T}$ and $\nu^{V,T}$ are quite easy since the expression (12) is monotone with respect to σ. In practical cases, these volatilities exist, and can be seen as some sort of means of time-averages of $\sigma\sqrt{x_s^\alpha}$ (resp. $\nu\sqrt{y_s^\beta}$) on $[0,T]$. We then adopt the following approximations to estimate the impact of correlation:

$$\mathbb{E}\left[\exp\left(-\int_0^T (x_s^\alpha + y_s^\beta)ds\right)\right] \approx \mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha_T,V} + y_s^{\beta_T,V})ds\right)\right] \quad (13)$$

$$\mathbb{E}\left[\exp\left(-\int_0^T (x_s^\alpha + y_s^\beta)ds\right) y_T^\beta\right] \approx \mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha_T,V} + y_s^{\beta_T,V})ds\right) y_T^{\beta_T,V}\right] + \Delta \quad (14)$$

where

$$\Delta = \mathbb{E}\left[\exp\left(-\int_0^T x_s^\alpha ds\right)\right] \mathbb{E}\left[\exp\left(-\int_0^T y_s^\beta ds\right) y_T^\beta\right]$$
$$-\mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha,T,V} ds\right)\right] \mathbb{E}\left[\exp\left(-\int_0^T y_s^{\beta,T,V} ds\right) y_T^{\beta,T,V}\right]$$

and where we use the known analytical expressions for the right-hand sides.

3.7 Numerical Tests

We perform numerical tests for formulae (13) and (14) and for the related CDS prices, based on Monte Carlo simulations of the left-hand sides. We take the α and β parameters as from Section 3.3, and assume $T = 5y$. We obtain the results of Tables 1 and 2. The Vasicek mapped volatilities are $\sigma^{V,5y} = 0.016580$ and $\nu^{V,5y} = 0.0025675$.

To check the quality of the approximation under stress, we multiply all parameters k, θ, σ and κ, μ, ν by three and check again the approximation. We obtain the results shown in Tables 3 and 4, and now the Vasicek mapped volatilities are $\sigma^{V,5y} = 0.108596$ and $\nu^{V,5y} = 0.0060675$.

	$\rho =$ -1	$\rho =$1
LHS of (13)	0.86191 (0.861815 0.862004)	0.8624 (0.862272 0.862529)
RHS of (13)	0.861762,	0.862554

Table 1: MC simulation for the quality of the approximation (13)

If the values in Table 1 were interpreted as bond prices, the corresponding continuously compounded spot rates would be $-\ln(0.86191)/5 = 0.02972$ and $-\ln(0.861762)/5 = 0.029755$, respectively, giving a small difference.

	$\rho =$ -1	$\rho =$1
LHS of (14)	3.5848E-3 (3.57946 3.59014)	3.44852E-3 (3.44408 3.45295)
RHS of (14)	3.59831E-3	3.43174E-3

Table 2: MC simulation for the quality of the approximation (14)

If the values in Table 3 were interpreted as bond prices, the corresponding continuously compounded spot rates would be instead $-\ln(0.64232)/5 = 0.088534$ and

	ρ = -1	ρ =1
LHS of (13)	0.64232 (0.642106 0.642534)	0.644151 (0.643909 0.644393)
RHS of (13)	0.641989	0.643904

Table 3: MC simulation for the quality of the approximation (13) under stress

	ρ = -1	ρ =1
LHS of (14)	2.4757E-3 (2.46991 2.48149)	2.27465E-3 (2.27018 2.27913)
RHS of (14)	2.53527	2.24435

Table 4: MC simulation for the quality of the approximation (14) under stress

$-\ln(0.641989)/5 = 0.088637$, so that we see a larger difference than before, ranging around 1 basis point, which is however still contained.

So we may trust the approximation to work well within the typical market bid-ask spreads for CDS's. Indeed, we consider the valuation of CDS's both by Monte Carlo simulation and by the Gaussian dependence mapped model, where we apply formulae (13) and (14) each time with the most convenient maturity T for that part of the CDS payoff we are evaluating.

In Table 5 we give the results of the application of the approximations (13) and (14) to CDS valuation in presence of correlation $\rho \neq 0$ under the parameters given in Section 3.3. In Table 6 we give instead the corresponding Monte Carlo simulation for the extreme cases $\rho = -1$ and $\rho = 1$ and the known case $\rho = 0$, based on 140.000 paths with control variate variance reduction technique, both under the usual parameters of Section 3.3 and under some amplified λ parameters, increasing stochasticity. The Gaussian mapping approximation, even in the case of increased randomness, remains well within a small fraction of the CDS bid-ask spread (8).

ρ	-1	-0.5	0	0.5	1
cds	-1.12E-4	-0.554E-4	0.012E-4	0.578E-4	1.14E-4

Table 5: 5y CDS price as a function of ρ with Gaussian mapping

3.8 The impact of correlation

It can be interesting to study the main terms that appear in basic payoffs of the credit derivatives world from the point of view of the impact of the correlation ρ between interest rates r and stochastic default intensities λ. Precisely, we will study here the influence of the correlation ρ in the following payoffs

$$\begin{aligned} A &= L(T-1y,T)D(0,T)\mathbf{1}_{\{\tau<T\}}, \quad B = D(0,\tau)\mathbf{1}_{\{\tau<T\}} \qquad (15)\\ C &= D(0,\tau\wedge T), \quad D = D(0,T)L(T-1y,T)\mathbf{1}_{\{\tau\in[T-1y,T]\}}, \end{aligned}$$

CDS prices	Gaussian Mapping	Monte Carlo value and 95% window
$\rho = -1$	-1.12E-4	-1.48625E-4 (-1.79586 -1.17664)
$\rho = 0$	0.012E-4	0.17708E-4 (-0.142444 0.496605)
$\rho = 1$	1.14E-4	1.25475E-4 (0.922997 1.5865)

Same run with κ, ν increased by a factor 5 and μ by a factor 3 :

CDS prices	Gaussian Mapping	Monte Carlo value and 95% window
$\rho = -1$	-1.03E-4	-1.77E-4 (-2.02 -1.51)
$\rho = 0$	0.021E-4	0.143E-4 (-0.138 0.424)
$\rho = 1$	1.07E-4	1.08E-4 (0.78 1.37)

Table 6: 5y CDS prices as a function of ρ with MC simulation

under the SSRD correlated model. We will see that in all cases even high correlations between r and λ induce a small effect on the particular functional forms of $D(0, \cdot)$ in r and of indicators of the default times τ in λ. Higher effects are observed, in relative terms, when terms such as $L(T - 1y, T)$ and $1_{\{\tau \in [T-1y,T]\}}$ are included in the payoff. Indeed, the indicator isolates λ between $T - 1y$ and T, while L isolates r between $T - 1y$ and T. Thus we have a sort of more direct correlation between r and λ in the same interval, and this explains the highest percentage influcence of correlation observed in this case. Results are summarized in Table 7. As expected, D is the case where the correlation influence is most visible in relative terms. We have used the same paths for W and Z when changing ρ from -1 to 1, and we have taken $T = 5y$ and the same parameters in the dynamics as in Section 3.3.

To check that indeed it is the "localization" of λ and r in the same interval $[T - 1y, T] = [4y, 5y]$ that generates the high relative influence of ρ, we consider also the terms

$$E = D(0,5)L(4,5)\mathbf{1}_{\{\tau\in[3,4]\}},\ F = D(0,5)L(4,5)\mathbf{1}_{\{\tau\in[2,3]\}}, \qquad (16)$$
$$G = D(0,5)L(4,5)\mathbf{1}_{\{\tau\in[1,2]\}},\ H = D(0,5)L(4,5)\mathbf{1}_{\{\tau\in[0,1]\}}$$

and check that the correlation decreases as τ gets far from the 4y LIBOR reset date. This is indeed the case, as one can see from Table 8.

	$\rho = -1$	$\rho = 1$	relative variation	absolute variation
A	30.3672E-4	31.1962	+2.73%	+0.829E-4
B	679.197E-4	676.208	-0.44%	-2.989E-4
C	8207.23E-4	8209.61	+0.03%	+2.38E-4
D	2.77376E-4	3.10889	+12.08%	+0.34E-4

Table 7: Influence of ρ on the terms A,B,C and D defined in (15)

	$\rho = -1$	$\rho = 1$	relative variation	absolute variation
E	5.6E-4	5.88E-4	+5.010%	+0.281E-4
F	7.16E-4	7.31 E-4	+2.09%	+0.149E-4
G	7.41E-4	7.44E-4	+0.36%	2.66E-6
H	7.55E-4	7.56E-4	+0.056%	4.26 E-7

Table 8: Influence of ρ on the terms E,F,G and H defined in (16)

4 Pricing with the calibrated SSRD model.

In this final section we present examples of payoffs that can be valued with the calibrated (λ, r) model. The first example we consider is a sort of cancellable swap with a recovery value.

4.1 A Cancellable Structure

A first company "A" owns a bond issued by Merrill Lynch (ML), and receives from ML once an year at time T_i a payment consisting of $L(T_i - 1, T_i) + s$, where s is a spread ($s = 50$ basis points), up to a final date $T = T_n = 5y$. We assume unit year fractions for simplicity.

$$\text{ML (until possible default)} \rightarrow L(T_i - 1y, T_i) + s \rightarrow \text{"A"},$$

In turn, "A" has a swap with a bank "B", where "A" turns the payment $L(T_i - 1y, T_i) + s$ to "B",

$$\text{"A"} \rightarrow L(T_i - 1, T_i) + s \rightarrow \text{"B"},$$

and, in exchange for this, the bank "A" receives from "B" some fixed payments that we express as the percentages of the unit nominal value given in (17).

$$\text{"A"} \leftarrow \begin{array}{|c|c|}\hline \text{Year} & \% \\ \hline T_1 = 1 & \alpha_1 = 4.20 \\ T_2 = 2 & \alpha_2 = 3.75 \\ T_3 = 3 & \alpha_3 = 3.25 \\ T_4 = 4 & \alpha_4 = 0.50 \\ T_5 = T_n = T = 5 & \alpha_5 = 0.50 \\ \hline \end{array} \leftarrow \text{"B"} \qquad (17)$$

However, if ML defaults, "A" receives a recovery rate $\tilde{Z}$ from ML (typically one recovers from $\tilde{Z} = 0$ to 0.5 out of 1), and still has to pay the remaining payments $L(T_i - 1, T_i) + s$ to "B".

"A" wishes to have the possibility to cancel the swap with "B" in case both ML defaults and the recovery rate $\tilde{Z}$ is not enough to close the swap with "B" without incurring in a loss.

Continuing the swap after the default τ implies for "A" to pay cash flows whose total discounted value at time τ is (including the recovery rate $\tilde{Z}$):

$$-\tilde{Z} + \sum_{i=\beta(\tau)}^{n} P(\tau, T_i)\left(-\alpha_i + s + F(\tau; T_{i-1}, T_i)\right) \tag{18}$$

where $F(\tau; T_{i-1}, T_i) = (P(\tau, T_{i-1})/P(\tau, T_i) - 1)/(T_i - T_{i-1})$ is the forward LIBOR rate at time τ between T_{i-1} and T_i. "A" wishes to cancel this payment when it is positive. By simple algebra, and substituting the definition of F, this cancellation has the following value at time τ:

$$\left[\sum_{i=\beta(\tau)}^{5} \left(P(\tau, T_i)(s - \alpha_i) + P(\tau, T_{i-1}) - P(\tau, T_i)\right) - \tilde{Z}\right]^+ .$$

Thus we need computing

$$\mathbb{E}\left\{D(0,\tau)1_{\{\tau < T_n\}}\left[\sum_{i=\beta(\tau)}^{5} \left(P(\tau, T_i)(s - \alpha_i) + P(\tau, T_{i-1}) - P(\tau, T_i)\right) - \tilde{Z}\right]^+\right\}. \tag{19}$$

By a joint simulation of (λ, r) this payoff can be easily valued. Indeed, from the simulation of Λ and $\xi = \Lambda(\tau)$ one obtains a simulation of τ, and thus, through the joint simulation of r, is able to build scenarios of r_τ. Since all bonds $P(\tau, T)$ are known functions of r_τ in the SSRD CIR++ model, we simply have to discount these scenarios from τ to 0 and then average along scenarios.

Our results, with the same interest-rate and default-intensity dynamics (r, λ) as in Section 3.3 are reported in Tables 9 (recovery $\tilde{Z} = 0.1$), 10 (recovery $\tilde{Z} = 0$) and 11 (recovery $\tilde{Z} = 0$ and stressed parameters, κ and ν increased by a factor 5 and μ by a factor 3).

Results show that for this nonlinear payoff correlation may have a relevant impact. It is interesting to notice that the correlation pattern is inverted when randomness increases as in the last table, since the value decreases as the correlation increases, contrary to the earlier cases. This may be explained qualitatively as follows. The indicator term $\mathbf{1}_{\{\tau < T_5\}}$ selects relatively high values of λ. In case of positive correlation ρ, high λ's correspond to high r's (and thus a low discount factor $D(0, \tau)$). So in (18) the F term is "dominating" the remaining terms and selects a high value for the inner payoff in (19). In turn, $D(0, \tau)$ is low, and the combined effect depends on the dynamic parameters of the model, which is what we observe in our examples.

Again in the case with amplified randomness in intensities, in Table 11, we observe possible excursions of about 15 basis points due to correlation. So cancellable swaps turn out to be more sensitive to correlation than the almost insensitive CDS's.

$s \downarrow \quad \rho \rightarrow$	-1	0	1	Det
-100	0.59 (0.56, 0.62)	0.78 (0.74, 0.82)	1.09 (1.05, 1.12)	0
-50	1.075 (1.03, 1.12)	1.45 (1.40, 1.50)	1.92 (1.86, 1.98)	0
0	2.1 (2.04, 2.17)	2.68 (2.61, 2.75)	3.40 (3.31, 3.48)	0
+50	4.56 (4.47, 4.65)	5.53 (5.43, 5.63)	6.63 (6.52, 6.75)	2.35
+100	11.61 (11.47, 11.75)	12.92 (12.77 13.07)	14.45 (14.28, 14.62)	11.87

Table 9: Cancellable swap price in basis points (10^{-4}) as a function of ρ and s with MC simulation, $\tilde{Z} = 0.1$, "Det" for deterministic model

$s \downarrow \quad \rho \rightarrow$	-1	0	1	Det
-100	32.56 (32.15, 32.97)	34.26 (33.83, 34.69)	36.24 (35.78, 36.70)	34.38
-50	43.48 (42.96, 44.00)	45.19 (44.65, 45.74)	47.03 (46.46, 47.59)	45.08
0	54.351 (53.71, 54.99)	55.59 (54.94, 56.25)	57.40 (56.72, 58.08)	55.79
+50	64.91 (64.15, 65.67)	66.26 (65.48, 67.04)	68.25 (67.45, 69.05)	66.49
+100	75.64 (74.76, 76.53)	76.78 (75.88 77.68)	78.81 (77.89, 79.73)	77.20

Table 10: Cancellable swap price in basis points (10^{-4}) as a function of ρ and s with MC simulation, $\tilde{Z} = 0$, "Det" for deterministic model

$s \downarrow \quad \rho \rightarrow$	-1	0	1	Det
-100	59.06 (58.63, 59.49)	50.23 (49.86, 50.60)	44.92 (44.58, 45.26)	34.38
-50	74.11 (73.59, 74.63)	65.58 (65.12, 66.03)	60.17 (59.75, 60.60)	45.08
0	89.60 (88.99, 90.22)	80.97 (80.41, 81.52)	75.56 (75.04, 76.08)	55.79
+50	104.76 (104.04, 105.48)	96.55 (95.89, 97.20)	91.21 (90.58, 91.83)	66.49
+100	119.99 (119.18, 120.81)	111.50 (110.75 112.26)	106.40 (105.68, 107.13)	77.20

Table 11: Cancellable swap price in basis points (10^{-4}) as a function of ρ with stressed parameters and s with MC simulation, $\tilde{Z} = 0$, "Det" for deterministic model

4.2 CDS Options and Jamshidian's Decomposition

We developed this formula by an initial hint of Ouyang (2003). Consider the option to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, receiving protection Z against default up to time T_b, in exchange for a fixed rate K. At T_a there is the option of entering a CDS paying a fixed rate K at times $\mathcal{T}_{a,b} = T_{a+1}, \dots, T_b$ or until default, in exchange for protection against a possible default in $[T_a, T_b]$. If default occurs a protection payment Z is received. By noticing that the market CDS rate $R_f(T_a, T_b)$ at T_a will set the CDS value in T_a to 0, the payoff can be written as the discounted difference between said CDS and the corresponding CDS with rate K. We have that the payoff at T_a reads

$$\begin{aligned}
\Pi_a &:= [\mathrm{CDS}(T_a, \mathcal{T}_{a,b}, T_b, R_f(T_a, T_b), Z) - \mathrm{CDS}(T_a, \mathcal{T}_{a,b}, T_b, K, Z)]^+ \\
&\qquad = [-\mathrm{CDS}(T_a, \mathcal{T}_{a,b}, T_b, K, Z)]^+ = \\
&1_{\{\tau > T_a\}} \Bigg(\mathbb{E}\Bigg\{ - D(T_a, \tau)(\tau - T_{\beta(\tau)-1}) K \mathbf{1}_{\{\tau < T_b\}} \\
&- \sum_{i=a+1}^{b} D(T_a, T_i) \alpha_i K \mathbf{1}_{\{\tau > T_i\}} + \mathbf{1}_{\{\tau < T_b\}} D(T_a, \tau)\ Z | \mathcal{G}_{T_a} \Bigg\} \Bigg)^+ \\
= 1_{\{\tau > T_a\}} \Bigg\{ &- K \int_{T_a}^{T_b} \mathbb{E}\left[\exp\left(- \int_{T_a}^{u} (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_{T_a} \right] (u - T_{\beta(u)-1}) du \\
&- K \sum_{i=a+1}^{b} \alpha_i \mathbb{E}\left[\exp\left(- \int_{T_a}^{T_i} (r_s + \lambda_s) ds \right) | \mathcal{F}_{T_a} \right] \\
&+ Z \int_{T_a}^{T_b} \mathbb{E}\left[\exp\left(- \int_{T_a}^{u} (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_{T_a} \right] du \Bigg\}^+
\end{aligned}$$

If we take deterministic interest rates r this reads

$$\begin{aligned}
\Pi_a = 1_{\{\tau > T_a\}} \Bigg\{ &- K \int_{T_a}^{T_b} \mathbb{E}\left[\exp\left(- \int_{T_a}^{u} \lambda_s ds \right) \lambda_u | \mathcal{F}_{T_a} \right] P(T_a, u)(u - T_{\beta(u)-1}) du \\
&- K \sum_{i=a+1}^{b} \alpha_i P(T_a, T_i) \mathbb{E}\left[\exp\left(- \int_{T_a}^{T_i} \lambda_s ds \right) | \mathcal{F}_{T_a} \right] \\
&+ Z \int_{T_a}^{T_b} P(T_a, u) \mathbb{E}\left[\exp\left(- \int_{T_a}^{u} \lambda_s ds \right) \lambda_u | \mathcal{F}_{T_a} \right] du \Bigg\}^+
\end{aligned}$$

Define

$$H(t, T; y_t^\beta) := \mathbb{E}\left[\exp\left(- \int_t^T \lambda_s ds \right) | \mathcal{F}_t \right]$$

and notice that

$$\mathbb{E}\left[\exp\left(- \int_t^T \lambda_s ds \right) \lambda_T | \mathcal{F}_t \right] = -\frac{d}{dT} \mathbb{E}\left[\exp\left(- \int_t^T \lambda_s ds \right) | \mathcal{F}_t \right] = -\frac{d}{dT} H(t, T)$$

Write then

$$\Pi_a = 1_{\{\tau>T_a\}}\Big\{K\int_{T_a}^{T_b} P(T_a,u)(u-T_{\beta(u)-1})\frac{d}{du}H(T_a,u)du$$
$$-K\sum_{i=a+1}^{b}\alpha_i P(T_a,T_i)H(T_a,T_i) - Z\int_{T_a}^{T_b} P(T_a,u)\frac{d}{du}H(T_a,u)du\Big\}^+$$

Note that the first two summations add up to a positive quantity, since they are expectations of positive terms.

By integrating by parts in the first and third integral, we obtain, by defining $q(u) := -dP(T_a,u)/du$,

$$\Pi_a = 1_{\{\tau>T_a\}}\Big\{Z - \int_{T_a}^{T_b}\big[Zq(u) + KP(T_a,T_{\beta(u)})\delta_{T_{\beta(u)}}(u) - K(u-T_{\beta(u)-1})q(u)$$
$$-KP(T_a,T_{\beta(u)})\delta_{T_{\beta(u)}}(u) + Z\delta_{T_b}(u)P(T_a,u) + KP(T_a,u)\big]H(T_a,u)du\Big\}^+$$

Define

$$h(u) := Zq(u) - K(u-T_{\beta(u)-1})q(u) + Z\delta_{T_b}(u)P(T_a,u) + KP(T_a,u)$$

so that

$$\Pi_a = 1_{\{\tau>T_a\}}\Big\{Z - \int_{T_a}^{T_b} h(u)H(T_a,u;y^\beta_{T_a})du\Big\}^+ \tag{20}$$

It is easy to check, by remembering the signs of the terms of which the above coefficients are expectations, that

$$h(u) > 0 \quad \text{for all} \quad u.$$

Now we look for a term y^* such that

$$\int_{T_a}^{T_b} h(u)H(T_a,u;y^*)du = Z. \tag{21}$$

It is easy to see that in general $H(t,T;y)$ is decreasing in y for all t,T. This equation can be solved, since $h(u)$ is known and deterministic and since H is given in terms of the CIR bond price formula. Furthermore, either a solution exists or the option valuation is not necessary. Indeed, consider first the limit of the left hand side for $y^*\to\infty$. We have

$$\lim_{y^*\to\infty}\int_{T_a}^{T_b} h(u)H(T_a,u;y^*)du = 0 < Z,$$

which shows that for y^* large enough we always go below the value Z. Then consider the limit of the left hand side for $y^*\to 0$:

$$\lim_{y^*\to 0+}\int_{T_a}^{T_b} h(u)H(T_a,u;y^*)du =$$

$$= Z + \int_{T_a}^{T_b} [ZP(T_a, u)\frac{\partial H(T_a, u; 0)}{\partial u} + (K(u - T_{\beta(u)-1})q(u) + KP(T_a, u))H(T_a, u; 0)]du$$

Now if the integral in the last expression is positive then we have that the limit is larger than Z and by continuity and monotonicity there is always a solution y^* giving Z. If instead the integral in the last expression is negative, then the limit is smaller than Z and we have that (21) admits no solution, in that its left hand side is always smaller than the right hand side. However, this implies in turn that the expression inside curly brackets in the payoff (20) is always positive and thus the contract loses its optionality and can be valued by taking the expectation without positive part, giving as option price simply $-\text{CDS}(t, \mathcal{T}_{a,b}, T_b, K, Z) > 0$, the opposite of a forward start CDS. In case y^* exists, instead, we may rewrite our discounted payoff as

$$\Pi_a = 1_{\{\tau > T_a\}} \left\{ \int_{T_a}^{T_b} h(u)(H(T_a, u; y^*) - H(T_a, u; y^\beta_{T_a}))du \right\}^+$$

Since $H(t, T; y)$ is decreasing in y for all t, T, all terms $(H(T_a, u; y^*) - H(T_a, u; y^\beta_{T_a}))$ have the same sign, which will be positive if $y^\beta_{T_a} > y^*$ or negative otherwise. Since all such terms have the same sign, we may write

$$\Pi_a =: 1_{\{\tau > T_a\}} Q_a = 1_{\{\tau > T_a\}} \left\{ \int_{T_a}^{T_b} h(u)(H(T_a, u; y^*) - H(T_a, u; y^\beta_{T_a}))^+ du \right\}$$

Now compute the price as

$$\mathbb{E}[D(0, T_a)\Pi_a] = P(0, T_a)\mathbb{E}[1_{\{\tau > T_a\}} Q_a] = P(0, T_a)\mathbb{E}[\exp(-\int_0^{T_a} \lambda_s ds)Q_a] =$$

$$= \int_{T_a}^{T_b} h(u)\mathbb{E}[\exp(-\int_0^{T_a} \lambda_s ds)(H(T_a, u; y^*) - H(T_a, u; y^\beta_{T_a}))^+]du$$

From a structural point of view, $H(T_a, u; y^\beta_{T_a})$ are like zero coupon bond prices in a CIR++ model with short term interest rate λ, for maturity T_a on bonds maturing at u. Thus, each term in the summation is $h(u)$ times a zero-coupon bond like call option with strike $K^*_u = H(T_a, u; y^*)$. A formula for such options is given for example in (3.78) p. 94 of Brigo and Mercurio (2001b).

If one maintains stochastic interest rates with possibly non-null ρ, then a possibility is to use the Gaussian mapped processes x^V and y^V introduced earlier and to reason as for pricing swaptions with the G2++ model through Jamshidian's decomposition and one-dimensional Gaussian numerical integration, along the lines of the procedures leading to (4.31) in Brigo and Mercurio (2001b). Clearly the resulting formula has to be tested against Monte Carlo simulation.

5 Conclusions and further research

In this work we have introduced a two-dimensional shifted square-root diffusion (SSRD) model for interest rate derivatives and single-name credit derivatives, in a stochastic intensity framework. This model offers the only known case, to the best of our knowledge, allowing for an automatic calibration of the term structure of interest rates and of credit default swaps (CDS's). Additional parameters can be set so as to calibrate option data from the interest rate market and option data on the credit market, although we do not use the latter, due to the fact that the related products appeared very recently and are characterized by wide bid-ask spreads. The interest-rate calibration and the credit market calibration are separate, which can be helpful in splitting competence. We discussed numerically the impact of interest-rate and default-intensity correlation on calibration and pricing. We also introduced an analytical approximation based on a Gaussian dependence mapping for some basic credit derivatives terms involving correlated CIR processes. We used a variant of Jamshidian's decomposition to derive an analytical formula for CDS options under CIR++ stochastic intensity. Further research includes checking future default structures implied by a calibration and related diagnostics, analysis of the impact of correlation on more involved payoffs, possible extensions to multiname credit derivatives, and analyzing hedging strategies associated with the model, for example in the framework of Blanchet-Scaillet and Jeanblanc (2001).

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