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Credit Default Swap Calibration and Counterparty Risk Valuation with a Scenario based First Passage Model*

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Abstract

In this work we develop a tractable structural model with analytical default probabilities depending on a random default barrier and possibly random volatility ideally associated with a scenario based underlying firm debt. We show how to calibrate this model using a chosen number of reference Credit Default Swap (CDS) market quotes. In general this model can be seen as a possible extension of the time-varying AT1P model in Brigo and Tarenghi (2004). The calibration capability of the Scenario Volatility/Barrier model (SVBAT1P), when keeping time-constant volatility, appears inferior to the one of AT1P with time-varying deterministic volatility. The SVBAT1P model, however, maintains the benefits of time-homogeneity and can lead to satisfactory calibration results, as we show in a case study where we compare different choices on scenarios and parameters.

Similarly to AT1P, SVBAT1P is suited to pricing hybrid equity/credit derivatives and to evaluate counterparty risk in equity payoffs, and more generally to evaluate hybrid credit/equity payoffs. We consider the equity return swap in Brigo and Tarenghi (2004) and show its valuation under SVBAT1P with the same CDS and equity calibration input used earlier for AT1P, and further we hint at equity default swap valuation in the conclusions.

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Duke of Alencon, this was your default,
That, being captain of the watch to-night,
Did look no better to that weighty charge.

King Henry VI, Part 1, Act 2, Scene I.

1 Introduction

Modelling firms default is becoming more and more important, especially in recent times where the market is experiencing a large development in credit derivatives trading. In this paper we develop a tractable structural model with analytical default probabilities depending on some dynamics parameters and on a random default barrier ideally associated with the underlying firm debt. The scenario based default barrier models our possible uncertainty on the debt level of the firm. This uncertainty may be due to the fact that, for example, we do not trust completely the information from Balance Sheets.

We show how to calibrate the model using a chosen number of Credit Default Swap (CDS) market quotes. We apply the structural model to a concrete calibration case. This model can be seen as a variant or even a possible extension of the time-varying Analytically Tractable 1st Passage model (AT1P) in Brigo and Tarenghi (2004). There are a number of approaches for extending this earlier work. In Brigo and Tarenghi (2004) we essentially used information from equity volatility and CDS quotes to determine a time-varying value of the deterministic firm-volatility and a reference level for the default barrier implying exact calibration of a given number of CDS quotes. In this paper we consider a scenario based default barrier/volatility version of the basic model, which is a step forth, but take time-constant volatilities (in each scenario), which is a step backwards. Notwithstanding the time-constancy of the volatility scenarios, which is a restriction with respect to the earlier deterministic time-varying coefficients AT1P model, we refer to the random coefficients model as to an "extension", that we term Scenario Volatility/Barrier Analytically Tractable 1st Passage model (SVBAT1P). The calibration of the CDS term structure in SVBAT1P is left to the random default barrier, to the volatility scenarios parameters and to the probabilities of different scenarios. These can all be considered as calibrating parameters, or else some can be fixed arbitrarily and the calibration can be left to the remaining ones. We illustrate different possible choices with a case study.

With the first extension where we use scenarios only on the barrier (taking instead the same volatility in all scenarios) the model is not very flexible and can calibrate a low number of CDS, typically 3 or 4. Indeed, default barrier scenarios appear not to be natural fitting parameters for a large number of CDS quotes. In this case it can be best to associate scenarios also to volatilities, or to take a time varying volatility to be partially fitted to CDS quotes even in the case of random barrier. In this note we explore the former solution, and while time-constant volatility scenarios do not allow a perfect fit of all the five CDS maturities, they still allow a small calibration error, which can be deemed to be acceptable when compared to the market induced bid ask spreads. What is more, having a model like SVBAT1P with constant parameters, even if under different scenarios, may help with respect to robustness, typically endangered by all-fitting time-varying functions as in the basic AT1P model. Nonetheless, the time-varying volatility scenarios extension will be addressed in further work.

Finally we notice that, as for the earlier deterministic barrier AT1P model, the CDS calibrated SVBAT1P model is ideally suited to price hybrid equity/credit derivatives and to evaluate counterparty risk in equity payoffs. Given the same CDS and equity calibration inputs, we compare the price of an equity return swap under the deterministic time varying volatility model AT1P with the same price under the scenarios model SVBAT1P addressed in this paper.

2 Calibrating the deterministic barrier AT1P model

The fundamental hypothesis of the model we resume here is that the underlying firm value process is a Geometric Brownian Motion (GBM), which is also the kind of process commonly used for equity stocks in the Black Scholes model.

Classical structural models (Merton (1974), Black Cox (1976)) postulate a GBM (Black and Scholes) lognormal dynamics for the value of the firm V. This lognormality assumption is considered to be acceptable. Crouhy et al (2000) report that "this assumption is quite robust and, according to KMVs own empirical studies, actual data conform quite well to this hypothesis.".

In these models the value of the firm V is the sum of the firm equity value S and of the firm debt value D. The firm equity value S, in particular, can be seen as a kind of (vanilla or barrier-like) option on the value of the firm V. Merton typically assumes a zero-coupon debt at a terminal maturity $\bar{T}$. Black Cox assume, besides a possible zero coupon debt, safety covenants forcing the firm to declare bankruptcy and pay back its debt with what is left as soon as the value of the firm itself goes below a "safety level" barrier. This is what introduces the need for barrier option technology in structural models for default.

In Brigo and Tarenghi (2004) the following proposition is proved:

Proposition 2.1. (Analytically-Tractable First Passage (AT1P) Model) *Assume the risk neutral dynamics for the value of the firm V is characterized by a risk free rate $r(t)$, a payout ratio $q(t)$ and an instantaneous volatility $\sigma(t)$, according to*

$$dV(t) = V(t)\,(r(t) - q(t))\,dt + V(t)\,\sigma(t)\,dW(t)$$

and assume a safety barrier $\widehat{H}(t)$ of the form

$$\widehat{H}(t) = H \exp\left(-\int_0^t \left(q(s) - r(s) + (1+2\beta)\frac{\sigma(s)^2}{2}\right) ds\right) \tag{1}$$

where β is a parameter that can be used to shape the safety barrier, H is a reference level, and let τ be defined as the first time where V hits the safety covenants barrier $\widehat{H}$ from above, starting from $V_0 > H$,

$$\tau = \inf\{t \geq 0 : V(t) \leq \widehat{H}(t)\}.$$

Then the survival probability is given analytically by

$$\mathbb{Q}\{\tau > T\} = \left[\Phi\left(\frac{\log\frac{V_0}{H} + \beta\int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}}\right) - \left(\frac{H}{V_0}\right)^{2\beta}\Phi\left(\frac{\log\frac{H}{V_0} + \beta\int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}}\right)\right]. \tag{2}$$

In presence of zero coupon debt for the maturity $\bar{T}$, the actual default time is τ if $\tau < \bar{T}$, and is $\bar{T}$ if $\tau \geq \bar{T}$ and the value of the firm at $\bar{T}$ is below the debt face value. Otherwise, if $\tau \geq \bar{T}$ and the value of the firm at $\bar{T}$ is above the debt face value, there is no default.

In Brigo and Tarenghi (2004) we calibrate CDS quotes by means of the above formula inserted in CDS valuation, backing out $t \mapsto \sigma(t)$ from increasing-maturity CDS quotes through a cascade inversion method. We now recall briefly the CDS payoff and its risk neutral pricing formula.

One of the most representative protection instruments that can be used against default is the Credit Default Swap (CDS). Consider two companies "A" (the *protection buyer*) and "B" (the *protection seller*) who agree on the following.

If a third reference company "C" (*the reference credit*) defaults at a time $\tau_C \in (T_a, T_b]$, "B" pays to "A" at time $\tau = \tau_C$ itself a certain "protection" cash amount L_{GD} (Loss Given the Default of "C"), supposed to be deterministic in the present paper. This cash amount is a *protection* for "A" in case "C" defaults. A typical stylized case occurs when "A" has bought a corporate bond issued from "C" and is waiting for the coupons and final notional payment from this bond: If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the bond notional in case "C" defaults. Typically L_{GD} is equal to a notional amount, or to a notional amount minus a recovery rate. We denote the recovery rate by "R_{EC}".

In exchange for this protection, company "A" agrees to pay periodically to "B" a fixed "running" amount R, at a set of times $\{T_{a+1}, \ldots, T_b\}$, $\alpha_i = T_i - T_{i-1}$, $T_0 = 0$. These payments constitute the "premium leg" of the CDS (as opposed to the L_{GD} payment, which is termed the "protection leg"), and R is fixed in advance at time 0; the premium payments go on up to default time τ if this occurs before maturity T_b, or until maturity T_b if no default occurs.

"B"	$\rightarrow$	protection L_{GD} at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	"A"
"B"	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$	"A"

Formally, we may write the RCDS ("R" stands for running) discounted value at time t seen from "A" as

$$\Pi_{\text{RCDS}_{a,b}}(t) := -D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} - \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau \geq T_i\}} + \mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\text{L}_{\text{GD}} \qquad (3)$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t, and where α_i is the year fraction between T_{i-1} and T_i. The pricing formula for this payoff depends on the assumptions on the interest rates dynamics and on the default time τ. Let $\mathcal{F}_t$ denote the basic filtration without default, typically representing the information flow of interest rates and possibly other default-free market quantities (and also intensities in the case of reduced form models), and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$ the extended filtration including explicit default information. In our earlier "structural model" AT1P framework with deterministic default barrier the two sigma-algebras coincide by construction, i.e. $\mathcal{G}_t = \mathcal{F}_t$, because here the default is completely driven by default-free market information.

This is not the case with intensity models, where the default is governed by an external random variable and $\mathcal{F}_t$ is strictly included in $\mathcal{G}_t$, i.e. $\mathcal{F}_t \subset \mathcal{G}_t$.

If we include barrier scenarios and take the barrier independent of W, we are back in a situation where $\mathcal{F}_t \subset \mathcal{G}_t$. Indeed, if for example we consider the random barrier version with deterministic volatility and we know that default occurs at a given time, say now, we also know that V has hit one of the barriers in the current scenario. By observing V, we know which barrier scenario has realized itself, and by knowing that default is now we know that the value of V now is telling us the value of H. But observation of V alone (as happens with $\mathcal{F}$) until "now" would not have allowed us to know H now or to know that default is "now". Thus, contrary to the basic AT1P model, $\mathcal{G}_t$ tells us more than $\mathcal{F}_t$ under scenario based default barrier. The situation becomes more involved if we allow for a random volatility σ, independent of all observable quantities. We might assume the random volatility is realized after a small time ϵ, as we do in smile modeling for example in Brigo, Mercurio and Rapisarda (2004), but this leads to undesirable features such as unrealistic future dynamics. While in smile modeling keeping σ random "forever" would have led us to an underlying equity or FX rate that is not observed, unacceptably leading to payoffs that remain random even at maturity, in case of structural models we may argue that the current value of the firm is not necessarily known with certainty. This would cut us some slack in keeping the volatility in V random "forever". The mechanics of iterated expectation in computing prices does not change, one just has to make sure to increase the inner conditioning filtration with the unobservable variable and everything works fine. Here, however, even knowledge of τ is not necessarily implying knowledge of H or V, given V's uncertainty. At τ we would only know that V hit H, but since V is now random, depending on the volatility scenario, we cannot be sure of which H scenario has been hit. Thus the $\mathcal{G}$ filtration is less informative in this random volatility case.

More generally, a discussion on the role of filtrations under random default barriers and partial observation is in many papers by Giesecke, see for example Giesecke and Goldberg (2004) (who introduce the I^2 model).

We denote by $\text{CDS}_{a,b}(t, R, \text{L}_{\text{GD}})$ the price at time t of the above standard running CDS. In general we can compute the CDS price according to risk-neutral valuation (see for example Bielecki and Rutkowski (2001)):

$$\text{CDS}_{a,b}(t, R, \text{L}_{\text{GD}}) = \mathbb{E}\{\Pi_{\text{RCDS}_{a,b}}(t)|\mathcal{G}_t\} =: \mathbb{E}_t\{\Pi_{\text{RCDS}_{a,b}}(t)\} \tag{4}$$

in our structural model setup. Straightforward computations lead to the price at initial time 0 as

$$\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}) = R\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})d\mathbb{Q}(\tau > t) \tag{5}$$
$$-R\sum_{i=a+1}^{b} P(0,T_i)\alpha_i\mathbb{Q}(\tau \geq T_i) - \text{L}_{\text{GD}}\int_{T_a}^{T_b} P(0,t)d\mathbb{Q}(\tau > t)$$

so that if one has a formula for the curve of survival probabilities $t \mapsto \mathbb{Q}(\tau > t)$, as in our AT1P structural model, one also has a formula for CDS.

A CDS is quoted through its "fair" R, in that the rate R that is quoted by the market at time t satisfies $\text{CDS}_{a,b}(t, R, \text{L}_{\text{GD}}) = 0$. The fair rate R strongly depends on the default probabilities. The idea is to use quoted values of these fair R's with different maturities to derive the default probabilities assessed by the market.

While in simple intensity models the survival probabilities can be interpreted as discount factors (with credit spreads as discounting rates), and as such can be easily stripped from CDS's or corporate bonds, in structural models the situation is much more complicated. In fact, here, it is not possible to find a simple "credit spread" formulation for $d\mathbb{Q}\{\tau > t\}$ starting from (2).

To calibrate the AT1P model to quoted market R's for different CDS, we insert the quoted R's in Formula (5) and find the σ and H values that set said formula to zero. See Brigo and Tarenghi (2004) for several numerical examples and a case study on Parmalat CDS data.

2.1 Numerical example

In this section we present some results of the calibration performed with the AT1P structural model. We borrow from Brigo and Tarenghi (2004). We consider CDS contracts having Vodafone as underlying with recovery rate $\text{R}_{\text{EC}} = 40\%$ ($\text{L}_{\text{GD}} = 0.6$). In Table 1 we report the maturities T_b of the contracts and the corresponding "mid" CDS rates $R_{0,b}^{\text{MID}}(0)$ (quarterly paid) on the date of March 10th, 2004, in basis points ($1bp = 10^{-4}$). We take $T_a = 0$ in all cases.

	CDS maturity T_b	$R_{0,b}^{\text{BID}}(0)$ (bps)	$R_{0,b}^{\text{ASK}}(0)$	$R_{0,b}^{\text{MID}}(0)$
1y	20-mar-05	19	24	21.5
3y	20-mar-07	32	34	33
5y	20-mar-09	42	44	43
7y	20-mar-11	45	53	49
10y	20-mar-14	56	66	61

Table 1: Maturities of quoted CDS's with their corresponding spreads on March 10, 2004.

CDS mat	CDS value bid (bps)	CDS value ask (bps)
1y	2.56	-2.56
3y	2.93	-2.93
5y	4.67	-4.67
7y	24.94	-24.94
10y	41.14	-41.14

Table 2: CDS values computed with deterministic default intensities stripped from mid R quotes but with bid and ask rates R in the premium legs.

In Table 2 we report the values (in basis points) of the CDS's computed inserting the bid and ask premium rate R quotes into the payoff and valuing the CDS with deterministic intensities stripped by mid quotes. This way we transfer the bid offer spread in the rates R on a bid offer spread on the CDS present value. In Table 3 we present the results of the calibration performed with the structural model and, as a comparison, of the calibration performed with a deterministic intensity (credit spread) model (using piecewise linear intensity). In this first example the parameters used for the structural model have been selected on qualitative considerations, and are $\beta = 0.5$ and $H/V_0 = 0.4$ (this is a significant choice since this value is in line with the expected value of the random H, completely determined by market quotes, in the Scenario based model presented later on).

We report the values of the calibrated parameters in the two models (volatilities and intensities) and the survival probabilities.

T_b	σ	Surv.	Intensity	Surv.
0	36.625%	100.000%	0.357%	100.000%
1y	36.625%	99.627%	0.357%	99.627%
3y	17.311%	98.316%	0.952%	98.316%
5y	17.683%	96.355%	1.033%	96.355%
7y	17.763%	94.206%	1.189%	94.206%
10y	21.861%	89.650%	2.104%	89.604%

Table 3: Results of the calibrations performed with both models.

Comments on the realism of short term credit spreads and on the robustness of default probabilities with respect to CDS are in Brigo and Tarenghi (2004).

The parameter β has not a direct economic meaning and in principle can be chosen arbitrarily, and can help in setting more or less stringent safety covenants, depending on one's attitude towards default risk. As for H, in Brigo and Tarenghi (2004) we present some methods to estimate it based on market considerations.

3 The scenario volatility/barrier SVBAT1P model

In some cases it can be interesting to keep the volatility of the process V as an exogenous input coming from the equity and debt worlds (for example it could be related to an historical or implied volatility). Or we might retain a time-varying volatility to be used only partly as a fitting parameter. Or, also, we might wish to remove time-varying volatility to avoid an all-fitting approach that is dangerous for robustness.

In all cases we would need to introduce other fitting parameters into the model. One such possibility comes from introducing a random default barrier. This corresponds to the intuition that the balance sheet information is not certain, possibly because the company is hiding some information. In this sense, assuming a random default barrier can model this uncertainty. This means that we retain the same model as before, but now the default barrier level H is replaced by a random variable H assuming different scenarios with given risk neutral probabilities. At a second stage, we may introduce volatility scenarios as well. By imposing time-constant volatility scenarios, in each single scenario we loose flexibility with respect to AT1P, but regain flexibility thanks to the multiple scenarios on otherwise too simple time-constant volatilities. Even so, the scenario based model results in a less flexible structure than the old deterministic volatility and barrier AT1P model as far as CDS calibration is concerned.

In detail:

Definition 3.1. (Scenario Volatility and Barrier Analytically Tractable 1st Passage model, SVBAT1P) *Let the firm value process risk neutral dynamics be given by*

$$dV(t) = (r(t) - q(t))V(t)dt + \nu(t)V(t)dW(t),$$

same notation as earlier in the paper. This time, however, let the safety barrier parameter H in (1) and the firm value volatility function $t \mapsto \nu(t)$ assume scenarios $(H_1, t \mapsto \sigma^1(t)), \ldots, (H_{N-1}, t \mapsto \sigma^{N-1}(t)), (H_N, t \mapsto \sigma^N(t))$ with $\mathbb{Q}$ probability $p_1, \ldots, p_{N-1}, p_N$ respectively. The safety barrier will thus be random and equal to

$$\widehat{H}^i(t) := H_i \exp\left(-\int_0^t \left(q(s) - r(s) + (1+2\beta)\frac{(\sigma^i(s))^2}{2}\right) ds\right)$$

with probability p_i. The random variables H and ν are assumed to be independent of the driving Brownian motion W of the value of the firm V risk neutral dynamics.

This definition has very general consequences. Indeed, if we are to price a payoff Π based on V, by iterated expectation we have

$$\mathbb{E}[\Pi] = \mathbb{E}\{\mathbb{E}[\Pi|H,\nu]\} = \sum_{i=1}^N p_i \mathbb{E}[\Pi|H = H_i, \nu = \sigma^i]$$

Now, thanks to independence, the term $\mathbb{E}[\Pi|H = H_i, \nu = \sigma^i]$ is simply the price of the payoff Π under the model with deterministic barrier and volatility seen earlier in the paper, when the barrier parameter H is set to H_i and the volatility to $t \mapsto \sigma^i(t)$, so that the safety barrier is $\widehat{H}^i$. This means that, in particular, for CDS payoffs we obtain

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L{\scriptsize GD}}) = \sum_{i=1}^N \mathrm{CDS}_{a,b}(t, R, \mathrm{L{\scriptsize GD}}; H_i, \sigma^i) \cdot p_i \qquad (6)$$

where $\mathrm{CDS}_{a,b}(t, R, \mathrm{L{\scriptsize GD}}; H_i, \sigma^i)$ is the CDS price (5) computed according to survival probabilities (2) when the barrier H is set to H_i and the volatility to σ^i. Let us consider now a set of natural maturities for CDS quotes. This is to say that we assume $T_a = 0$ and T_b ranging a set of standard maturities, $T_b = 1y, 3y, 5y, 7y, 10y$. Let us set

$$\mathrm{CDS}^i_k := \mathrm{CDS}_{0,k}(0, R, \mathrm{L{\scriptsize GD}}; H_i, \sigma_i),$$

i.e. the CDS with first reset in 0, final maturity T_k and valued under the deterministic barrier model with the barrier set to $\widehat{H}^i$ and the deterministic volatility set to $t \mapsto \sigma^i(t)$, which from now on we assume to be constant $\sigma^i(t) = \sigma^i$ for all t.

Now assume we aim at calibrating the scenario based barrier/volatility model to a term structure of CDS data. Our first attempt is only with a scenario based barrier model, in that we take a common, time constant volatility $\sigma^i = \bar{\sigma}$ in all scenarios, so that the only uncertain quantity in the model is the safety level barrier parameter H.

3.1 CDS Calibration: Scenario based Barrier with Linear Algebra

For this particular case we could reason with some linear algebra. Assume we have N CDS maturities T_b to calibrate and consider the parameters we might use. If we resort to a scenario based barrier and fix the volatility exogenously, with N barrier scenarios we have N values for H and $N-1$ values for the probabilities p, with a total of $2N-1$ parameters to fit N quotes. Thus we have a system with more unknowns than equations. One possible way of proceeding would be to fix $N-1$ parameters (some barriers and some probabilities) and then looking for the remaining free parameters. This procedure, when tried, is not very stable and the calibration is quite difficult. A better way to face the problem is the following. Let $C_{k,i} := \text{CDS}_k^i = \text{CDS}_{0,k}(0, R, \text{L}_{\text{GD}}; H_i, \bar{\sigma})$

If we write the relationship equating (6) in correspondence of market R's to zero, as should be, for N maturities and with N barriers, we have a linear system of the form

$$\mathbf{C} \cdot \underline{\mathbf{p}} = \underline{\mathbf{0}} \tag{7}$$

where $\underline{\mathbf{p}}$ is the $(N \times 1)$ column vector of the probabilities and $\underline{\mathbf{0}}$ is a $(N \times 1)$ null vector. It is well known from basic algebra that if the determinant of $\mathbf{C}$ is different from zero, then the system (7) admits a unique solution and this solution is identically equal to zero (the kernel of C is the trivial null space), which is not good as a set of probabilities $\mathbf{p}$. Since we aim at a meaningful result for $\mathbf{p}$ we ask that $\det(\mathbf{C}) = 0$, so that the infinite solutions of the system will be given by all the eigenvectors relative to the null eigenvalue. Of all eigenvectors, we consider unitary norm versions. Hence the first aim is to find those values of the barriers that make the determinant of $\mathbf{C}$ equal to zero. This is a single equation with N unknowns. So we can fix $N-1$ barrier parameters H and look for the last one solving this "vanishing determinant" equation. Obviously we must be careful in the choice of the parameters, since in principle given the first $N-1$ barriers, the equation could admit no solutions. Also, even if we find an admissible value for the last barrier, there is no guarantee that the probabilities be all positive, since the eigenvectors in principle could have negative components.

Actually, as one could guess beforehand, these considerations result in a calibration technique that does not seem to be very powerful. Let us analyze the particular case of the Vodafone telecommunication company at the date of March 10th, 2004 (the data are the mid quotes reported in Table 1; the recovery rate is $\text{R}_{\text{EC}} = 40\%$, the common and time-constant volatility in all scenarios is $\sigma = 24\%$, β is set equal to 0.5 and the spreads are expressed in basis points). In the case of deterministic barrier we find a term structure of deterministic firm value volatility to calibrate exactly the data, as we have seen in detail in Section 2.1.

Here instead we keep the volatility fixed as an exogenous constant and look for the barrier parameters H and the probabilities p.

First of all we try to calibrate the first two maturities. We have to fix the value for one of the barriers. We choose a high value, i.e. $H_1 = 0.8$. The results of the calibration are reported in Table 4.

We see that the calibration assigns a low probability to the high barrier and a high probability to the low barrier: This is consistent with the fact that the CDS rates are very low, indicating a quite good credit quality of the underlying. In fact the high barrier

i	H_i	p_i
1	0.3710*	98.86%
2	0.8000	1.14%

Table 4: Results of the calibration with the first two maturities. The asterisk indicates that the corresponding value is the value obtained from the calibration while the other barrier value has been fixed in advance.

(corresponding to high debt) indicates proximity to default, while the low barrier (low debt) indicates a distant default possibility.

Now let us add a maturity, i.e. let us move from two maturities to three maturities, namely the first three dates in Table 1. It would be somehow natural to try a configuration of barriers which is an extension of the previous one, i.e. to fix the two barriers just found and to look for a third one. However, this procedure does not lead to a solution: To calibrate we need to change one of the barriers. We decided to keep fixed $H = 0.8$ and change the other one, choosing $H = 0.2$. The results are reported in Table 5.

i	H_i	p_i
1	0.2000	61.06%
2	0.4303*	37.83%
3	0.8000	1.10%

Table 5: Results of the calibration with the first three maturities. The asterisk indicates that the corresponding value is the value obtained from the calibration while the other values have been fixed in advance.

We notice that a splitting has occurred: From a single barrier set at 0.3710 we obtained two barriers, one higher and one lower. The highest barrier has still a low probability, almost equal to the corresponding one obtained earlier, when calibrating only two maturities. The other probability of the two-maturities case has split, following the splitting of the H parameters, and now the lowest H is the most likely.

Again, moving to the first four maturities in Table 1 we would try simply to add a new barrier parameter while retaining the H's calibrated under three quotes. As before, this calibration does not work. Then we substitute the last barrier found (0.4303) with 0.5500 before trying the calibration. In Table 6 we report the result of the calibration.

i	H_i	p_i
1	0.2000	27.65%
2	0.3347*	66.75%
3	0.5500	7.71%
3	0.8000	0.89%

Table 6: Results of the calibration with the first four maturities. The asterisk indicates that the corresponding value is the value obtained from the calibration while the other values have been fixed in advance.

As when moving from two to three maturities, when moving from three to four maturities we observe a splitting of one barrier parameter H into two values, one higher and one lower. We see that the greatest part of the probability is divided among the three "low" barriers while the highest barrier has still a low probability. More precisely, we see that adding information (i.e. adding maturities), the high barrier seems to become less probable. In general adding information leads obviously to a refinement of the barrier configuration.

It may be interesting to see what happens in case of a worse credit quality. To see this let us arbitrarily change the CDS rates (all other things being equal), in particular let us double them, as shown in Table 7, and repeat the calibration as before. The results are reported in Tables 8 and 9.

Maturity T_b	Doubled Rate $R_{0,b}^{\text{MID}}(0)$ (bps)
March 21st, 2005	43.0
March 20th, 2007	66.0
March 20th, 2009	86.0

Table 7: Maturities of quoted CDS's with their corresponding doubled spreads, valuation date March 10, 2004.

i	H_i	p_i
1	0.4123*	97.76%
2	0.8000	2.24%

Table 8: Results of the calibration with the first two maturities. The asterisk indicates that the corresponding value is the value obtained from the calibration while the other barrier value has been fixed in advance.

i	H_i	p_i
1	0.2000	19.81%
2	0.4277*	77.98%
3	0.8000	2.21%

Table 9: Results of the calibration with the first three maturities. The asterisk indicates that the corresponding value is the value obtained from the calibration while the other values have been fixed in advance.

We see that by increasing the CDS rates, i.e. decreasing the credit quality of the underlying firm, the calibrated barriers parameters H become higher and also the probabilities to have high barriers increase, as one would expect, since high barriers increase the proximity of default.

With this method we have not been able, by adding a further scenario to H, to calibrate all of the five CDS of Table 1. This is due probably to the time-homogeneity of the model. To have a larger calibration capability we may have to use a time varying ν.

To sum up, as one could have expected, this "linear algebra" procedure and the scenario based barrier approach in general are not very robust and are quite sensitive to

CDS rates: For high rates the calibration works only for a low number of maturities, and the refinement is limited. This is one of the main drawbacks of the model which makes it less powerful than the standard calibration with deterministic barrier AT1P of Brigo and Tarenghi (2004), at least if one insists in having an exact calibration. One possibility to improve the model is to include a time dependency for the parameters (for example in the volatility $\bar{\sigma}$), as hinted at before. This idea is currently under investigation. A different possibility is moving in two different directions: first, abandoning the "linear algebra" approach and using a numerical optimization within the same framework, and secondly abandoning the single time-constant volatility scenario and allowing several time-constant volatility scenarios to come into play, which we explore in Section 3.3.

3.2 CDS Calibration: Scenario based Barrier with Numerical Optimization

Here we abandon the requirement to obtain a perfect calibration and allow for a (small) calibration error, while trying to restore a one to one correspondence between model parameters and market quotes.

We start by the first three quotes in Table 1, but this time calibrate them with an optimization, abandoning the "linear algebra" approach. We minimize the function expressing the sum of the squares of the CDS's prices in the model, since each CDS price should be zero in correspondence of the market quoted R. Thus we actually solve numerically

$$\begin{aligned}[H_1^*, H_2^*, p_1^*] &= \text{argmin}_{H,p} \sum_{k=1}^{3} [p_1 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_1) \\ &\quad + (1-p_1)\text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_2)]^2\end{aligned}$$

We find the following results:

H_i	p_i
0.3188	94.83%
0.6592	5.17%

Table 10: SBAT1P calibrated parameters. Exact calibration of the first three CDS quotes of Table 1. The expected value of the barrier is $\mathbb{E}[H] = 0.3364$.

Luckily, in this case the target function is practically zero, so that the calibration is exact. While this could be somehow hoped, since the number of free parameters matches the number of quotes to be calibrated, in general it is not guaranteed. Indeed, we will see shortly that even if the parameters are in the right number they often do not have enough flexibility to account for the market quotes exactly.

Let us move to optimizing over the five quotes of Table 1. To this end, we can choose the version with three H scenarios and two probabilities. We run an optimization to solve

$$[H_1^*, H_2^*, H_3^*, p_1^*, p_2^*] = \text{argmin}_{H,p} \sum_{k=1}^{5} \Big[p_1 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_1) + p_2 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_2) + p_3 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_3) \Big]^2$$

where in the optimization we impose the p to take values that are allowed for probabilities, i.e. in $[0, 1]$ and adding up to one. This time we obtain

H_i	p_i
0.7296	1.24%
0.3384	97.52%
0.7296	1.24%

Table 11: SBAT1P calibrated parameters. Calibration of the first five CDS quotes of Table 1. The expected value of the barrier is $\mathbb{E}[H] = 0.3481$.

The calibration error is not zero this time, but amounts to $915 bps^2$. We will discuss the calibration error shortly. One barrier scenario H remains close to 0.3188, with an even higher probability than before. Notice an important point: the two other scenarios of H more in proximity of default are identical, being both 0.7296, with equal probabilities 0.0124. This suggests the parametrization to be not effective in describing a dynamics consistent with the cross sectional data we are observing, since the parameters collapse to a sub-parameterization with just two barrier scenarios (three parameters in total) when fitting five quotes. This seems to suggest the new parameters we added with respect to the previous case add nothing to explain the increase of data. And indeed, if one tries the model with two barriers and one probability to fit the five quotes above, consistently with what we just obtained one finds the same calibration error and the parameters

barrier	probability
0.7296	2.48%
0.3384	97.52%

Table 12: SBAT1P calibrated parameters. Calibration of the first five CDS quotes of Table 1 with only three unknowns parameters.

Let us have a look at the calibration error in single CDS's. If we compute the five CDS prices using the R mid quotes and with the parameters H and p above coming out of the optimization, we obtain the values reported in Table 13.

We can compare these calibration errors on the single CDS's present values with the errors induced in the CDS present value by the market R bid-ask spreads, reported in Table 2. The first two CDS are out of the bid ask windows with our calibration, the worst case being the second, giving 9.99 bps of present value against a corresponding bid ask window $[-2.93, 2.93]$.

This situation cannot be considered to be fully satisfactory, which leads us to the following extension.

CDS maturity T_k	$\text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}, H_1, H_2, H_3; p_1, p_2, p_3)$ (bps)
1y	-2.77
3y	9.99
5y	-1.47
7y	-22.99
10y	16.63

Table 13: CDS values obtained using the parameters resulting from the calibration.

3.3 Scenario based Barrier and Volatility: Numerical Optimization

Now we consider again all of the five quotes of Table 1 and we use the general model with two scenarios on the barrier/volatility parameters (H, ν) and one probability (the other one being determined by normalization to one), with a total of five parameters for five quotes. Here, by trial and error we decided to set $\beta = 0$.

$$[H_1^*, H_2^*; \sigma_*^1, \sigma_*^2; p_1^*] = \text{argmin}_{H,p,\nu} \sum_{k=1}^{5} \Big[p_1 \text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}(0), \text{L}_{\text{GD}}; H_1, \sigma^1) + (1 - p_1)\text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}(0), \text{L}_{\text{GD}}; H_2, \sigma^2) \Big]^2$$

H_i	σ_i	p_i
0.3721	17.37%	93.87%
0.6353	23.34%	6.13%

Table 14: SVBAT1P model calibrated to Vodafone CDS. The expected value of the barrier is $\mathbb{E}[H] = 0.3882$.

We obtain a much lower optimization error than before, i.e. $147 bps^2$, corresponding to the following calibration errors on single CDS present values:

CDS maturity T_k	$\text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}, H_1, H_2; \sigma^1, \sigma^2; p_1)$ (bps)
1y	1.38
3y	-3.89
5y	8.16
7y	-7.56
10y	2.41

Table 15: CDS values obtained using the parameters resulting from the calibration.

Now the single CDS calibration errors are much lower than before, being the CDS present values corresponding to market R closer to zero, with the exception of the five years maturity, which could be adjusted by introducing weights in the target function.

In particular we introduce weights that are inversely proportional to the bid-ask spread and repeat the calibration for the SVBAT1P model finding the results presented in Tables 16 and 17 (compare with Tables 14 and 15).

H_i	σ_i	p_i
0.3713	17.22%	92.63%
0.6239	22.17%	7.37%

Table 16: SVBAT1P model calibrated to Vodafone CDS using weights in the objective function which are inversely proportional to the bid ask-spread. The expected value of the barrier is $\mathbb{E}[H] = 0.3899$.

CDS maturity T_k	$\mathrm{CDS}_{0,k}(0, R^{\mathrm{MID}}_{0,k}, H_1, H_2; \sigma^1, \sigma^2; p_1)$ (bps)
1y	5.85
3y	-3.76
5y	4.92
7y	-10.46
10y	1.47

Table 17: CDS values obtained using the parameters resulting from the weighted calibration.

We see that now the 1y CDS value is outside the bid-ask spread, but conversely the 5y CDS value has decreased, assuming a size more in line with the spread. This is what we aimed at, since the 5y CDS is probably the most liquid one and the model needs to reproduce it well.

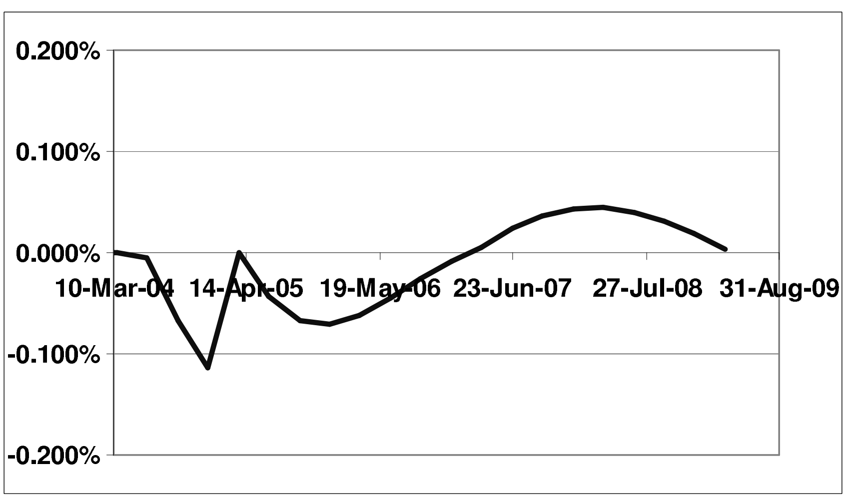

Figure 1: SBAT1P-AT1P survival probabilities difference. The data for the SBAT1P are those in Table 10 and refer to an exact calibration of 3 CDS quotes.

Furthermore, to see the goodness of the calibration, we can compare the survival probabilities of the S(V)BAT1P model with those of the perfectly calibrated AT1P model.

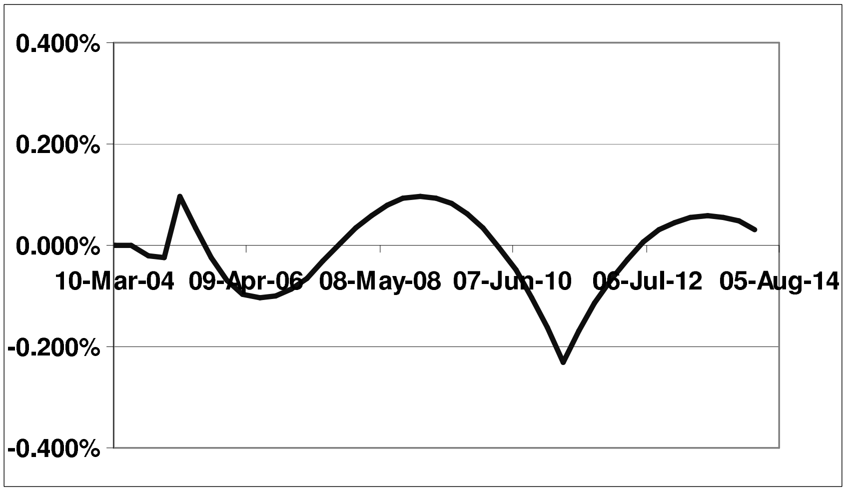

Figure 2: SVBAT1P-AT1P survival probabilities difference. The data for the SVBAT1P are those in Table 16 and refer to an optimal minimum calibration of 5 CDS quotes using weights in the objective function.

In Figure 1 we plot the difference between the survival probabilities computed with the SBAT1P model exact calibration (see Table 10) and the AT1P calibration (on a 5 years horizon). In Figure 2 we plot the difference between the survival probabilities computed with the SVBAT1P model weighted calibration (see Table 16) and the AT1P calibration (on a 10 years horizon). The two figures show a good agreement between the models in terms of probabilities, and the main difference is that the survival probabilities nearly coincide at CDS maturities in the case of exact calibration (Figure 1), while a difference is found in the minimization case (Figure 2). Anyway, it is clear that the fit is quite good, indicating two main results: (i) the SVBAT1P model returns a good fit of CDS quotes, even if the calibration is not perfect; (ii) as already observed in Brigo and Tarenghi (2004), CDS are a robust source for survival (default) probabilities, whatever model is used.

More in general, when combining default barrier and volatility scenarios, scenarios on ν and H can be taken jointly (as we did so far) or separately, but one needs to keep the combinatorial explosion under control. The pricing formula remains easy, giving linear combination of formulas in each basic scenario. Taking time-varying parametric forms for the σ^i's can add flexibility and increase the calibrating power of the model, and will be addressed in future work.

A final important remark is in order. We see that the parameters resulting from the scenario versions calibration are more credible than those obtained in the AT1P framework. In particular we notice that in all cases we have an expected H which is comparable with the fixed $H = 0.4$ used in the deterministic case. What is more, even if the H are similar, the volatilities involved are quite smaller than before. As previously hinted at, this fact is essentially due to volatility scenarios inducing a mixture of lognormal distributions for the firm value, implying fatter tails, and allowing for the same default probabilities with smaller volatilities.

4 Counterparty risk in equity swap pricing: Reprise

"Of course it does.
A conservative estimate puts material reality as ten thousand million light years across. Even using the combined computational function of all the processors in the united planets capable of dealing with numerical scale, it would take at least ninety years to..."

Brainiac 5, "Legion Lost", 2000, DC Comics

This section summarizes the results on counterparty risk pricing in Equity Return Swaps under AT1P in Brigo and Tarenghi (2004) and presents a short analysis based on the SBVAT1P models introduced in this paper. This is an example of counterparty risk pricing with the calibrated structural model.

Let us consider an equity return swap payoff. Assume we are a company "A" entering a contract with company "B", our counterparty. The reference underlying equity is company "C". The contract, in its prototypical form, is built as follows. Companies "A" and "B" agree on a certain amount K of stocks of a reference entity "C" (with price $S = S^C$) to be taken as nominal ($N = K\,S_0$). The contract starts in $T_a = 0$ and has final maturity $T_b = T$. At $t = 0$ there is no exchange of cash (alternatively, we can think that "B" delivers to "A" an amount K of "C" stock and receives a cash amount equal to KS_0). At intermediate times "A" pays to "B" the dividend flows of the stocks (if any) in exchange for a periodic rate (for example a semi-annual LIBOR or EURIBOR rate L) plus a spread X. At final maturity $T = T_b$, "A" pays KS_T to "B" (or gives back the amount K of stocks) and receives a payment KS_0. This can be summarized as follows:

$$\begin{array}{c}
\text{Initial Time 0: no flows, or} \\
\text{A} \longrightarrow KS_0^C \text{ cash} \longrightarrow \text{B} \\
\text{A} \longleftarrow K \text{ equity of ``C''} \longleftarrow \text{B} \\
\cdots\cdot \\
\text{Time } T_i\text{:} \\
\text{A} \longrightarrow \text{equity dividends of ``C''} \longrightarrow \text{B} \\
\text{A} \longleftarrow \text{Libor + Spread} \longleftarrow \text{B} \\
\cdots\cdot \\
\text{Final Time } T_b\text{:} \\
\text{A} \longrightarrow \text{K equity of ``C''} \longrightarrow \text{B} \\
\text{A} \longleftarrow KS_0^C \text{ cash} \longleftarrow \text{B}
\end{array}$$

The price of this product can be derived using risk neutral valuation, and the (fair) spread is chosen in order to obtain a contract whose value at inception is zero. We ignore default of the underlying "C", thus assuming it has a much stronger credit quality than the counterparty "B". It can be proved that if we do not consider default risk for "B" either, the fair spread is identically equal to zero. But when taking into account counterparty default risk in the valuation the fair spread is no longer zero. In case an early default of the counterparty "B" occurs, the following happens. Let us call $\tau = \tau_B$ the default instant. Before τ everything is as before, but if $\tau \leq T$, the net present value (NPV) of the position at time τ is computed. If this NPV is negative for us, i.e. for "A", then its opposite is completely paid to "B" by us at time τ itself. On the contrary, if it is positive for "A", it is not received completely but only a recovery fraction R_{EC} of that

NPV is received by us. It is clear that to us ("A") the counterparty risk is a problem when the NPV is large and positive, since in case "B" defaults we receive only a fraction of it.

The risk neutral expectation of the discounted payoff is given in the following proposition (see e.g. Brigo and Tarenghi (2004), $L(S,T)$ is the simply compounded rate at time S for maturity T):

Proposition 4.1. (Equity Return Swap price under Counterparty Risk). *The fair price of the Equity Return Swap defined above can be simplified as follows:*

$$ERS(0) = KS_0 X \sum_{i=1}^{b} \alpha_i P(0,T_i) - L_{GD}\mathbb{E}_0\left\{ \mathbf{1}_{\{\tau \le T_b\}} D(0,\tau)(NPV(\tau))^+ \right\}.$$

where

$$\begin{aligned} NPV(\tau) \;&=\; \mathbb{E}_\tau\Big\{ -K\, NPV^{\tau \div T_b}_{dividends}(\tau) + KS_0 \sum_{i=\beta(\tau)}^{b} D(\tau,T_i)\alpha_i \left(L(T_{i-1},T_i)+X\right) \\ &\quad + \left(KS_0 - KS_{T_b}\right) D\left(\tau,T_b\right) \Big\}. \end{aligned} \tag{8}$$

and where we denote by $NPV^{s \div t}_{dividends}(u)$ the net present value of the dividend flows between s and t computed in u.

The first term in Π_{ES} is the equity swap price in a default-free world, whereas the second one is the optional price component due to counterparty risk.

If we try and find the above price by computing the expectation through a Monte Carlo simulation, we have to simulate both the behavior of S_t for the equity "C" underlying the swap, and the default of the counterparty "B". In particular we need to know exactly $\tau = \tau_B$. Obviously the correlation between "B" and "C" could have a relevant impact on the contract value. Here the structural model can be helpful: Suppose to calibrate the underlying process V to CDS's for name "B", finding the appropriate default barrier and volatilities according to the procedure outlined earlier in this paper with the AT1P model. We could set a correlation between the processes V_t^B for "B" and S_t for "C", derived for example through historical estimation directly based on equity returns, and simulate the joint evolution of $[V_t^B, S_t]$. As a proxy of the correlation between these two quantities we may consider the correlation between S_t^B and S_t^C, i.e. between equities.

Going back to our equity swap, now it is possible to run the Monte Carlo simulation, looking for the spread X that makes the contract fair.

We performed some simulations under different assumptions on the correlation between "B" and "C". We considered five cases: $\rho = -1$, $\rho = -0.2$, $\rho = 0$, $\rho = 0.5$ and $\rho = 1$. In Table 18 we present the results of the simulation, together with the error given by one standard deviation (Monte Carlo standard error). For counterparty "B" we used the Vodafone CDS rates seen earlier. For the reference stock "C" we used a hypothetical stock with initial price $S_0 = 20$, volatility $\sigma = 20\%$ and constant dividend yield $q = 0.80\%$. The contract has maturity $T = 5y$ and the settlement of the LIBOR rate has a semi-annual frequency. Finally, we included a recovery rate $R_{EC} = 40\%$. The starting date is the same we used for the calibration, i.e. March 10th, 2004. Since the reference number of stocks K is just a constant multiplying the whole payoff, without losing generality we set it equal to one.

In order to reduce the errors of the simulations, we have adopted a variance reduction technique using the default indicator (whose expected value is the known default probability) as a control variate. In particular we have used the default indicator $1_{\{\tau \le T\}}$ at the maturity T of the contract, which has a large correlation with the final payoff. Even so, a large number of scenarios is needed to obtain errors with a lower order of magnitude than X. In our simulations we have used $N = 2000000$.

We notice that X increases together with ρ, and in Brigo and Tarenghi (2004) we explain why this is natural.

ρ	X	ES payoff	MC error
-1	0	0	0
-0.2	2.45	-0.02	1.71
0	4.87	-0.90	2.32
0.5	14.2	-0.53	2.71
1	24.4	-0.34	0.72

Table 18: Spread X (in bps) under five correlations, $S_0 = 20$, basic AT1P model. We also report the value of the average of the simulated payoff (times 10000) across the 2000000 scenarios and its standard error, thus showing that X is fair (leads to a zero NPV).

To check the impact of the scenarios barrier, we have re-priced with the same X's found in Table 18 for AT1P our equity swap under the SBAT1P model calibrated to the same CDS data, i.e. with the parameters given in Table 10. We have the outputs given in Table 19. If the models are close in terms of equity swap pricing, this payoff should be zero. We find actually values ranging from -28bps to 165 bps. Recalling that $S_0 = 20$, for a unit notional we would get a maximum discrepancy of about $165/20 \approx 8$ bps. By comparing with the bid-offer window induced by market bid-offer R's on CDS's with unit notional, as shown for example in Table 2, we realize that this difference, if not completely negligible, is small. Besides, for the largest payoff value, i.e. $\rho = 0.5$ (where the AT1P model gives -0.53 bps and the SBAT1P gives 165.5 bps) we notice that with $X = 16$ (instead of 14.2) the AT1P model would give a payoff price of 166.9, so that our price difference between SBAT1P and AT1P, when translated back in AT1P spread X, is less than two basis points.

ρ	X	ES payoff	MC error estimate
-0.2	2.45	-28.44	1.49
0	4.87	3.45	2.04
0.5	14.2	165.50	2.29

Table 19: Equity swap valuation under the SBAT1P model calibrated to the same CDS data as the AT1P model leading to Table 18.

We can try the same considerations with the SVBAT1P model, with the weighted calibration given in Table 16. If we consider the case with $\rho = 0.5$, the Monte Carlo method gives us the payoff expected value as 292.03 bps, with a Monte Carlo error of about 1.67 bps. Again, recalling that $S_0 = 20$ we can consider $292.03/20 \approx 14.6$, and compare with CDS payoff bid ask values, as in Table 2. We see that we are within the 7y and 10y CDS bid ask spreads but not within the 1y, 3y and 5y spreads. Also in this case we tried to see which value for X in the AT1P model with $\rho = 0.5$ would produce an expected payoff close to 290. We obtained that a spread of $X = 17.3$ would give an expected payoff value of 289, so that we see that the difference between AT1P and SVBAT1P in terms of AT1P spread is about 17.3-14.2= 3.1 bps. By comparing with bid-ask spreads in CDS rates R, as given in Table 1, we see again that this difference is inside the 1y, 7y and 10y spreads on the R quotes (which are respectively of 5, 8 an 10 bps). Also, even if a little larger, it is comparable to the 3y and 5y spreads (both of 2 bps), which are very narrow being related to the most traded CDS maturities. Thus we see that the difference is nearly negligible.

In general we have deviated further, since we started from the 0 payoff expected value under AT1P and the 165.5 expected payoff value under SBAT1P. Since AT1P, SBAT1P and SVBAT1P are calibrated to the same CDS data up to five years (but SVBAT1P is also calibrated to 7y and 10y CDS), we are seeing here that the different dynamics assumptions in the three models lead to different counterparty risk valuations in the equity return swap. The difference is not large when compared to bid ask spreads of CDS. We may expect more significant deviations in hedging. We have to keep in mind an important consideration, though. SVBAT1P is calibrated not exactly and not only on 1y, 3y and 5y CDS as the earlier AT1P and SBAT1P models. Probably some of the difference between the price obtained with the SBAT1P and the SVBAT1P is to be attributed to this fact.

Remark 4.2. (Nested calibration with AT1P but not with S(V)BAT1P). *A further important remark is the "nested calibration" aspect. With AT1P the calibration is "nested", in that adding one CDS with a larger maturity does not change the earlier σ parameters found with the calibration up to that point. In a way, this is a "cascade calibration". This may be helpful with sensitivities and bucketing. Instead, in the S(V)BAT1P model the parameters assume a "global" role: if we add a CDS quote and recalibrate, all the parameters change again. In principle, every CDS quote has an impact on* all *the S(V)BAT1P model parameters at the same time. This is less desirable in computing sensitivities to market inputs and can lead to numerical problems, especially with SVBAT1P where a further problem is that the calibration is not exact.*

We will address these matters in future work.

5 Conclusions

In general the link between default probabilities and credit spreads is best described by intensity models. The credit spread to be added to the risk free rate represents a good measure of a bond credit risk for example. Yet, intensity models present some drawbacks: They do not link the default event to the economy but rather to an exogenous jump process whose jump component remains economically unexplained. Also, when dealing with default correlation, a copula function must be introduced between the jump processes thresholds in a way that has no clear immediate relation with equity correlation, a source of correlation that can be used for practical purposes.

In this paper we introduced an analytically tractable structural first passage model (SVBAT1P) based on scenarios on the value of the firm volatility and on the default barrier. This model allows for a solution to the above points. In this model the default has an economic cause, in that it is caused by the value of the firm hitting the default safety barrier value, and all quantities have a clear economic interpretation. Also, when dealing with multi-name products, the model allows for the introduction of the correlation in a very natural way, by simply correlating shocks in the different values of the firms by means of equity correlation.

We showed how to calibrate the model parameters to actual market data: Starting from CDS quotes and the scenario based barrier model, we first devised a calibration method based on the kernel of a matrix built by CDS with different maturities and under different scenarios. This approach involves a model with more parameters than market quotes, so that part of the parameters have to be fixed arbitrarily. After showing the limits of this approach, we considered a number of parameters equal to the number of quotes and calibrated via an optimization. This approach, examined in a case study, pointed out that the scenario based barrier parameterization is not describing well the implicit dynamics and debt structure given in CDS quotes for different maturities. An improvement is obtained when allowing for different scenarios also on the firm value volatility and also when introducing weights in the objective function, leading to a calibration error essentially within the bid ask spread. Pricing of counterparty risk in equity payoffs shows a partial consistency with the deterministic coefficients AT1P model in Brigo and Tarenghi (2004) calibrated to the same data.

Finally, the S(V)BAT1P model can be used to build a relationship between the firm value V and the firm equity S (perceived as a suitable barrier payoff in terms of V itself), for example along the lines of Jones et al (1984) and Hull, Nelken and White (2004). This approach can be used to price an equity default swap. To do so we need to find an expression for the debt (and thus the equity) within the chosen structural model. Debt and equity expressions are known in closed form for time-constant and standard (exponential) barrier Black Cox models (see for example Bielecki and Rutkowski (2001), Chapter 3). Under SVBAT1P we simply obtain a linear combination of said formulas under each scenario and we have a closed form expression for the equity. Then we can price an equity default swap by means of Monte Carlo simulation of the firm value, from which, scenario by scenario, we deduce analytically the equity value by means of the found formula. This is currently under investigation.

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