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Candidate Market Models and the Calibrated CIR++ Stochastic Intensity Model for Credit Default Swap Options and Callable Floaters *

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Abstract

We consider the standard Credit Default Swap (CDS) payoff and some alternative approximated versions, stemming from different conventions on the premium and protection legs. We consider standard running CDS (RCDS), upfront CDS and postponed-payments running CDS (PRCDS). Each different definition implies a different definition of forward CDS rate, which we consider with some detail. We introduce defaultable floating rate notes (FRN)'s. We point out which kind of CDS payoff produces a forward CDS rate that is equal to the fair spread in the considered defaultable FRN. An approximated equivalence between CDS's and defaultable FRN's is established, which allows to view CDS options as structurally similar to the optional component in defaultable callable notes. We briefly investigate the possibility to express forward CDS rates in terms of some basic rates and discuss a possible analogy with the LIBOR and swap default-free models. Finally, we discuss the change of numeraire approach for deriving a Black-like formula for CDS options or, equivalently, defaultable callable FRN's. We also introduce an analytical formula for CDS option prices under the CDS-calibrated SSRD stochastic-intensity model, and discuss the impact of the different CIR++ dynamics parameters on the related CDS options implied volatilities. Hints on possible methods for smile modeling of CDS options are given for possible future developments of the CDS option market.

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Contents

Thou know'st that all my fortunes are at sea;
Neither have I money nor commodity
To raise a present sum: therefore go forth;
Try what my credit can in Venice do: [...]

The Merchant of Venice, Act 1, Scene I.

1 Introduction

We consider some alternative expressions for CDS payoffs, stemming from different conventions on the payment flows and on the protection leg for these contracts. We consider standard running CDS (RCDS), postponed payments running CDS (PPCDS), and briefly upfront CDS (UCDS). Each different running CDS definition implies a different definition of forward CDS rate, which we consider with some detail.

We introduce defaultable floating rate notes (FRN)'s. We point out which kind of CDS payoff produces a forward CDS rate that is equal to the fair spread in the considered DFRN. An approximated equivalence between CDS's and DFRN's is established, which allows to view CDS options as structurally similar to the optional component in defaultable callable notes. Equivalence of CDS and DFRN's has been known for a while in the market, see for example Schönbucher (1998), where the simpler case with continuous flows of payments is considered. Here we consider a discrete set of flows, as in real market contracts, and find that the equivalence holds only after postponing or anticipating some relevant default indicators or discount factors.

We briefly investigate the possibility to express forward CDS rates in terms of some basic rates and discuss a possible analogy with the LIBOR and swap default-free models. We then discuss the change of numeraire approach to deriving a Black-like formula for CDS options, allowing us to quote CDS options through their implied volatilities. The foundations of this work are indeed in the earlier papers by Schönbucher (2000) and especially Jamshidian (2002). Interesting considerations are also in Hull and White (2003). Here, using Jamshidian's approach as a guide, and based on a result by Jeanblanc and Rutkowski (2000), we derive CDS option prices for CDS payoffs given in the market and for the new approximated CDS payoffs. We do so by means of a rigorous change of numeraire technique. We consider the standard market model for CDS options resulting from this approach. In doing so we point out some analogies with the default free LIBOR and swap market models. This approach allows also for writing a dynamics for CDS forward rates leading to a CDS options volatility smile.

In the final part of the paper we introduce a formula for CDS option pricing under the CDS-calibrated CIR++ stochastic intensity model. We give patterns of implied volatilities as functions of the CIR++ model parameters.

The paper is structured as follows: Section 2 introduces notation, different kind of CDS discounted payoffs, and the main definition of CDS forward rate. The notion of CDS implied hazard function and its possible use as quoting mechanism is recalled.

Notation	Description
$\tau = \tau^C$	Default time of the reference entity "C"
$T_a, (T_{a+1}, \ldots, T_{b-1}), T_b$	initial and final dates in the protection schedule of the CDS
$T_{\beta(\tau)}, T_{\beta(t)}$	First of the T_i's following τ and t, respectively
α_i	year fraction between T_{i-1} and T_i
$L(S, T)$	LIBOR rate at time S for maturity T
$R_{a,b}$	Rate in the premium leg of a CDS, paid by "A", the protection buyer
REC	Recovery fraction on a unit notional
$\text{LGD} = 1 - \text{REC}$	Protection payment against a Loss (given default of "C" in $[T_a, T_b]$)
$\Pi\text{RCDS}_{a,b}(t)$	Discounted payoff of a running CDS to "B", the protection seller
$\text{CDS}(t, T_a, T_b, R, \text{LGD})$	Price of a running CDS to "B", protecting against default of "C" in $[T_a, T_b]$
$\Pi\text{DFRN}_{a,b}(t)$	Discounted payoff of a floating rate note (FRN) spanning $[T_a, T_b]$
$\text{DFRN}_{a,b}(t, X, \text{REC})$	Price of the floating rate note with spread X and recovery REC
$X_{a,b}$	Par spread in a prototypical floating rate note spanning $[T_a, T_b]$.
$\Pi\text{CallCDS}_{a,b}(t; K)$	Discounted payoff of a payer CDS option to enter at T_a a CDS at strike rate K
$\text{CallCDS}_{a,b}(t, K, \text{LGD})$	Price of payer CDS option to enter at T_a a CDS with strike rate K
$1_{\{\tau>T\}}$	Survival indicator, is one if default occurs after T and zero otherwise
$1_{\{\tau\leq T\}}$	Default indicator, is one if default occurs before or at T, and zero otherwise
$D(t, T)$	Stochastic discount factor at time t for maturity T
$P(t, T)$	Zero coupon bond at time t for maturity T
$1_{\{\tau>t\}} \bar{P}(t, T)$	Defaultable Zero coupon bond at time t for maturity T
$\widehat{C}_{a,b}(t),\ \widehat{\mathbb{Q}}^{a,b}$	Defaultable "Present value per basis point" numeraire and associated measure
$\mathcal{F}_t$	Default free market information up to time t
$\mathcal{G}_t$	Default free market information plus explicit monitoring of default up to time t

Table 1: Main notation in the paper. The postponed versions "PR" and "PR2" of the payoffs are omitted.

Upfront CDS's are hinted at.

Section 3 examines some possible variant definitions of CDS rates. Furthermore, we examine the relationship between CDS rates on different periods and point out some parallels with the default free LIBOR and swap market rates.

Section 4 introduces defaultable floating rate notes and explores their relationship with CDS payoffs, finding equivalence under some payment schedules.

Section 5 describes the payoffs and structural analogies between CDS options and callable DFRN.

Section 6 introduces the market formula for CDS options and callable DFRN, based on a rigorous change of numeraire technique.

Section 7 discusses possible developments towards a compete specifications of the vector dynamics of CDS forward rates under a single pricing measure, based on one period or co-terminal CDS rates.

Section 8 gives some hints on modeling of the volatility smile for CDS options, based on the general framework introduced earlier.

Finally, Section 9 introduces a formula for CDS option pricing under the CDS-calibrated CIR++ stochastic intensity model. The formula is based on Jamshidian's decomposition. We investigate patterns of implied volatilities as functions of the CIR++ model parameters.

2 Credit Default Swaps: Different Formulations

2.1 CDS payoffs

We recall briefly some basic definitions for CDS's. Consider a CDS where we exchange protection payment rates R at times $T_{a+1}, \ldots, T_b$ (the "premium leg") in exchange for a single protection payment L_{GD} (loss given default, the "protection leg") at the default time τ of a reference entity "C", provided that $T_a < \tau \leq T_b$. This is called a "running CDS" (RCDS) discounted payoff. Formally, we may write the RCDS discounted value at time t as

$$\Pi_{\text{RCDS}_{a,b}}(t) := D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\geq T_i\}} + \\ -\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ \text{L}_{\text{GD}} \tag{1}$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t, and where α_i is the year fraction between T_{i-1} and T_i. The stochastic discount factor at time t for maturity T is denoted by $D(t,T) = B(t)/B(T)$, where $B(t) = \exp(\int_0^t r_u du)$ denotes the bank-account numeraire, r being the instantaneous short interest rate.

We explicitly point out that we are assuming the offered protection amount L_{GD} to be deterministic and, in particular, not to depend on the CDS rate but only on the reference entity and on the payment dates. Typically $\text{L}_{\text{GD}} = 1 - \text{R}_{\text{EC}}$, where the recovery rate R_{EC} is assumed to be deterministic and the notional is set to one.

Sometimes a slightly different payoff is considered for RCDS contracts. Instead of considering the exact default time τ, the protection payment L_{GD} is postponed to the first time T_i following default, i.e. to $T_{\beta(\tau)}$. If the grid is three-or six months spaced, this postponement consists in a few months at worst. With this formulation, the CDS discounted payoff can be written as

$$\Pi_{\text{PRCDS}_{a,b}}(t) := \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\geq T_i\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}D(t,T_i)\ \text{L}_{\text{GD}}, \tag{2}$$

which we term "Postponed payoffs Running CDS" (PRCDS) discounted payoff. Compare with the earlier discounted payout (1) where the protection payment occurs exactly at τ: The advantage of the postponed protection payment is that no accrued-interest term in $(\tau - T_{\beta(\tau)-1})$ is necessary, and also that all payments occur at the canonical grid of the T_i's. The postponed payout is better for deriving market models of CDS rates dynamics, as we shall see shortly. Yet, unless explicitly specified, in the following we consider the first payout (1) since this is the formulation most resembling market practice (Prampolini (2002)). When we write simply "CDS" we refer to the RCDS case.

A slightly different postponed discounted payoff would be more appropriated. Indeed, if we consider

$$\Pi_{\text{PR2CDS}_{a,b}}(t) := \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_{i-1}\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}D(t,T_i)\ \text{L}_{\text{GD}} \tag{3}$$

(notice the T_{i-1} in the indicators of the first summation), we see that we are including one more R-payment with respect to the earlier postponed case. This is appropriate, since by pretending default is occurring at $T_{\beta(\tau)}$ instead of τ we are in fact introducing one more whole interval we have to account for in the "premium leg".

From a different point of view, and since the protection leg, even if postponed, is discounted with the appropriate discount factor taking into account postponement, notice that in cases where τ is slightly larger than T_i then the first postponed payoff (2) is a better approximation of the actual one. Instead, in cases where τ is slightly smaller than T_i, the postponed payoff (3) represents a better approximation. We will see the different implications of these two payoffs.

Recently, there has been some interest in "upfront CDS" contracts (Perdichizzi and Veronesi (2003)). In this version, the present value of the protection leg is paid upfront by the party that is buying protection. In other terms, instead of exchanging a possible protection payment for some coupons, one exchanges it with an upfront payment.

The discounted payoff of the protection leg is simply

$$\Pi_{\text{UCDS}_{a,b}}(t) := \mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\ \text{L}_{\text{GD}} = \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} D(t,\tau)\ \text{L}_{\text{GD}}. \tag{4}$$

Alternatively, one can approximate this leg by a "postponed payment" version, where we postpone the protection payment until the first T_i following default τ:

$$\Pi_{\text{UPCDS}_{a,b}}(t) := \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} D(t,T_i)\ \text{L}_{\text{GD}}. \tag{5}$$

2.2 CDS pricing and Cox Processes

We denote by $\text{CDS}(t, [T_{a+1}, \ldots, T_b], T_a, T_b, R, \text{L}_{\text{GD}})$ the price at time t of the above standard running CDS. At times some terms are omitted, such as for example the list of payment dates $[T_{a+1}, \ldots, T_b]$. We add the prefixes "PR1" or "PR2" to denote, respectively, the analogous prices for the postponed payoffs (2) and (3). We add the prefix "U" (upfront) to denote the present value at t of the protection leg (4) of the CDS, and "UP" (upfront postponed) in case we are considering the present value of (5).

The pricing formulas for these payoffs depend on the assumptions on interest-rate dynamics and on the default time τ. Here we place ourselves in a stochastic intensity framework, where the intensity is an $\mathcal{F}_t$-adapted continuous positive process, $\mathcal{F}_t$ denoting the basic filtration without default, typically representing the information flow of interest rates, intensities and possibly other default-free market quantities. Default is modeled as the first jump time of a Cox process with the given intensity process. In the Cox process setting we have $\tau = \Lambda^{-1}(\xi)$, where Λ is the stochastic hazard function which we assume to be $\mathcal{F}_t$ adapted, absolutely continuous and strictly increasing, and ξ is exponentially distributed with parameter 1 and independent of $\mathcal{F}$. These assumptions imply the existence of a positive adapted process λ, which we assume also to be right continuous and limited on the left, such that $\Lambda(t) = \int_0^t \lambda_s ds$ for all t. We will not model the intensity directly in this paper, except in section 9. Rather, we model some market quantities embedding the impact of the relevant intensity model that is consistent with them. In general, we can compute the CDS price according to risk-neutral valuation (see for example Bielecki and Rutkowski (2001)):

$$\mathrm{CDS}(t, T_a, T_b, R, \mathrm{L{\scriptsize GD}}) = \mathbb{E}\left\{\Pi_{\mathrm{RCDS}_{a,b}}(t)|\mathcal{G}_t\right\} \tag{6}$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$, and $\mathbb{E}$ denotes the risk-neutral expectation in the enlarged probability space supporting τ. We will denote by $\mathbb{E}_t$ the expectation conditional on the sigma field $\mathcal{F}_t$.

This expected value can also be written as

$$\mathrm{CDS}(t, T_a, T_b, R, \mathrm{L{\scriptsize GD}}) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)}\mathbb{E}\left\{\Pi_{\mathrm{RCDS}_{a,b}}(t)|\mathcal{F}_t\right\} \tag{7}$$

(see again Bielecki and Rutkowski (2001) formula (5.1) p. 143, or more in particular Jeanblanc and Rutkowski (2000) for the most general results of this kind).

This second expression, and the analogous definitions with postponed payoffs, will be fundamental for introducing the market model for CDS options in a rigorous way.

For the time being, let us deal with the definition of (running) CDS forward rate $R_{a,b}(t)$. This can be defined as that R that makes the CDS value equal to zero at time t, so that

$$\mathrm{CDS}(t, T_a, T_b, R_{a,b}(t), \mathrm{L{\scriptsize GD}}) = 0.$$

The idea is then solving this equation in $R_{a,b}(t)$. In doing this one has to be careful. It is best to use expression (7) rather than (6). Equate this expression to zero and derive R correspondingly. Strictly speaking, the resulting R would be defined on $\{\tau > t\}$ only, since elsewhere the equation is satisfied automatically thanks to the indicator in front of the expression, regardless of R. Since the value of R does not matter when $\{\tau < t\}$, the equation being satisfied automatically, we need not worry about $\{\tau < t\}$ and may define, in general,

$$R_{a,b}(t) = \frac{\mathrm{L{\scriptsize GD}}\ \mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\leq T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i) + \mathbb{E}\left\{D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t\right\}}, \tag{8}$$

where $\bar{P}(t,T) := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t)$ is the "no survival-indicator" part of the defaultable T-maturity (no recovery) zero coupon bond, i.e.

$$\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t) = \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T)$$

is the price at time t of a defaultable zero-coupon bond maturing at time T. Notice that replacing $\{\tau > T\}$ by $\{\tau \geq T\}$, as we implicitly do in (8), does not change anything since we are assuming continuous processes for the short rate and the stochastic intensity. We will denote by $P(t,T)$ the default-free zero coupon bond at time t for maturity T.

This approach to define $R_{a,b}$ amounts to equating to zero only the expected value in (7), and in a sense is a way of privileging $\mathcal{F}_t$ expected values to $\mathcal{G}_t$ ones. The technical tool allowing us to do this is the above-mentioned Jeanblanc Rutkowski (2000) result, and this is the spirit of part of the work in Jamshidian (2002).

2.3 Market quoting mechanism and implied hazard functions

Now we explain shortly how the market quotes running and upfront CDS prices. First we notice that typically the T's are three- months spaced. Let us begin with running CDS's. Usually at time $t = 0$, provided default has not yet occurred, the market sets R to a value $R^{\text{MID}}_{a,b}(0)$ that makes the CDS fair at time 0, i.e. such that $\text{CDS}(0, T_a, T_b, R^{\text{MID}}_{a,b}(0), \text{L}_{\text{GD}}) = 0$. In fact, in the market running CDS's used to be quoted at a time 0 through a bid and an ask value for this "fair" $R^{\text{MID}}_{a,b}(0)$, for CDS's with $T_a = 0$ and with T_b spanning a set of canonical final maturities, $T_b = 1y$ up to $T_b = 10y$. As time moves on to say $t = 1day$, the market shifts the T's of t, setting $T_a = 0 + t, \ldots, T_b = 10y + t$, and then quotes $R^{\text{MID}}_{a,b}(t)$ satisfying $\text{CDS}(t, T_a, T_b, R^{\text{MID}}_{a,b}(t), \text{L}_{\text{GD}}) = 0$. This means that as time moves on, the maturities increase and the times to maturity remain constant.

Recently, the quoting mechanism has changed and has become more similar to the mechanism of the futures markets. Let 0 be the current time. Maturities $T_a, \ldots, T_b$ are fixed at the original time 0 to some values such as 1y, 2y, 3y etc and then, as time moves for example to $t = 1day$, the CDS maturities are not shifted correspondingly of 1 day as before but remain 1y,2y etc from the original time 0. This means that the times to maturity of the quoted CDS's decrease as time passes. When the quoting time approaches maturity, a new set of maturities are fixed and so on. A detail concerning the "constant maturities" paradigm is that when the first maturity T_a is less than one month away from the quoting time (say 0), the payoff two terms

$$T_a D(0, T_a) R \mathbf{1}_{\{\tau > T_a\}} + (T_{a+1} - T_a) D(0, T_{a+1}) R \mathbf{1}_{\{\tau > T_{a+1}\}}$$

are replaced by

$$T_{a+1} D(0, T_{a+1}) R \mathbf{1}_{\{\tau > T_{a+1}\}}$$

in determining the "fair" R. If we neglect this last convention, once we fix the quoting time (say to 0) the method to strip implied hazard functions is the same under the two quoting paradigms. For example, Brigo and Alfonsi (2003) present a more detailed section on the "constant time-to-maturity" paradigm, and illustrate the notion of implied deterministic intensity (hazard function), satisfying

$$\mathbb{Q}\{s < \tau \leq t\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)).$$

The market Γ's are obtained by inverting a pricing formula based on the assumption that τ is the first jump time of a Poisson process with deterministic intensity $\gamma(t) = d\Gamma(t)/dt$. In this case one can derive a formula for CDS prices based on integrals of γ, and on the initial interest-rate curve, resulting from the above expectation:

$$\text{CDS}(t, T_a, T_b, R, \text{L}_{\text{GD}}; \Gamma(\cdot)) = \mathbf{1}_{\{t<\tau\}} \Bigg[R \int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d(e^{-(\Gamma(u)-\Gamma(t))}) + \sum_{i=a+1}^{b} P(t, T_i) R \alpha_i e^{\Gamma(t)-\Gamma(T_i)} + \text{L}_{\text{GD}} \int_{T_a}^{T_b} P(t,u) d(e^{-(\Gamma(u)-\Gamma(t))}) \Bigg]. \quad (9)$$

By equating to zero the above expression in γ for $t = 0, T_a = 0$, after plugging in the relevant market quotes for R, one can extract the γ's corresponding to CDS market quotes for increasing maturities T_b and obtain market implied γ^{mkt} and Γ^{mkt}'s. It is important to point out that usually the actual model one assumes for τ is more complex and may involve stochastic intensity either directly or through stochastic modeling of the future R dynamics itself. Even so, the γ^{mkt}'s are retained as a mere quoting mechanism for CDS rate market quotes, and may be taken as inputs in the calibration of more complex models.

Upfront CDS are simply quoted through the present value of the protection leg. Under deterministic hazard rates γ, we have

$$\text{UCDS}(t, T_a, T_b, R, \text{L}_{\text{GD}}; \Gamma(\cdot)) = \mathbf{1}_{\{t<\tau\}}\text{L}_{\text{GD}} \int_{T_a}^{T_b} P(t,u) d(e^{-(\Gamma(u)-\Gamma(t))}).$$

As before, by equating to the corresponding upfront market quote the above expression in γ, one can extract the γ's corresponding to UCDS market quotes for increasing maturities and obtain again market implied γ^{mkt} and Γ^{mkt}'s.

Once the implied γ are estimated, it is easy to switch from the "running CDS quote" R to the "upfront CDS quote" UCDS, or vice versa. Indeed, we see that, without postponed payments, the two quotes are linked by

$$\text{UCDS}(t, T_a, T_b, R, \text{L}_{\text{GD}}; \Gamma^{\text{mkt}}(\cdot)) = R_{a,b}(t) \Big[\int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d(e^{-(\Gamma^{\text{mkt}}(u)-\Gamma^{\text{mkt}}(t))}) + \sum_{i=a+1}^{n} P(t,T_i)\alpha_i e^{\Gamma^{\text{mkt}}(t)-\Gamma^{\text{mkt}}(T_i)} \Big]$$

3 Different Definitions of CDS Forward Rates and Analogies with the LIBOR and SWAP Models

The procedure of equating to 0 the current price of a contract to derive a sensible definition of forward rate is rather common. For example, the default free forward LIBOR rate $F(t, S, T)$ is obtained as the rate at time t that makes the time-t price of a Forward Rate Agreement contract (FRA) vanish. This FRA contract locks in the interest rate between time S and T. An analogous definition of forward swap rate at time t is obtained as the rate in the fixed leg of the swap that makes the swap value at time t equal to 0. For a discussion on both the default free FRA and swap cases see for example Brigo and Mercurio (2001), Chapter 1.

In the current context, we can set a CDS price to zero to derive a forward CDS rate. Clearly, the obtained rate changes according to the different running CDS payoff we consider. For example, by equating to 0 expression (7) and solving in R, we have the standard running CDS forward rate given in (8). We may wonder about what we would have obtained as definition of forward CDS rates when considering CDS payoffs PRCDS with postponed protection payments (2) or even PR2CDS (3). By straightforwardly adapting the above derivation, we would have obtained a CDS forward rate defined as

$$R^{\text{PR}}_{a,b}(t) = \frac{\text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t]} = \frac{\text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)},$$

and

$$R^{\mathrm{PR2}}_{a,b}(t) = \frac{\mathrm{L_{GD}}\ \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau > T_{i-1}\}}|\mathcal{F}_t]}$$

(where "PR" and "PR2" stand for "postponed-running" payoffs of the first and second kind, respectively).

Can we use the forward CDS rate definition, limited to a one-period interval, to introduce defaultable one-period forward rates? A straightforward generalization of the definition of forward LIBOR rates to the defaultable case is given for example in Schönbucher (2000). This definition mimics the definition in the default free case, in that from zero-coupon bonds one builds a "defaultable forward LIBOR rate"

$$\bar F(t;T_{j-1},T_j) := (1/\alpha_j)(\bar P(t,T_{j-1})/\bar P(t,T_j) - 1)$$

on $\tau > t$. However, as noticed earlier, the default free F is obtained as the *fair rate* at time t for a Forward Rate Agreement contract (FRA). Can we see $\bar F$ as the fair rate for a sort of defaultable FRA? Since the most liquid credit instruments are CDS's, consider a running postponed CDS on a one-period interval, with $T_a = T_{j-1}$ and $T_b = T_j$. We obtain (take $\mathrm{L_{GD}} = 1$)

$$R^{PR}_j(t) := \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar P(t,T_j)} = \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau > T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau > T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar P(t,T_j)} \tag{10}$$

where we have set $R^{PR}_j := R^{PR}_{j-1,j}$. The analogous part of $\bar F_j = \bar F(\cdot, T_{j-1}, T_j)$ would be, after adjusting the conditioning to $\mathcal{F}_t$ ($\hat F_j(t) = \bar F_j(t)$ on $\tau > t$ but $\hat F$ is defined also on $\tau \le t$)

$$\hat F_j(t) = \frac{\mathbb{E}[D(t,T_{j-1})\mathbf{1}_{\{\tau > T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau > T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar P(t,T_j)}.$$

The difference is that in R^{PR}_j's numerator we are taking expectation of a quantity that vanishes in all paths where $\tau > T_j$, whereas in $\bar F$ the corresponding quantity does not vanish necessarily in paths with $\tau > T_j$. Moreover, while R_j comes from a financial contract, $\bar F$ remains an abstraction not directly linked to a financial payoff.

Schönbucher (2000) defines the discrete tenor credit spread, in general, to be

$$H_j(t) := \frac{1}{\alpha_j}\left(\frac{\bar P(t,T_{j-1})/P(t,T_{j-1})}{\bar P(t,T_j)/P(t,T_j)} - 1\right)$$

(in $\tau > t$), and it is easy to see that we get

$$H_j(t) = R^{PR}_j(t),$$

but under independence of the default intensity and the interest rates, and not in general. Again, in general R_j comes from imposing a one-period CDS to be fair whereas H_j does not.

A last remark concerns an analogy with the default-free swap market model, where we have a formula linking swap rates to forward rates through a weighted average:

$S_{a,b}(t) = \sum_{i=a+1}^{b}[\alpha_i P(t,T_i)/(\sum_{k=a+1}^{b}\alpha_k P(t,T_k))]F_i(t) = \sum_{i=a+1}^{b} w_i(t;F(t))\ F_i(t)$. This is useful since it leads to an approximated formula for swaptions in the LIBOR model, see for example Brigo and Mercurio (2001), Chapter 6. A similar approach can be obtained for CDS forward rates. It is easy to check that

$$R_{a,b}^{PR}(t) = \frac{\sum_{i=a+1}^{b}\alpha_i R_i^{PR}(t)\bar{P}(t,T_i)}{\sum_{i=a+1}^{b}\alpha_i\bar{P}(t,T_i)} = \sum_{i=a+1}^{b}\bar{w}_i(t)R_i^{PR}(t) \approx \sum_{i=a+1}^{b}\bar{w}_i(0)R_i^{PR}(t). \qquad (11)$$

A similar relationship for $R_{a,b}^{PR2}$ involving a weighted average of one-period rates is obtained when resorting to the second type of postponed payoff.

A possible lack of analogy with the swap rates is that the $\bar{w}$'s cannot be expressed as functions of the R_i's only, unless we make some particular assumptions on the correlation between default intensities and interest rates. However, if we freeze the $\bar{w}$'s to time 0, which we have seen to work in the default-free LIBOR model, we obtain easily a useful approximate expression for $R_{a,b}$ and its volatility in terms of R_i's and their volatilities/correlations.

More generally, when not freezing, the presence of stochastic intensities besides stochastic interest rates adds degrees of freedom. Now the $\bar{P}$'s (and thus the $\bar{w}$'s) can be determined as functions for example of one- and two-period rates. Indeed, it is easy to show that

$$\bar{P}(t,T_i) = \bar{P}(t,T_{i-1})\frac{\alpha_{i-1}(R_{i-1}^{PR}(t) - R_{i-2,i}^{PR}(t))}{\alpha_i(R_{i-2,i}^{PR}(t) - R_i^{PR}(t))}. \qquad (12)$$

We have to assume $R_{i-2,i}^{PR}(t) - R_i^{PR}(t) \neq 0$. Actually, if we assume this to hold for the initial conditions, i.e. $R_{i-2,i}^{PR}(0) - R_i^{PR}(0) \neq 0$, and then take a diffusion dynamics for the two rates, the probability of our condition to be violated at future times is the probability of a continuous random variable to be 0, i.e. it is zero in general.

We will see later how this formula will help us in obtaining a market model for CDS rates. For the time being let us keep in mind that the exact weights $\bar{w}(t)$ in (11) are completely specified in terms of $R_i(t)$'s and $R_{i-2,i}(t)$'s, so that if we include these two rates in our dynamics the "system" is closed in that we also know all the relevant $\bar{P}$'s. The difference with the LIBOR/Swap model is that here to close the system we need also two-period rates.

4 Defaultable Floater and CDS

Consider a prototypical defaultable floating rate note (FRN).

Definition 4.1. Prototypical defaultable floating-rate note. *A prototypical defaultable floating-rate note is a contract ensuring the payment at future times $T_{a+1},\dots,T_b$ of the LIBOR rates that reset at the previous instants $T_a,\dots,T_{b-1}$ plus a spread X, each payment conditional on the issuer having not defaulted before the relevant previous instant. Moreover, the note pays a last cash flow consisting of the reimbursement of the notional value of the note at final time T_b if the issuer has not defaulted earlier. We* *assume a deterministic recovery value* R_{EC} *to be paid at the first* T_i *following default if default occurs before* T_b*. The note is said to quote at par if its value is equivalent to the value of the notional paid at the first reset time* T_a *in case default has not occurred before* T_a*.*

Recall that if no default is considered, then the fair spread making the FRN quote at par is 0, see for example Brigo and Mercurio (2001b), p. 15.

When in presence of Default, the note discounted payoff, including the initial cash flow on 1 paid in T_a, is

$$\Pi_{\mathrm{DFRN}a,b} = -D(t,T_a)1_{\{\tau>T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)(L(T_{i-1},T_i)+X)\mathbf{1}_{\{\tau>T_i\}}$$

$$+D(t,T_b)1_{\{\tau>T_b\}} + \mathrm{R_{EC}} \sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}},$$

where $\mathrm{R_{EC}}$ is the recovery rate, i.e. the percentage of the notional that is paid in replacement of the notional in case of default, and it is paid at the first instant among $T_{a+1},\dots,T_b$ following default. This is the correct definition of DFRN, consistent with market practice. The problem with such definition is that it has no equivalent in terms of approximated CDS payoff. This is due to the fact that, in a Cox process setting, it is difficult to disentangle the LIBOR rate L from the indicator and stochastic discount factor in such a way to obtain expectations of pure stochastic discount factors times default indicators. This becomes possible if we replace $1_{\{\tau>T_i\}}$ in the first summation with $1_{\{\tau>T_{i-1}\}}$, as one can see from computations (13) below. The same computations, in case we keep T_i in the default indicator, even in the simplified case where $\mathrm{L_{GD}} = 1$ and interest rates are independent of default intensities, would lead us to a corresponding definition of CDS forward rate where protection is paid at the last instant T_i *before* default, which is not natural since one should anticipate default.

4.1 A first approximated DFRN payoff

We thus consider two alternative definitions of DFRN. The first one is obtained by moving the default indicator of $L(T_{i-1},T_i) + X$ from T_i to T_{i-1}. The related FRN discounted payoff is defined as follows:

$$\Pi_{\mathrm{DFRN2}a,b} = -D(t,T_a)1_{\{\tau>T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)(L(T_{i-1},T_i)+X)\mathbf{1}_{\{\tau>T_{i-1}\}}$$

$$+D(t,T_b)1_{\{\tau>T_b\}} + \mathrm{R_{EC}} \sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1}\leq\tau<T_i\}},$$

Recall that, in the CDS payoff, $\mathrm{L_{GD}} = 1 - \mathrm{R_{EC}}$. We may now value the above discounted payoff at time t and derive the value of X that makes it 0. Define

$$\mathrm{DFRN2}_{a,b}(t,X,\mathrm{R_{EC}}) = \mathbb{E}\{\Pi_{\mathrm{DFRN2}a,b}|\mathcal{G}_t\} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\{\Pi_{\mathrm{DFRN2}a,b}|\mathcal{F}_t\}/\mathbb{Q}(\tau>t|\mathcal{F}_t)$$

and solve $\mathbb{E}\{\Pi_{\text{DFRN2}a,b}|\mathcal{F}_t\} = 0$ in X. The only nontrivial part is computing

$$\alpha_i \mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] = \alpha_i \mathbb{E}[\mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t] = \ldots$$

Under our Cox process setting for τ we can write

$$\begin{aligned} \ldots &= \alpha_i \mathbb{E}[\mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\xi>\Lambda(T_{i-1})\}}|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t] = \\ &= \alpha_i \mathbb{E}[D(t,T_{i-1})L(T_{i-1},T_i)\exp(-\Lambda(T_{i-1}))\mathbb{E}[D(T_{i-1},T_i)|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t] = \\ &= \alpha_i \mathbb{E}[\exp(-\Lambda(T_{i-1}))D(t,T_{i-1})L(T_{i-1},T_i)P(T_{i-1},T_i)|\mathcal{F}_t] = \\ &= \mathbb{E}[\exp(-\Lambda(T_{i-1}))D(t,T_{i-1})(1-P(T_{i-1},T_i))|\mathcal{F}_t] = \\ &= \mathbb{E}[D(t,T_{i-1})(1-P(T_{i-1},T_i))\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] = \\ &= \mathbb{E}[D(t,T_{i-1})\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] \end{aligned} \tag{13}$$

Now the LIBOR flow has vanished from the above payoff and we have expressed everything in terms of pure discount factor and default indicators. Some of these computations could have been performed more simply by means of standard and model independent arguments, but we carried them out in the explicit intensity case so that the reader may try them when not replacing $1_{\{\tau>T_i\}}$ to see what goes wrong.

We may write also

$$\begin{aligned} \text{DFRN2}_{a,b}(t,X,\text{R}_{\text{EC}}) = (1_{\{\tau>t\}}/\mathbb{Q}(\tau>t|\mathcal{F}_t))\Big[-\mathbb{E}_t[D(t,T_a)1_{\{\tau>T_a\}}] + \mathbb{E}_t[D(t,T_b)1_{\{\tau>T_b\}}] \\ -\sum_{i=a+1}^{b}\mathbb{E}_t[(D(t,T_i)-D(t,T_{i-1}))\mathbf{1}_{\{\tau>T_{i-1}\}}] + X\sum_{i=a+1}^{b}\alpha_i\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}] \\ +\text{R}_{\text{EC}}\sum_{i=a+1}^{b}\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}]\Big]. \end{aligned}$$

We may simplify terms in the summations and obtain

$$\begin{aligned} \text{DFRN2}_{a,b}(t,X,\text{R}_{\text{EC}}) = \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\Big[-\text{L}_{\text{GD}}\sum_{i=a+1}^{b}\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}] \\ +X\sum_{i=a+1}^{b}\alpha_i\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]\Big], \end{aligned}$$

from which we notice en passant that

$$\text{DFRN2}_{a,b}(t,X,\text{R}_{\text{EC}}) = \text{PR2CDS}(t,T_a,T_b,X,1-\text{R}_{\text{EC}}). \tag{14}$$

By taking into account this result, the expression for X that makes the DFRN quote at par is clearly the running "postponed of the second kind" CDS forward rate

$$X^{(2)}_{a,b}(t) = R^{PR2}_{a,b}(t),$$

i.e. *the fair spread in a defualtable floating rate note is equal to the running postponed CDS forward rate.*

4.2 A second approximated DFRN payoff

The second alternative definition of DFRN, leading to a useful relationship with approximated CDS payoffs, is obtained by moving the default indicator of $L(T_{i-1}, T_i) + X$ from T_i to T_{i-1} but only for the LIBOR flow, not for the spread X. This payoff is closer to the original Π_{DFRN} payoff than the approximated Π_{DFRN2} payoff considered above. Set

$$\Pi_{\text{DFRN1}a,b} = -D(t, T_a)1_{\{\tau > T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t, T_i)(L(T_{i-1}, T_i)\mathbf{1}_{\{\tau > T_{i-1}\}} + X\mathbf{1}_{\{\tau > T_i\}})$$

$$+D(t, T_b)1_{\{\tau > T_b\}} + \text{R}_{\text{EC}} \sum_{i=a+1}^{b} D(t, T_i)\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}}.$$

By calling $\text{DFRN1}_{a,b}(t, X, \text{R}_{\text{EC}})$ the t-value of the above payoff and by going through the computations we can see easily that this time

$$\text{DFRN1}_{a,b}(t, X, \text{R}_{\text{EC}}) = \text{PRCDS}(t, T_a, T_b, X, 1 - \text{R}_{\text{EC}}). \tag{15}$$

and that, as far as fair spreads are concerned,

$$X^{(1)}_{a,b}(t) = R^{PR}_{a,b}(t).$$

5 CDS Options and Callable Defaultable Floaters

Consider the option to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, paying a fixed rate K at times $T_{a+1}, \ldots, T_b$ or until default, in exchange for protection against possible default in $[T_a, T_b]$. If default occurs a protection payment L_{GD} is received. By noticing that the market CDS rate $R_{a,b}(T_a)$ will set the CDS value in T_a to 0, the payoff can be written as the discounted difference between said CDS and the corresponding CDS with rate K. We will see below that this is equivalent to a call option on the future CDS fair rate $R_{a,b}(T_a)$. The discounted CDS option payoff reads, at time t,

$$\Pi_{\text{CallCDS}a,b}(t; K) = D(t, T_a)[\text{CDS}(T_a, T_a, T_b, R_{a,b}(T_a), \text{L}_{\text{GD}}) - \text{CDS}(T_a, T_a, T_b, K, \text{L}_{\text{GD}})]^+, \tag{16}$$

leading to two possible expressions, depending on whether we explicit the CDS values, given respectively by

$$\begin{aligned}\Pi_{\text{CallCDS}a,b}(t; K) &= \frac{1_{\{\tau > T_a\}}}{\mathbb{Q}(\tau > T_a | \mathcal{F}_{T_a})} D(t, T_a) \left[\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > T_a | \mathcal{F}_{T_a}) \bar{P}(T_a, T_i) + \right. \\ &+ \left. \mathbb{E}\left\{ D(T_a, \tau)(\tau - T_{\beta(\tau)-1}) \mathbf{1}_{\{\tau < T_b\}} | \mathcal{F}_{T_a} \right\} \right] (R_{a,b}(T_a) - K)^+\end{aligned} \tag{17}$$

or, by remembering that by definition $\text{CDS}(T_a, T_a, T_b, R_{a,b}(T_a), \text{L}_{\text{GD}}) = 0$, as

$$\Pi_{\text{CallCDS}a,b}(t; K) = D(t, T_a)[-\text{CDS}(T_a, T_a, T_b, K, \text{L}_{\text{GD}})]^+. \tag{18}$$

This last expression points out that in holding this CDS option we would be interested in the option to buy protection L_{GD} at time T_a against default in $[T_a, T_b]$ in exchange for a premium leg characterized by the strike-rate K. These options can be introduced for every type of underlying CDS. We have illustrated the standard CDS case, and we will consider the postponed CDS cases below.

The quantity inside square brackets in (17) will play a key role in the following. We will often neglect the accrued interest term in $(\tau - T_{\beta(\tau)-1})$ and consider the related simplified payoff: in such a case the quantity between square brackets is denoted by $\widehat{C}_{a,b}(T_a)$ and is called ("no survival-indicator"-) "defaultable present value per basis point (DPVBP) numeraire" (and sometimes "annuity"). Actually the real DPVBP would have a $1_{\{\tau > \cdot\}}$ term in front of the summation, but by a slight abuse of language we call DPVBP the expression without indicator. More generally, at time t, we set

$$\widehat{C}_{a,b}(t) := \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{C}_{a,b}(t), \quad \bar{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \bar{P}(t, T_i).$$

When including as a factor the indicator $1_{\{\tau > t\}}$, this quantity is the price, at time t, of a portfolio of defaultable zero-coupon bonds with zero recovery and with different maturities, and as such it is the price of a tradable asset. The original work of Schönbucher (2000) is in this spirit, in that the "numeraire" is taken with the indicator, so that it may vanish and the measure it defines is not equivalent to the risk neutral measure. If we keep the indicator away, following in spirit part of the work in Jamshidian (2002), this quantity maintains a link with said price and is always strictly positive, so that we are allowed to take it as numeraire.

The related probability measure, equivalent to the risk neutral measure, is denoted by $\widehat{Q}^{a,b}$ and the related expectation by $\widehat{\mathbb{E}}^{a,b}$.

Neglecting the accrued interest term, the option discounted payoff simplifies to

$$1_{\{\tau > T_a\}} D(t, T_a) \left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a, T_i) \right] (R_{a,b}(T_a) - K)^+ \tag{19}$$

but this is only an approximated payoff and not the exact one.

5.1 First equivalence: PRCDS and DFRN1

Let us follow the same derivation under the postponed CDS payoff of the first kind. Consider thus

$$\Pi_{\mathrm{CallPRCDS}_{a,b}}(t; K) = D(t, T_a)[\mathrm{PRCDS}(T_a, T_a, T_b, R^{PR}_{a,b}(T_a), \mathrm{L}_{\mathrm{GD}}) - \mathrm{PRCDS}(T_a, T_a, T_b, K, \mathrm{L}_{\mathrm{GD}})]^+, \tag{20}$$

or, since $\mathrm{PRCDS}(T_a, T_a, T_b, R^{PR}_{a,b}(T_a), \mathrm{L}_{\mathrm{GD}}) = 0$,

$$\Pi_{\mathrm{CallPRCDS}_{a,b}}(t; K) = D(t, T_a)[-\mathrm{PRCDS}(T_a, T_a, T_b, K, \mathrm{L}_{\mathrm{GD}})]^+,$$

which, by (15), is equivalent to

$$D(t, T_a)[-\mathrm{DFRN1}_{a,b}(T_a, K, \mathrm{R}_{\mathrm{EC}})]^+, \tag{21}$$

with $\text{L}_{\text{GD}} = 1 - \text{R}_{\text{EC}}$, or, since $\text{DFRN1}_{a,b}(T_a, X_{a,b}(T_a), \text{R}_{\text{EC}}) = 0$, is equivalent to

$$D(t,T_a)[\text{DFRN1}_{a,b}(T_a, X_{a,b}(T_a), \text{R}_{\text{EC}}) - \text{DFRN1}_{a,b}(T_a, K, \text{R}_{\text{EC}})]^+.$$

By expanding the expression of PRCDS we obtain as *exact* discounted payoff the quantity

$$\Pi_{\text{CallPRCDS}_{a,b}}(t,K) = 1_{\{\tau > T_a\}} D(t,T_a) \left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a, T_i) \right] (R^{PR}_{a,b}(T_a) - K)^+, \tag{22}$$

which is structurally identical to the *approximated* payoff (19) for the standard CDS case. Thus we have created a link between the first postponed CDS payoff and an option on an approximated DFRN. Notice in particular that the quantity in front of the optional part is the same as in the earlier standard approximated discounted payoff, i.e. the DPVBP.

5.2 Second equivalence: PR2CDS and DFRN2

We may also consider the postponed running CDS of the second kind. The related discounted CDS option payoff reads, at time t,

$$\Pi_{\text{CallPR2CDS}_{a,b}}(t,K) = D(t,T_a)[\text{PR2CDS}(T_a, T_b, R^{PR2}_{a,b}(T_a), \text{L}_{\text{GD}}) - \text{PR2CDS}(T_a, T_b, K, \text{L}_{\text{GD}})]^+, \tag{23}$$

and given (14), this is equivalent to

$$D(t,T_a)[\text{DFRN2}_{a,b}(T_a, X_{a,b}(T_a), \text{R}_{\text{EC}}) - \text{DFRN2}_{a,b}(T_a, K, \text{R}_{\text{EC}})]^+, \tag{24}$$

with $\text{R}_{\text{EC}} = 1 - \text{L}_{\text{GD}}$, or, by expanding the expression for PR2CDS, as

$$(1_{\{\tau > T_a\}}/\mathbb{Q}(\tau > T_a | \mathcal{F}_{T_a})) D(t,T_a) \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_{T_a}[D(T_a, T_i) \mathbf{1}_{\{\tau > T_{i-1}\}}] (R^{PR2}_{a,b}(T_a) - K)^+. \tag{25}$$

Again we have equivalence between CDS options and options on the defaultable floater.

5.3 Callable defaultable floaters

The option on the floater can be seen as the optional component of a callable DFRN. A DFRN with final maturity T_b is issued at time 0 with a fair rate $X_{0,b}(0)$ in such a way that $\text{DFRN}_{0,b}(0, X_{0,b}(0), \text{R}_{\text{EC}}) = 0$. Suppose that this FRN includes a callability feature: at time T_a the issuer has the right to take back the subsequent FRN flows and replace them with the notional 1. The issuer will do so only if the present value in T_a of the subsequent FRN flows is larger than 1 in T_a. This is equivalent, for the note holder, to receive $1_{\{\tau > T_a\}} + \min(\text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{R}_{\text{EC}}), 0) = 1_{\{\tau > T_a\}} + \text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{R}_{\text{EC}}) - (\text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{R}_{\text{EC}}))^+$ at time T_a if no default has occurred by then (recall that in our notation $\text{DFRN}_{a,b}$ includes a negative cash flow of 1 at time T_a).

The component $1_{\{\tau > T_a\}} + \text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{R}_{\text{EC}})$ when valued at time 0 is simply the residual part of the original DFRN without callability features from T_a on, so that when added to the previous payments in $0 \div T_a$ its present value is 0. This happens because $X_{a,b}(0)$ is the fair rate for the total DFRN at time 0. The component $(\mathrm{DFRN}_{a,b}(T_a, X_{0,b}(0), \mathrm{R}_{\mathrm{EC}}))^+ = (\mathrm{DFRN}_{a,b}(T_a, X_{0,b}(0), \mathrm{R}_{\mathrm{EC}}) - \mathrm{DFRN}_{a,b}(T_a, X_{a,b}(T_a), \mathrm{R}_{\mathrm{EC}}))^+$ is structurally equivalent to a CDS option, provided we approximate its payoff with $DFRN1$ or $DFRN2$, as we have seen earlier, and we may value it if we have a model for CDS options. We are deriving a market model for such options in the next section, so that we will be implicitly deriving a model for callable defaultable floaters.

6 A market formula for CDS options and callable defaultable floaters

As usual, one may wish to introduce a notion of implied volatility for CDS options. This would be a volatility associated to the relevant underlying CDS rate R. In order to do so rigorously, one has to come up with an appropriate dynamics for $R_{a,b}$ directly, rather than modeling instantaneous default intensities explicitly. This somehow parallels what we find in the default-free interest rate market when we resort to the swap market model as opposed for example to a one-factor short-rate model for pricing swaptions. In a one-factor short-rate model the dynamics of the forward swap rate is a byproduct of the short-rate dynamics itself, through Ito's formula. On the contrary, the market model for swaptions directly postulates, under the relevant numeraire a (lognormal) dynamics for the forward swap rate.

6.1 Market formulas for CDS Options

In the case of CDS options the market model is derived as follows. First, let us ignore the accruing term in $(\tau - T_{\beta(\tau)-1})$, by replacing it with zero. It can be seen that typically the order of magnitude of this term is negligible with respect to the remaining terms in the payoff. Failing this negligibility, one may reformulate the payoff by postponing the default payment to the first date among the T_i's following τ, i.e. to $T_{\beta(\tau)}$. This amounts to considering as underlying a payoff corresponding to (2) and eliminates the accruing term altogether, even though it slightly modifies the option payoff. Take as numeraire the DPVBP $\widehat{C}_{a,b}$, so that

$$R_{a,b}(t) = \frac{\mathrm{L}_{\mathrm{GD}}\, \mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\le T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)} = \frac{\mathrm{L}_{\mathrm{GD}}\, \mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\le T_b\}}|\mathcal{F}_t]}{\widehat{C}_{a,b}(t)}, \quad t \le T_a, \quad (26)$$

having as numerator the price of an upfront CDS, can be interpreted as the ratio between a tradable asset and our numeraire. As such, it is a martingale under this numeraire's measure and can be modeled as a Black-Scholes driftless geometric Brownian motion, leading to a Black and Scholes formula for CDS options (notice that, when $R_{a,b}(0)$ is not quoted directly by the market, we may infer it by the market implied γ^{mkt} according to

$$R_{a,b}(0) = \frac{-\mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} P(0,u)\, d(e^{-\Gamma^{\mathrm{mkt}}(u)})}{\sum_{i=a+1}^{b} \alpha_i P(0,T_i) e^{-\Gamma^{\mathrm{mkt}}(T_i)}} \quad)$$

Indeed, by resorting to the change of numeraire starting from (19) (thus ignoring the accruing term or working with the postponed payoff) we see that

$$
\begin{aligned}
&\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{G}_t\}\\
&=\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\}\\
&=\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\big[\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{F}_{T_a}\}|\mathcal{F}_t\big]\\
&=\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\big[D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+\mathbb{E}\{1_{\{\tau>T_a\}}|\mathcal{F}_{T_a}\}|\mathcal{F}_t\big]\\
&=\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\big[D(t,T_a)\sum_{i=a+1}^{b}\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\big]\\
&=\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\big[D(t,T_a)\widehat{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\big]\\
&=\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\widehat{C}_{a,b}(t)\widehat{\mathbb{E}}^{a,b}[(R_{a,b}(T_a)-K)^+|\mathcal{F}_t]\\
&=1_{\{\tau>t\}}\bar{C}_{a,b}(t)\widehat{\mathbb{E}}^{a,b}[(R_{a,b}(T_a)-K)^+|\mathcal{F}_t]
\end{aligned}
$$

and we may take

$$dR_{a,b}(t)=\sigma_{a,b}R_{a,b}(t)dW^{a,b}(t), \tag{27}$$

where $W^{a,b}$ is a Brownian motion under $\widehat{\mathbb{Q}}^{a,b}$, leading to a market formula for the CDS option. We have

$$\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\bar{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+|\mathcal{G}_t\}=1_{\{\tau>t\}}\bar{C}_{a,b}(t)[R_{a,b}(t)N(d_1(t))-KN(d_2(t))] \tag{28}$$

$$d_{1,2}=\left(\ln(R_{a,b}(t)/K)\pm(T_a-t)\sigma_{a,b}^2/2\right)/(\sigma_{a,b}\sqrt{T_a-t}).$$

As happens in most markets, this formula could be used as a quoting mechanism rather than as a real model formula. That is, the market price can be converted into its implied volatility matching the given price when substituted in the above formula.

6.2 Market Formula for callable DFRN

Since we are also interested in the parallel with DFRN's, let us derive the analogous market model formula under running CDS's postponed payoffs of the first kind. The derivation goes trough as above and we obtain easily the same model as in (27) and (28) with R^{PR} replacing R everywhere.

If we consider the second kind of approximation for FRN's, the option price is obtained as the price of a CDS option, where the CDS is a postponed CDS of the second kind. Compute then

$$\mathbb{E}\left\{ D(t,T_a)[\text{PR2CDS}(T_a,T_a,T_b,R^{PR2}_{a,b}(T_a),\text{L}_{\text{GD}}) - \text{PR2CDS}(T_a,T_a,T_b,K,\text{L}_{\text{GD}})]^+ | \mathcal{G}_t \right\}$$

$$= \mathbb{E}\left\{ \frac{1_{\{\tau>T_a\}}}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} D(t,T_a) \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_{T_a}[D(T_a,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}](R^{PR2}_{a,b}(T_a)-K)^+ | \mathcal{G}_t \right\}$$

$$= \mathbb{E}\left\{ 1_{\{\tau>T_a\}} D(t,T_a) \sum_{i=a+1}^{b} \alpha_i \frac{\mathbb{E}_{T_a}[D(T_a,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} (R^{PR2}_{a,b}(T_a)-K)^+ | \mathcal{G}_t \right\} = \dots$$

This time let us take as numeraire

$$\check{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}], \quad \text{(notice} \quad \widehat{C}_{a,b}(t) = \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}])$$

This quantity is positive, and when including the indicator $1_{\{\tau>t\}}$ this is, not surprisingly, a multiple of the premium leg of a PR2CDS at time t. We may also view it as a ("no survival-indicator"-) portfolio of defaultable bonds where the default maturity is one-period-displaced with respect to the payment maturity. Thus this quantity is only approximately a numeraire. Compute

$$\dots = \mathbb{E}\left\{ 1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} (R^{PR2}_{a,b}(T_a)-K)^+ | \mathcal{G}_t \right\}$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{ 1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} (R^{PR2}_{a,b}(T_a)-K)^+ \right\}$$

$$\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{ \mathbb{E}_{T_a}\left[1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} (R^{PR2}_{a,b}(T_a)-K)^+ \right] \right\}$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{ D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} (R^{PR2}_{a,b}(T_a)-K)^+ \mathbb{E}_{T_a}\left[1_{\{\tau>T_a\}} \right] \right\}$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{ D(t,T_a)\check{C}_{a,b}(T_a)(R^{PR2}_{a,b}(T_a)-K)^+ \right\}$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \check{C}_{a,b}(t)\check{\mathbb{E}}^{a,b}_t\left\{ (R^{PR2}_{a,b}(T_a)-K)^+ \right\}$$

Now notice that R^{PR2} can be also written as

$$R^{PR2}_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\check{C}_{a,b}(t)},$$

so that it is a martingale under $\check{\mathbb{Q}}^{a,b}$. As such, we may model it as

$$dR^{PR2}_{a,b}(t) = \sigma_{a,b} R^{PR2}_{a,b}(t) d\check{W}^{a,b}(t) \tag{29}$$

and compute the above expectation accordingly. We obtain, as price of the option,

$$\mathbb{E}\{1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} (R^{PR2}_{a,b}(T_a)-K)^+ | \mathcal{G}_t\} = \tag{30}$$

$$= 1_{\{\tau>t\}} \frac{\check{C}_{a,b}(t)}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} [R^{PR2}_{a,b}(t)N(d_1) - KN(d_2)]$$

where d_1 and d_2 are defined as usual in terms of $R_{a,b}^{PR2}(t)$, K and σ.

Which model should one use between DFRN1 and DFRN2 when dealing with DFRN options? DFRN1 has the advantage of better approximating the real DFRN; further, the related market model is derived under a numeraire; DFRN2 is derived only under and approximated numeraire and is a worse approximation of the real DFRN, but the related CDS payoff PR2CDS is in some cases a better approximation of a real CDS than PRCDS.

6.3 Examples of Implied Volatilities from the Market

We present now some CDS options implied volatilities obtained with the postponed payoff of the first and second kind. We consider three companies C1, C2 and C3 on the Euro market and the related CDS options quotes as of March 26, 2004; the recovery is $\text{R}_{\text{EC}} = 0.4$; C1 and C3 are in the telephonic sector, whereas C2 is a car industry; $\text{L}_{\text{GD}} = 1 - 0.4 = 0.6$; T_0 = March 26 2004 (0); We consider two possible maturities T_a =June 20 2004 (86d≈3m) and T'_a =Dec 20 2004 (269d≈9m); T_b = june 20 2009 (5y87d); we consider receiver option quotes (puts on R) in basis points (i.e. 1E-4 units on a notional of 1). We obtain the results presented in Table 2.

	Option: bid	mid	ask	$R_{0,b}(0)$	$R_{a,b}^{PR}(0)$	$R_{a,b}^{PR2}(0)$	K	$\sigma_{a,b}^{PR}$	$\sigma_{a,b}^{PR(2)}$
C1(T_a)	14	24	34	60	61.497	61.495	60	50.31	50.18
C2	32	39	46	94.5	97.326	97.319	94	54.68	54.48
C3	18	25	32	61	62.697	62.694	61	52.01	51.88
C1(T'_a)	28	35	42	60	65.352	65.344	61	51.45	51.32

Table 2: CDS forward rates and implied volatilites on three companies on March 26, 2004. Rates are in basis points and volatilities are percentages.

Implied volatilities are rather high when compared with typical interest-rate default free swaption volatilities. However, the values we find have the same order of magnitude as some of the values found by Hull and White (2003) via historical estimation. Further, we see that while the option prices differ considerably, the related implied volatilities are rather similar. This shows the usefulness of a rigorous model for implied volatilities. The mere price quotes could have left one uncertain on whether the credit spread variabilities implicit in the different companies were quite different from each other or similar.

We analyze also the implied volatilities and CDS forward rates under different payoff formulations and under stress. Table 2 shows that the impact of changing postponement from PR to PR2 (maintaining the same $R_{0,b}(0)$'sand re-stripping Γ's) leaves both CDS forward rates and implied volatilities almost unchanged.

In Table 3 we check the impact of the recovery rate on implied volatilities and CDS forward rates. Every time we change the recovery we re-calibrate the Γ's, since the only direct market quotes are the $R_{0,b}(0)$'s, which we cannot change, and our uncertainty is on the recovery rate that might change. As we can see from the table the impact of the recovery rate is rather small, but we have to keep in mind that the CDS option payoff is

	REC = 20%	REC = 30%	REC = 40%	REC = 50%	REC = 60%
$\sigma_{a,b}^{PR}$:					
C1(T_a)	50.02	50.14	50.31	50.54	50.90
C2	54.22	54.42	54.68	55.05	55.62
C3	51.71	51.83	52.01	52.25	52.61
C1(T'_a)	51.13	51.27	51.45	51.71	52.10
$R_{a,b}^{PR}$:					
C1(T_a)	61.488	61.492	61.497	61.504	61.514
C2	97.303	97.313	97.326	97.346	97.374
C3	62.687	62.691	62.697	62.704	62.716
C1(T'_a)	65.320	65.334	65.352	65.377	65.415

Table 3: Impact of recovery rates on the implied volatility and on the CDS forward rates for the PR payoff. Vols are expressed as percentages and rates as basis points

built in such a way that the recovery direct flow in LGD cancels and the recovery remains only implicitly inside the initial condition $R_{a,b}(0)$ for the dynamics of $R_{a,b}$, as one can see for example from (22), where LGD does not appear explicitly. It is $R_{a,b}(0)$ that depends on the stripped Γ's which, in turn, depend on the recovery.

	shift −0.5%	0	+0.5%
C1(T_a)	49.68	50.31	50.93
C2	54.02	54.68	55.34
C3	51.36	52.01	52.65

shift −0.5%	0	+0.5%
61.480	61.497	61.514
97.294	97.326	97.358
62.677	62.697	62.716

Table 4: Implied volatilities $\sigma_{a,b}$ (left, as percentages) and forward CDS rates $R_{a,b}^{PR}$ (right, as basis points) as the simply compounded rates are shifted uniformly for all maturities.

In Table 4 we check the impact of a shift in the simply compounded rates of the zero coupon interest rate curve on CDS forward rates and implied volatilities. Every time we shift the curve we recalibrate the Γ's, while maintaining the same $R_{0,b}(0)$'s. We see that the shift has a more relevant impact than the recovery rate, an impact that remains small.

We also include the zero coupon curve we used in Table 5 and the CDS market quotes we used in Table 6.

Date	Discount	Date	Discount	Date	Discount
26-mar-04	1	30-dec-04	0.985454616	28-mar-13	0.701853679
29-mar-04	0.999829196	30-mar-06	0.956335676	31-mar-14	0.665778313
31-mar-04	0.9997158	30-mar-07	0.9261161	30-mar-15	0.630686684
06-apr-04	0.999372341	31-mar-08	0.891575268	30-mar-16	0.597987523
30-apr-04	0.99806645	30-mar-09	0.85486229	30-mar-17	0.566052224
31-may-04	0.996398755	30-mar-10	0.816705705	29-mar-18	0.535085529
30-jun-04	0.994847843	30-mar-11	0.777867013	29-mar-19	0.505632535
30-sep-04	0.99014189	30-mar-12	0.739273058		

Table 5: Euro curve for Zero coupon bonds $P(0, T)$ as of march 26, 2004.

Maturity T_b	$R_{0,b}$ (C1)	$R_{0,b}$ (C2)	$R_{0,b}$ (C3)
1y	30	38.5	27
3y	49	72.5	49
5y	60	94.5	61
7y	69	104.5	73

Table 6: Quoted CDS rates for the three names in basis points as of march 26, 2004

7 Towards a Completely Specified Market Model

J'onn: "I fear the Justice League's greatest challenge lies just ahead..."
Kal: "Doesn't it always, J'onn?"

"Death Star", DC One Million 4, 1998, DC Comics

So far we have been able to rigorously justify the market CDS option formula. However, to completely specify the market model we need to show how the dynamics of $R_{a,b}$ changes when changing numeraire. We describe the essential steps briefly in two important cases, and we refer to the PR payoff.

7.1 One- and Two-period CDS rates market model

The first case we address is a family of one-period rates. This is to say that we are trying to build a sort of forward LIBOR model for CDS rates. As the LIBOR model is based on one-period forward rates, our first choice of a market model for CDS options will be based on one-period rates. The fundamental components of our numeraires $\widehat{C}$ are the $\bar{P}$'s. The $\bar{P}$'s, through (12), can be reduced to a function of a common initial $\bar{P}$ (that cancels when considering the relevant ratios) and of one- and two-period rates R_k, $R_{k-2,k}$ in the relevant range. We start then by writing the (martingale) dynamics of one- and two-period rates R_k $R_{k-2,k}$ each under its canonical numeraire $\widehat{\mathbb{Q}}^{k-1,k}$ and $\widehat{\mathbb{Q}}^{k-2,k}$. At this point we use the change of numeraire technique on each of this rates to write their dynamics under a single preferred $\widehat{\mathbb{Q}}^{\cdot,\cdot}$. This is possible in terms of quadratic covariations between the one- and two-period rates being modeled and the analogous rates concurring to form the $\bar{P}$'s entering the relevant $\hat{C}$. We detail this scheme in the following subsection. Now we have the dynamics for all the relevant R_k, $R_{k-2,k}$'s under a common measure $\widehat{\mathbb{Q}}^{\cdot,\cdot}$. Since $R_{a,b}$ is completely specified in terms of one and two-period rates through (11) and (12), we have indirectly also $R_{a,b}$'s dynamics.

Notice that if first we assign the dynamics of one period rates, then the dynamics of the two-period rates has to be selected carefully. For example, two-period rates will have to be selected into a range determined by one-period rates to avoid $\bar{P}(t,T_k)/\bar{P}(t,T_{k-1})$ to be negative or larger than one. The use of suitable martingale dynamics for each $R_{k-2,k}$ under $\widehat{\mathbb{Q}}^{k-2,k}$ ensuring this property is currently under investigation.

If we are concerned about lognormality of R's, leading to Black-like formulas for CDS options, one of the possible choices is to impose one-period rates R_k to have a lognormal distribution under their canonical measures. It suffices to postulate a driftless geometric Brownian motion dynamics for each such rate under its associated measure. The resulting $R_{a,b}$ will only be approximately lognormal, especially under the freezing approximation for the weights $\bar{w}$, but this is the case also with LIBOR vs SWAP models, since lognormal one-period swap rates (i.e. forward LIBOR rates) and multi period swap rates cannot be all lognormal (each under its canonical measure). The important difference with the LIBOR model is that here we need also two-period rates to close the system. The need for two-period rates stems from the additional degrees of freedom coming from stochastic intensity whose "maturities", in rates like R's, are not always temporally aligned with the stochastic interest rates maturities. More precisely, the fact that in the numerator of the last term in (10) we have not only $\bar{P}$ (second term in the numerator) but also a term in $D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}$ (first term in the numerator, notice the misaligned T_{j-1} and T_j) adds degrees of freedom that are accounted for by considering two period rates.

A final remark is that the freezing approximation is typically questionable when volatilities are very large. Since, as we will see below, at the moment implied volatilities in the CDS option market are rather large, the freezing approximation has to be considered with care.

7.1.1 Detailed scheme for the change of numeraire technique

Let us postulate the following dynamics for one- and two- period CDS forward rates. Recall that $R_j = R_{j-1,j}$.

$$dR_j(t) = \sigma_j(t) R_j(t) dZ_j^j(t)$$
$$dR_{j-2,j}(t) = \nu_j(t;R) R_{j-2,j}(t) dV_j^{j-2,j}(t)$$

In the Brownian shocks Z and V the upper index denotes the measure (i.e. the measure associated with the numeraires $\widehat{C}_{j-1,j}, \widehat{C}_{j-2,j}$ in the above case) and the lower index denotes to which component of the one- and two- period rate vectors the shock refers. **The volatilities σ are deterministic, whereas the ν's depend on the one-period R's.** We assume correlations

$$dZ_i dZ_j = \rho_{i,j} dt, \quad dV_i dV_j = \eta_{i,j} dt, \quad dZ_i dV_j = \theta_{i,j} dt$$

and $R_{i-2,i}(t) \in (\min(R_{i-1}(t), [R_{i-1}(t) + R_i(t)]/2), \max(R_{i-1}(t), [R_{i-1}(t) + R_i(t)]/2))$**. This latter condition ensures that the resulting** $\bar{P}$ **from formula (12) be positive and decreasing with respect to the maturity, i.e.** $0 < \bar{P}(t,T_i)/\bar{P}(t,T_{i-1}) < 1$**. The specific definition of** ν **ensuring this property is currently under investigation.**

We aim at finding the drift of a generic R_j under the measure associated with $\widehat{C}_{i-1,i}$, let us say for $j \geq i$.

The change of numeraire toolkit provides the formula relating shocks under $\widehat{C}_{i-1,i}$ to shocks under $\widehat{C}_{j-2,j}$, see for example Formula (2.13) in Brigo and Mercurio (2001), Chapter 2. We can write

$$d\begin{bmatrix} Z^{j-2,j} \\ V^{j-2,j} \end{bmatrix} = d\begin{bmatrix} Z^i \\ V^i \end{bmatrix} - \text{CorrMatrix} \times \text{VectorDiffusionCoefficient}\left(\ln\left(\frac{\widehat{C}_{j-2,j}}{\widehat{C}_{i-1,i}}\right)\right)' dt$$

Let us abbreviate "Vector Diffusion Coefficient" by "DC".

This is actually a sort of operator for diffusion processes that works as follows. $DC(X_t)$ is the row vector $\mathbf{v}$ in

$$dX_t = (...)dt + \mathbf{v}\; d\begin{bmatrix} Z_t \\ V_t \end{bmatrix}$$

for diffusion processes X with Z and V column vectors Brownian motions common to all relevant diffusion processes. This is to say that if for example $dR_1 = \sigma_1 R_1 dZ_1^1$, then

$$DC(R_1) = [\sigma_1 R_1,\ 0,\ 0, \ldots,\ 0].$$

Let us call Q the total correlation matrix including ρ, η and θ. We have

$$d\begin{bmatrix} Z^{j-2,j} \\ V^{j-2,j} \end{bmatrix} = d\begin{bmatrix} Z^i \\ V^i \end{bmatrix} - Q\ \text{DC}\left(\ln\left(\frac{\widehat{C}_{j-2,j}}{\widehat{C}_{i-1,i}}\right)\right) dt$$

Now we need to compute

$$\mathrm{DC}\left(\ln\left(\frac{\widehat{C}_{j-2,j}}{\widehat{C}_{i-1,i}}\right)\right) = \mathrm{DC}\left(\ln\left(\frac{\alpha_{j-1}\bar{P}(t,T_{j-1}) + \alpha_j \bar{P}(t,T_j)}{\alpha_i \bar{P}(t,T_i)}\right)\right) =$$

$$= \mathrm{DC}\left(\ln\left(\frac{\alpha_{j-1}}{\alpha_i}\frac{\alpha_i}{\alpha_{j-1}}\prod_{k=i+1}^{j-1}\frac{R_{k-1}-R_{k-2,k}}{R_{k-2,k}-R_k} + \frac{\alpha_j}{\alpha_i}\frac{\alpha_i}{\alpha_j}\prod_{k=i+1}^{j}\frac{R_{k-1}-R_{k-2,k}}{R_{k-2,k}-R_k}\right)\right)$$

$$= \mathrm{DC}\left(\ln\left(\left[\prod_{k=i+1}^{j-1}\frac{R_{k-1}-R_{k-2,k}}{R_{k-2,k}-R_k}\right]\left[1+\frac{R_{j-1}-R_{j-2,j}}{R_{j-2,j}-R_j}\right]\right)\right)$$

$$= \mathrm{DC}\left(\sum_{k=i+1}^{j-1}\ln\left(\frac{R_{k-1}-R_{k-2,k}}{R_{k-2,k}-R_k}\right)\right) + \mathrm{DC}\left(\ln\left(\frac{R_{j-1}-R_j}{R_{j-2,j}-R_j}\right)\right)$$

$$= \sum_{k=i+1}^{j-1}\mathrm{DC}\left(\ln\left(\frac{R_{k-1}-R_{k-2,k}}{R_{k-2,k}-R_k}\right)\right) + \mathrm{DC}\left(\ln\left(\frac{R_{j-1}-R_j}{R_{j-2,j}-R_j}\right)\right) =$$

$$= \sum_{k=i+1}^{j-1}\left[\mathrm{DC}(\ln(R_{k-1}-R_{k-2,k})) - \mathrm{DC}(\ln(R_{k-2,k}-R_k))\right] +$$

$$+\mathrm{DC}(\ln(R_{j-1}-R_j)) - \mathrm{DC}(\ln(R_{j-2,j}-R_j))$$

$$= \sum_{k=i+1}^{j-1}\frac{\mathrm{DC}(\ R_{k-1}-R_{k-2,k})}{R_{k-1}-R_{k-2,k}} - \sum_{k=i+1}^{j-1}\frac{\mathrm{DC}(\ R_{k-2,k}-R_k)}{R_{k-2,k}-R_k} +$$

$$+\frac{\mathrm{DC}(\ R_{j-1}-R_j)}{R_{j-1}-R_j} - \frac{\mathrm{DC}(\ R_{j-2,j}-R_j)}{R_{j-2,j}-R_j} =$$

$$= \sum_{k=i+1}^{j-1}\frac{(\ \mathrm{DC}(R_{k-1}) - \mathrm{DC}(R_{k-2,k}))}{R_{k-1}-R_{k-2,k}} - \sum_{k=i+1}^{j-1}\frac{(\ \mathrm{DC}(R_{k-2,k}) - \mathrm{DC}(R_k))}{R_{k-2,k}-R_k}$$

$$+\frac{\mathrm{DC}(R_{j-1}) - \mathrm{DC}(R_j)}{R_{j-1}-R_j} - \frac{\mathrm{DC}(R_{j-2,j}) - \mathrm{DC}(R_j)}{R_{j-2,j}-R_j}$$

It follows that

$$dZ_m^{j-2,j} - dZ_m^i = -\sum_{k=i+1}^{j-1}\frac{(\rho_{k-1,m}\sigma_{k-1}R_{k-1} - \theta_{m,k}\nu_k R_{k-2,k})}{R_{k-1}-R_{k-2,k}}dt + \sum_{k=i+1}^{j-1}\frac{(\theta_{m,k}\nu_k R_{k-2,k} - \rho_{k,m}\sigma_k R_k)}{R_{k-2,k}-R_k}dt$$

$$-\frac{\rho_{j-1,m}\sigma_{j-1}R_{j-1} - \rho_{j,m}\sigma_j R_j}{R_{j-1}-R_j}dt + \frac{\theta_{m,j}\nu_j R_{j-2,j} - \rho_{j,m}\sigma_j R_j}{R_{j-2,j}-R_j}\ dt$$

and

$$dV_m^{j-2,j} - dV_m^i = -\sum_{k=i+1}^{j-1}\frac{(\theta_{k-1,m}\sigma_{k-1}R_{k-1} - \eta_{m,k}\nu_k R_{k-2,k})}{R_{k-1}-R_{k-2,k}}dt + \sum_{k=i+1}^{j-1}\frac{(\eta_{m,k}\nu_k R_{k-2,k} - \theta_{k,m}\sigma_k R_k)}{R_{k-2,k}-R_k}dt +$$

$$-\frac{\theta_{j-1,m}\sigma_{j-1}R_{j-1} - \theta_{j,m}\sigma_j R_j}{R_{j-1}-R_j}dt + \frac{\eta_{j,m}\nu_j R_{j-2,j} - \theta_{j,m}\sigma_j R_j}{R_{j-2,j}-R_j}dt =: \bar{\phi}_m^{i,j}\ dt$$

Therefore, by subtracting from the first equation, taking $h > i$:

$$dZ_m^h - dZ_m^i = dZ_m^{j-2,j} - dZ_m^i - (dZ_m^{j-2,j} - dZ_m^h) =$$

$$= - \sum_{k=i+1}^{h} \frac{(\rho_{k-1,m}\sigma_{k-1}R_{k-1} - \theta_{m,k}\nu_k R_{k-2,k})}{R_{k-1} - R_{k-2,k}} dt + \sum_{k=i+1}^{h} \frac{(\theta_{m,k}\nu_k R_{k-2,k} - \rho_{k,m}\sigma_k R_k)}{R_{k-2,k} - R_k} dt =: \bar{\mu}_m^{i,h} dt$$

so that we finally obtain (taking $h = j$)

$$dR_j(t) = \sigma_j R_j(t)(\bar{\mu}_j^{i,j} dt + dZ_j^i(t))$$
$$dR_{j-2,j}(t) = \nu_j R_{j-2,j}(t))(\bar{\phi}_j^{i,j} dt + dV_j^i(t)),$$

or, by setting

$$\mu_j^i := \bar{\mu}_j^{i,j} \ \sigma_j, \quad \phi_j^i := \bar{\phi}_j^{i,j} \ \nu_j,$$

we have

$$dR_j(t) = R_j(t)(\mu_j^i dt + \sigma_j dZ_j^i(t)), \quad dR_{j-2,j}(t) = R_{j-2,j}(t)(\phi_j^i dt + \nu_j dV_j^i(t)),$$

and since μ and ϕ are completely determined by one- and two- period rates vectors $R = [R_{i-1,i}]_i$ and $R^{(2)} = [R_{i-2,i}]_i$, **the system is closed.** We can write a vector SDE which is a vector diffusion for all the one- and two- period rates under any of the $\widehat{C}_{i-1,i}$ measures:

$$d\begin{bmatrix} R \\ R^{(2)} \end{bmatrix} = \mathrm{diag}(\mu(R, R^{(2)}), \phi(R, R^{(2)})) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} dt + \mathrm{diag}(\sigma, \nu) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} d \begin{bmatrix} Z^i \\ V^i \end{bmatrix}$$

At this point a Monte Carlo simulation of the process, based on a discretization scheme for the above vector SDE is possible. One only needs to know the initial CDS rates $R(0), R^{(2)}(0)$, which if not directly available one can build by suitably stripping spot CDS rates. Given the volatilities and correlations, one can easily simulate the scheme by means of standard Gaussian shocks.

If C is the Cholesky decomposition of the correlation Q ($Q = CC'$ with "C" lower triangular matrix) and W is a standard Brownian motion under $\widehat{C}_{i-1,i}$, we can write

$$d\begin{bmatrix} R \\ R^{(2)} \end{bmatrix} = \mathrm{diag}(\mu(R, R^{(2)}), \phi(R, R^{(2)})) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} dt + \mathrm{diag}(\sigma, \nu) \begin{bmatrix} R \\ R^{(2)} \end{bmatrix} C \ dW \quad (31)$$

The log process can be easily simulated with a Milstein scheme.

7.2 Co-terminal and one-period CDS rates market model

Our second choice is based on co-terminal CDS rates. Indeed, let us take a family of CDS rates $R_{a,b}, R_{a+1,b}, R_{b-1,b}$. Keep in mind that we are always referring to PR rates. Can we write the dynamics of all such rates under (say) $\widehat{\mathbb{Q}}^{a,b}$? The answer is affirmative if we take into account the following equality, which is not difficult to prove with some basic algebra:

$$\widehat{C}_{i,b}(t) = \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_b) \prod_{k=i+1}^{b-1} \frac{R_{k,b}(t) - R_k(t)}{R_{k-1,b}(t) - R_k(t)}, \quad i = a, \ldots, b-2. \qquad (32)$$

Notice that the term in front of the product is just $\widehat{C}_{b-1,b}(t)$. Notice also that the one-period rates canonical numeraires $\bar{P}$ can be obtained from the above numeraires via $\widehat{C}_{i,b} - \widehat{C}_{i-1,b}$'s. Take into account that we need to assume $R_{k-1,b}(t) \neq R_k(t)$. Analogously to what seen previously for the one- and two- period rates case, we can assume this to hold at time 0 and then the probability that this condition is violated at future times will be zero in general under a diffusion dynamics for the relevant rates.

As before, the set of rates and of different-numeraires ratios is not a closed system. To close the system we need to include one-period rates $R_{a+1}, \ldots, R_b$. In this framework we may derive the joint dynamics of $R_{a,b}, R_{a+1,b}, R_{a+2,b}, \ldots, R_{b-1,b}; R_{a+1}, R_{a+2}, R_{b-1}$ under a common measure (say $\widehat{\mathbb{Q}}^{a,b}$) as follows. First assume a lognormal driftless geometric Brownian motion dynamics for $R_{a,b}$ under $\widehat{\mathbb{Q}}^{a,b}$ and suitable martingale dynamics for every other rate under its canonical measure. These different dynamics have to be chosen so as to enforce the needed constraints on the $\widehat{C}_{k,b}(t)$, such as for example $\widehat{C}_{k-1,b}(t) > \widehat{C}_{k,b}(t)$ and similar inequalities implying the correct behavior of the embedded $\bar{P}$'s. Take then a generic rate in the family and write its dynamics under $\widehat{\mathbb{Q}}^{a,b}$ with the following method. Thanks to the change of numeraire technique, the drift of this rate dynamics under $\widehat{\mathbb{Q}}^{a,b}$ will be a function of the quadratic covariation between the rate being modeled and the ratio of $\widehat{C}_{a,b}(t)$ with the canonical numeraire of the selected rate itself. Thanks to (32), this ratio is a function of the rates in the family and therefore the relevant quadratic covariation can be expressed simply as a suitable function of the volatilities and correlation ("diffusion coefficients" and "instantaneous Brownian covariations" are more precise terms) of the one- and multi-period rates in our family. As before no inconsistency is introduced, thanks to the additional degrees of freedom stemming from stochastic intensity. Under this second "co-terminal" formulation we can obtain the Black-like market formula above for the $T_a \div T_b$ tenor in the context of a consistent and "closed" market model. The definition of suitable martingale dynamics for CDS rates with different tenor is under investigation.

8 Hints at Smile Modeling

Finally, we consider the possibility of including a volatility smile in our CDS options model. Since the derivation is general, we may replace the dynamics (27) or (29) by a different "local volatility" dynamics

$$dR^{PR}_{a,b}(t) = \nu_{a,b}(t, R^{PR}_{a,b}(t))R^{PR}_{a,b}(t)d\check{W}^{a,b}(t)$$

with ν a suitable deterministic function of time and state. We might choose the CEV dynamics, a displaced diffusion dynamics, an hyperbolic sine densities mixture dynamics or a lognormal mixture dynamics. Several tractable choices are possible already in the local volatility diffusion setup, and one may select a smile dynamics for the LIBOR or swap model and use it to model R. There are several possible choices. For example, one may select $\nu_{a,b}$ from Brigo and Mercurio (2003) or Brigo Mercurio and Sartorelli (2003). Also, the uncertain volatility dynamics from Brigo, Mercurio and Rapisarda (2004) can be adapted to this context.

9 CDS Option pricing with the SSRD stochastic intensity model

In this final section we move to explicit modelling of the stochastic intensity process λ driving the Cox process whose first jump-time represents the default time τ. We consider the shifted square root diffusion (SSRD) model introduced in Brigo and Alfonsi (2003).

9.1 The SSRD intensity and interest rates model

We now describe our assumptions on the short-rate process r and on the intensity λ dynamics. For more details on shifted r diffusion dynamics see also Brigo and Mercurio (2001, 2001b).

CIR++ interest-rate model (Brigo and Mercurio (2001))

We write the short-rate r_t as the sum of a deterministic function φ and of a Markovian process x^α_t:

$$r_t = x^\alpha_t + \varphi(t;\alpha) \ , \quad t \geq 0, \tag{33}$$

where φ depends on the parameter vector α (which includes x^α_0) and is integrable on closed intervals. Notice that x^α_0 is indeed one more parameter at our disposal: we are free to select its value as long as $\varphi(0;\alpha) = r_0 - x_0$. We take as reference model for x the Cox-Ingersoll-Ross (CIR) process:

$$dx^\alpha_t = k(\theta - x^\alpha_t)dt + \sigma\sqrt{x^\alpha_t}dW_t,$$

where W is a $\mathcal{F}_t$ Brownian motion, and the parameter vector is $\alpha = (k, \theta, \sigma, x^\alpha_0)$, with k, θ, σ, x^α_0 positive deterministic constants. The condition $2k\theta > \sigma^2$ ensures that the origin is inaccessible to the reference model, so that the process x^α is well defined and remains positive. As is well known, this process x^α features a noncentral *chi-square* distribution, and yields an affine term-structure of interest rates. Denote by f instantaneous forward rates, i.e. $f(t,T) = -\partial \ln P(t,T)/\partial T$. The initial market zero-coupon interest-rate curve $T \mapsto P^M(0,T)$ is automatically calibrated by our model if we set $\varphi(t;\alpha) = \varphi^{\text{CIR}}(t;\alpha)$ where $\varphi^{\text{CIR}}(t;\alpha) = f^M(0,t) - f^{\text{CIR}}(0,t;\alpha)$,

$$f^{\text{CIR}}(0,t;\alpha) = \frac{2k\theta(e^{th}-1)}{2h+(k+h)(e^{th}-1)} + x_0 \frac{4h^2 e^{th}}{[2h+(k+h)(e^{th}-1)]^2}$$

with $h = \sqrt{k^2 + 2\sigma^2}$. For restrictions on the α's that keep r positive see Brigo and Mercurio (2001, 2001b). Moreover, the price at time t of a zero-coupon bond maturing at time T is

$$P(t,T) = \frac{P^M(0,T)A(0,t;\alpha)\exp\{-B(0,t;\alpha)x_0\}}{P^M(0,t)A(0,T;\alpha)\exp\{-B(0,T;\alpha)x_0\}} P^{\text{CIR}}(t,T,r_t - \varphi^{\text{CIR}}(t;\alpha);\alpha) \tag{34}$$

where $P^{\text{CIR}}(t,T,x_t;\alpha) = \mathbb{E}(e^{-\int_t^T x^\alpha(u)du}|\mathcal{F}_t) = A(t,T;\alpha)\exp\{-B(t,T;\alpha)x_t\}$ is the bond price formula for the basic CIR model with the classical expressions for A and B given for example in (3.25) of Brigo and Mercurio (2001b). From P's the spot LIBOR rate $L(t,T)$ at t for maturity T, the forward LIBOR rates $F(t,T,S)$ at t for maturity T and expiry S, and all other rates can be computed as explicit functions of r_t.

The cap price formula for the CIR++ model can be derived in closed form from the corresponding formula for the basic CIR model. This formula is a function of the parameters α. One may calibrate the parameters α to cap prices, by inverting the analytical CIR++ formula, so that the interest rate model is calibrated to the initial zero coupon curve through φ and to the cap market through α, as in Brigo and Mercurio (2001, 2001b).

CIR++ intensity model (Brigo and Alfonsi (2003))

For the intensity model we adopt a similar approach, in that we set

$$\lambda_t = y_t^\beta + \psi(t) \ , \quad t \geq 0, \tag{35}$$

where ψ is a positive deterministic function that is integrable on closed intervals. As before, the parameter vector is $\beta = (\kappa, \mu, \nu, y_0^\beta)$, with κ, μ, ν, y_0^β positive deterministic constants such that $2\kappa\mu > \nu^2$, and we take y again of the form:

$$dy_t^\beta = \kappa(\mu - y_t^\beta)dt + \nu\sqrt{y_t^\beta}dZ_t,$$

where the process Z is a $\mathcal{F}_t$-Brownian motion. This ensures that λ be strictly positive, as should be for an intensity process. Notice incidentally that this basically forces the choice of a tractable y to the CIR model among all one factor short-rate diffusion models. Dependence of ψ on β and possibly on other parameters will be specified later when dealing with CDS calibration. We will often use the integrated process, that is $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y^\beta(t) = \int_0^t y_s^\beta ds$ and $\Psi(t) = \int_0^t \psi(s)ds$. We assume the short rate r and the intensity λ processes to be correlated, by assuming the driving Brownian motions W and Z to be instantaneously correlated according to $dW_t \, dZ_t = \rho \, dt$.

9.2 Joint SSRD model calibration to CDS: Separability

The SSRD model is characterized by the terms $\mathcal{P} = (\alpha, \varphi, \beta, \psi, \rho)$ and can be seen as an extension of the CIR++ model for interest rates. As we explained before, φ is chosen to fit exactly the default-free zero-coupon bonds and α is then selected to have the better approximation of the cap prices. This procedure for the CIR++ interest rate part of the model r still remains valid in presence of a correlated λ since the products used for that calibration do not depend on the dynamics of λ and of (β, ρ, ψ).

Once α and φ are fixed, we would like to fit the three remaining terms to the credit derivatives market. To do so, we need first to calculate the price of CDS's in the SSRD model. We find easily, through iterated expectations and the definition of τ, that (see Brigo and Alfonsi (2003) for the details)

$$\text{CDS}(0, T_a, T_b, R, \text{L}_{\text{GD}}; \mathcal{P}) = R \int_{T_a}^{T_b} \mathbb{E}[\exp\left(-\int_0^u (r_s + \lambda_s) ds\right) \lambda_u](u - T_{\beta(u)-1}) du \quad (36)$$

$$+ \sum_{i=a+1}^{b} \alpha_i R \mathbb{E}[\exp\left(-\int_0^{T_i} (r_s + \lambda_s) ds\right)] - \text{L}_{\text{GD}} \int_{T_a}^{T_b} \mathbb{E}[\exp\left(-\int_0^u (r_s + \lambda_s) ds\right) \lambda_u] du.$$

We plan to use ψ to calibrate exactly the market CDS quotes (given for $T_a = 0$ and T_b spanning a set of increasing final maturities). More precisely, we want to find, for each (β, ρ), a function $\psi_\alpha(\cdot; \beta, \rho)$ that makes the CDS present values null, $\text{CDS}(0, 0, T_b, R_{0,b}^{MID}(0), \text{L}_{\text{GD}}, \mathcal{P}) = 0$. This could be done if we were able to calculate analytically the above expectations in general, taking as in the deterministic case a specific shape for ψ_α. Since these expectations are known only when $\rho = 0$, we first restrict ourselves to calibrate the subclass of models with $\rho = 0$. Interest rates and default intensities are independent with $\rho = 0$. By switching expectation and differentiation with respect to u and Fubini's theorem it is easy to see that the price of the CDS satisfies the deterministic case formula (9) when replacing terms such as $\exp(\Gamma(t) - \Gamma(u))$ by $\mathbb{E}(\exp(\Lambda(t) - \Lambda(u)))$ (with $u \geq t$). Therefore, at time $t = 0$, for any β we can calibrate automatically our model to the CDS by choosing ψ such that

$$e^{-\Gamma^{\text{mkt}}(u)} = \mathbb{E}(e^{-\Lambda(u)}) = e^{-\Psi(u)} \mathbb{E}(e^{-Y^\beta(u)}) = e^{-\Psi(u)} P^{\text{CIR}}(0, u, y_0; \beta).$$

The remarkable point is that ψ does not depend on α (the zero-coupon bonds have been calibrated exactly earlier), so that this calibration to CDS can be done independently of the interest rate calibration. This "separability" is of practical interest. We thus denote by $\psi(.; \beta)$ the obtained ψ function, given by

$$\psi(u; \beta) = \gamma^{\text{mkt}}(u) + \frac{d}{du} \ln(\mathbb{E}(e^{-Y^\beta(u)})) = \gamma^{\text{mkt}}(u) + \frac{d}{du} \ln(P^{\text{CIR}}(0, u, y_0; \beta)). \quad (37)$$

The shape of ψ is partly implicitly specified by our choice for γ^{mkt} (piecewise linear or otherwise).

So far we have described an analytical and exact calibration of the SSRD model in case $\rho = 0$. However, numerical tests in Brigo and Alfonsi (2003) show that ρ has practically a negligible impact on CDS prices computed under the SSRD model. Therefore, we may assume $\rho = 0$, even if this is not true, and calibrate the model with the above procedure. The error induced by this approximation will be negligible.

Now that CDS's are automatically calibrated, we would like to calibrate the parameters β to some *options* on the credit derivatives market in the same way as α is used to fit cap prices. To do this, we need a way to compute CDS options prices in the SSRD model. We will see a formula where this is possible under deterministic r (and stochastic CIR++ CDS-calibrated λ), and provide some hints on possible solutions in presence of stochastic r and nonzero ρ as well.

9.3 CDS options pricing with the calibrated CIR++ λ model

We developed this formula by an initial hint of Ouyang (2003). Consider the option to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, receiving protection L_{GD} against default up to time T_b, in exchange for a fixed rate K. We have that the payoff at T_a reads, as we have seen earlier, as

$$\begin{aligned}
\Pi_a &:= \Pi_{\text{CallCDS}a,b}(T_a) = [\text{CDS}(T_a,T_a,T_b,R_{a,b}(T_a),\text{L}_{\text{GD}}) - \text{CDS}(T_a,T_a,T_b,K,\text{L}_{\text{GD}})]^+ \\
&= [-\text{CDS}(T_a,T_a,T_b,K,\text{L}_{\text{GD}})]^+ = 1_{\{\tau>T_a\}}\Bigg(\mathbb{E}\Bigg\{ - D(T_a,\tau)(\tau - T_{\beta(\tau)-1})K\mathbf{1}_{\{\tau<T_b\}} \\
&\qquad - \sum_{i=a+1}^{b} D(T_a,T_i)\alpha_i K\mathbf{1}_{\{\tau>T_i\}} + \mathbf{1}_{\{\tau<T_b\}}D(T_a,\tau)\,\text{L}_{\text{GD}}|\mathcal{G}_{T_a}\Bigg\}\Bigg)^+ \\
&= 1_{\{\tau>T_a\}}\Bigg\{ - K\int_{T_a}^{T_b}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}(r_s+\lambda_s)ds\right)\lambda_u|\mathcal{F}_{T_a}\right](u - T_{\beta(u)-1})du \\
&\qquad -K\sum_{i=a+1}^{b}\alpha_i\mathbb{E}\left[\exp\left(-\int_{T_a}^{T_i}(r_s+\lambda_s)ds\right)|\mathcal{F}_{T_a}\right] \\
&\qquad +\text{L}_{\text{GD}}\int_{T_a}^{T_b}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}(r_s+\lambda_s)ds\right)\lambda_u|\mathcal{F}_{T_a}\right]du\Bigg\}^+
\end{aligned}$$

If we take deterministic interest rates r this reads

$$\begin{aligned}
\Pi_a = 1_{\{\tau>T_a\}}\Bigg\{ &- K\int_{T_a}^{T_b}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s ds\right)\lambda_u|\mathcal{F}_{T_a}\right]P(T_a,u)(u - T_{\beta(u)-1})du \\
&-K\sum_{i=a+1}^{b}\alpha_i P(T_a,T_i)\mathbb{E}\left[\exp\left(-\int_{T_a}^{T_i}\lambda_s ds\right)|\mathcal{F}_{T_a}\right] \\
&+\text{L}_{\text{GD}}\int_{T_a}^{T_b}P(T_a,u)\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s ds\right)\lambda_u|\mathcal{F}_{T_a}\right]du\Bigg\}^+
\end{aligned}$$

Define

$$H(t,T;y_t^\beta) := \mathbb{E}\left[\exp\left(-\int_t^T\lambda_s ds\right)|\mathcal{F}_t\right]$$

and notice that

$$\mathbb{E}\left[\exp\left(-\int_t^T\lambda_s ds\right)\lambda_T|\mathcal{F}_t\right] = -\frac{d}{dT}\mathbb{E}\left[\exp\left(-\int_t^T\lambda_s ds\right)|\mathcal{F}_t\right] = -\frac{d}{dT}H(t,T)$$

Write then

$$\Pi_a = 1_{\{\tau > T_a\}} \Big\{ K \int_{T_a}^{T_b} P(T_a,u)(u - T_{\beta(u)-1}) \frac{d}{du} H(T_a,u) du$$
$$-K \sum_{i=a+1}^{b} \alpha_i P(T_a,T_i) H(T_a,T_i) - \text{L}_{\text{GD}} \int_{T_a}^{T_b} P(T_a,u) \frac{d}{du} H(T_a,u) du \Big\}^+$$

Note that the first two summations add up to a positive quantity, since they are expectations of positive terms. By integrating by parts in the first and third integral, we obtain, by defining $q(u) := -dP(T_a,u)/du$,

$$\Pi_a = 1_{\{\tau > T_a\}} \Big\{ \text{L}_{\text{GD}} - \int_{T_a}^{T_b} \big[\text{L}_{\text{GD}} q(u) + K P(T_a, T_{\beta(u)}) \delta_{T_{\beta(u)}}(u) - K(u - T_{\beta(u)-1}) q(u)$$
$$-K P(T_a, T_{\beta(u)}) \delta_{T_{\beta(u)}}(u) + \text{L}_{\text{GD}} \delta_{T_b}(u) P(T_a,u) + K P(T_a,u) \big] H(T_a,u) du \Big\}^+$$

where δ_x denotes the Dirac delta function centered at x. Define

$$h(u) := \text{L}_{\text{GD}} q(u) - K(u - T_{\beta(u)-1}) q(u) + \text{L}_{\text{GD}} \delta_{T_b}(u) P(T_a,u) + K P(T_a,u)$$

so that

$$\Pi_a = 1_{\{\tau > T_a\}} \Big\{ \text{L}_{\text{GD}} - \int_{T_a}^{T_b} h(u) H(T_a,u; y^\beta_{T_a}) du \Big\}^+ \tag{38}$$

It is easy to check, by remembering the signs of the terms of which the above coefficients are expectations, that

$$h(u) > 0 \quad \text{for all} \quad u.$$

Now we look for a term y^* such that

$$\int_{T_a}^{T_b} h(u) H(T_a,u;y^*) du = \text{L}_{\text{GD}}. \tag{39}$$

It is easy to see that in general $H(t,T;y)$ is decreasing in y for all t,T. This equation can be solved, since $h(u)$ is known and deterministic and since H is given in terms of the CIR bond price formula. Furthermore, either a solution exists or the option valuation is not necessary. Indeed, consider first the limit of the left hand side for $y^* \to \infty$. We have

$$\lim_{y^* \to \infty} \int_{T_a}^{T_b} h(u) H(T_a,u;y^*) du = 0 < \text{L}_{\text{GD}},$$

which shows that for y^* large enough we always go below the value L_{GD}. Then consider the limit of the left hand side for $y^* \to 0$:

$$\lim_{y^* \to 0+} \int_{T_a}^{T_b} h(u) H(T_a,u;y^*) du =$$

$$= \text{L}_{\text{GD}} + \int_{T_a}^{T_b} [\text{L}_{\text{GD}} P(T_a,u) \frac{\partial H(T_a,u;0)}{\partial u} + (K(u - T_{\beta(u)-1})q(u) + KP(T_a,u))H(T_a,u;0)]du$$

Now if the integral in the last expression is positive then we have that the limit is larger than L_{GD} and by continuity and monotonicity there is always a solution y^* giving L_{GD}. If instead the integral in the last expression is negative, then the limit is smaller than L_{GD} and we have that (39) admits no solution, in that its left hand side is always smaller than the right hand side. However, this implies in turn that the expression inside curly brackets in the payoff (38) is always positive and thus the contract loses its optionality and can be valued by taking the expectation without positive part, giving as option price simply $-\text{CDS}(0,T_a,T_b,K,\text{L}_{\text{GD}}) > 0$, the opposite of a forward start CDS. In case y^* exists, instead, we may rewrite our discounted payoff as

$$\Pi_a = 1_{\{\tau>T_a\}}\left\{ \int_{T_a}^{T_b} h(u)(H(T_a,u;y^*) - H(T_a,u;y^\beta_{T_a}))du \right\}^+$$

Since $H(t,T;y)$ is decreasing in y for all t,T, all terms $(H(T_a,u;y^*) - H(T_a,u;y^\beta_{T_a}))$ have the same sign, which will be positive if $y^\beta_{T_a} > y^*$ or negative otherwise. Since all such terms have the same sign, we may write

$$\Pi_a =: 1_{\{\tau>T_a\}}Q_a = 1_{\{\tau>T_a\}}\left\{ \int_{T_a}^{T_b} h(u)(H(T_a,u;y^*) - H(T_a,u;y^\beta_{T_a}))^+du \right\}$$

Now compute the price as

$$\mathbb{E}[D(0,T_a)\Pi_a] = P(0,T_a)\mathbb{E}[1_{\{\tau>T_a\}}Q_a] = P(0,T_a)\mathbb{E}[\exp(-\int_0^{T_a} \lambda_s ds)Q_a] =$$

$$= \int_{T_a}^{T_b} h(u)\mathbb{E}[\exp(-\int_0^{T_a} \lambda_s ds)(H(T_a,u;y^*) - H(T_a,u;y^\beta_{T_a}))^+]du$$

From a structural point of view, $H(T_a,u;y^\beta_{T_a})$ are like zero coupon bond prices in a CIR++ model with short term interest rate λ, for maturity T_a on bonds maturing at u. Thus, each term in the summation is $h(u)$ times a zero-coupon bond like call option with strike $K^*_u = H(T_a,u;y^*)$. A formula for such options is given for example in (3.78) p. 94 of Brigo and Mercurio (2001b).

If one maintains stochastic interest rates with possibly non-null ρ, then a possibility is to use the Gaussian mapped processes x^V and y^V introduced in Brigo and Alfonsi (2003) and to reason as for pricing swaptions with the G2++ model through Jamshidian's decomposition and one-dimensional Gaussian numerical integration, along the lines of the procedures leading to (4.31) in Brigo and Mercurio (2001b). Clearly the resulting formula has to be tested against Monte Carlo simulation.

Finally, in the general SSRD model, one may compute the CDS option price by means of Monte Carlo simulations, equate this Monte Carlo price to Formula (28) applied to the same CDS option at $t = 0$, and solve in $\sigma_{a,b}$. This $\sigma_{a,b}$ is then the implied volatility corresponding to the SSRD pricing model. The first numerical results we found in a number of cases point out the following patterns of $\sigma_{a,b}$ in terms of SSRD model parameters:

Param :	$\kappa\uparrow$	$\mu\uparrow$	$\nu\uparrow$	$y_0\uparrow$	$\rho\uparrow$
$\sigma_{a,b}^{imp}$:	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$

For more details see Brigo and Cousot (2004). The patterns are reasonable. When κ increases (all other things being equal) the time-homogeneous core of the stochastic intensity has a higher speed of mean reversion and then randomness reduces more quickly in time, so that the implied volatility reduces. When μ increases, the asymptotic mean of the homogeneous part of the intensity increases, so that we have higher intensity and thus, since instantaneous volatility is proportional to the increased $\sqrt{y}$, more randomness. When ν increases, clearly randomness of λ increases so that it is natural for the implied volatility to increase. We also find that increasing y_0 (the initial point of the time-homogeneous part of the intensity) increases the implied volatility, while increasing the correlation ρ decreases the implied volatility.

10 Conclusions and Further Research

We considered several CDS payoff formulations. For some of them, we established equivalence with approximated defaultable floaters. We explained the CDS market quoting mechanisms and considered CDS pricing in an intensity framework. We derived a market model for CDS options, and thus for callable defaultable floaters, given the above equivalence. We hinted at possible CDS options smile models and at a comparison of the market models with classic stochastic intensity models, such as for example the CDS-calibrated CIR++ model in Brigo and Alfonsi (2003), a comparison that is under more detailed investigation in Brigo and Cousot (2004). We also gave a CDS option formula under CIR++ stochastic intensity based on Jamshidian's decomposition. Moreover, further investigation on the possibility to link different CDS forward rate models, based on a fundamental set of candidate liquid CDS rates, is to be investigated, starting from the observed parallels with the LIBOR vs SWAP market models in the default free interest rate derivatives setting.

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