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Credit Derivatives Pricing

with a Smile-Extended Jump Stochastic Intensity Model

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Abstract

We present a two-factor stochastic default intensity and interest rate model for pricing single-name default swaptions. The specific positive square root processes considered fall in the relatively tractable class of affine jump diffusions while allowing for inclusion of stochastic volatility and jumps in default swap spreads. The parameters of the short rate dynamics are first calibrated to the interest rates markets, before calibrating separately the default intensity model to credit derivatives market data. A few variants of the model are calibrated in turn to market data, and different calibration procedures are compared. Numerical experiments show that the calibrated model can generate plausible volatility smiles. Hence, the model can be calibrated to a default swap term structure and few default swaptions, and the calibrated parameters can be used to value consistently other default swaptions (different strikes and maturities, or more complex structures) on the same credit reference name.

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1 Introduction

Default swaptions are options on default swaps, hence they are often treated by drawing analogies with interest rate swaptions, especially as far as their Black-like market pricing formula is concerned. Indeed, the most widely discussed model for their valuation, the Log-normal Default Swap Market (LDSM) model is similar to the Log-normal Swap Market (LSM) model from interest rates derivatives markets, consistent with Black's formula for swaptions. Schönbucher (2000) introduced the notion of survival pricing measures by conditioning on no-default up to a given maturity, although in contrast with the interest rate models this measure is not equivalent to the risk neutral one. A first formulation involves modeling discrete forward credit spreads by taking as numéraire a defaultable zero coupon bond, leading to a credit risky equivalent of the Log-normal Forward (Libor) Market (LFM) model. The second formulation allowing the derivation of the standard market formula for default swaptions is obtained by modeling the default swap spread directly as a geometric brownian motion, as summarized in Schönbucher (2004). Here the convenient pricing measure is termed the survival swap measure. Again, this measure is not equivalent to the risk neutral one and the associated defaultable numéraire asset is the defaultable annuity, and may vanish. Jamshidian (2004) partly addresses this problem, presenting a more formalized setup of the LDSM model, and generalizes the theory to semi-martingales driven money market account and conditional survival probabilities. Hull and White (2003) present practical aspects for the implementation of the Black market formula and present empirical estimates of default swap spread volatilities for actively traded A-rated names that ranged from %67 to %130 in the period from end 1999 to mid-2002. Brigo (2005) introduces various different candidate formulations by using alternative definitions of defaultable forward rates and develops a market model leading to the standard Black formula under equivalent pricing measures, showing implied volatilities about %50 from market prices of default swaptions.

The variants of the Black (1976) formulae obtained for the LDSM model, by their inherent simplicity are particularly convenient for quoting single default swaptions by selecting an appropriate volatility parameter matching the market price. However, quoting default swaptions for different sets of maturities or more complex instruments consistently with just the implied volatilities given by the formula inversion becomes problematic. One is confronted with the need to develop a fully specified dynamical model to impose a structure on the joint dynamics of one-period rates or credit default swap spreads as it is done in the interest rates derivatives markets with the LFM, or the LSM. In particular, the valuation of more exotic instruments like Bermudan default swaptions requires the use of a model that accurately incorporates the term structure of default swap rates, as well as the dynamical deformations and movements of this term structure. Indeed, default swap rates are subject to large jumps and possibly stochastic volatility effects. In the interest rates derivatives models, these features are incorporated more or less successfully by specifying richer joint dynamics of the forward LIBOR rates, leading to an explosion of the number of parameters in the models. However, contrary to the interest rate markets with their huge number of caps/floors and swaptions, the single-name default swap markets are most famous by the very small number of traded instruments, rendering the calibration or estimation of any model with a large number of parameters unfeasible.

An alternative approach, more suitable for the current state of the default swaptions markets that is explored in this paper calls for modeling the default intensity instead. This is the approach followed in Brigo and Alfonsi (2005) who adopt a stochastic default intensity model, where both the risk-free interest rates and the default intensity are driven by shifted square root diffusion processes. Brigo and Cousot (2006) examined implied volatilities generated by this two-factor shifted square root model and characterized the qualitative behavior of the implied volatilities with respect to the stochastic intensity model parameters. The numerical experiments conducted with stylized parameter values suggested that this model might be unable to generate large enough implied volatilities. Modeling the intensity process automatically imposes a strong structure on the default swap spread joint dynamics across different maturities, simplifying the achievement of consistency across instruments. Judicious choices of the intensity process can also incorporate jumps and some stochastic volatility effects in default swap spreads, and possibly generate plausible defaultable term structure evolutions. In this paper, we extend the SSRD model of Brigo and Alfonsi (2005) and Brigo and Cousot (2006) by allowing for positive jumps in the process driving the default intensity, consistently with empirical evidence. We also suggest a calibration procedure to match the term structure of default swaps, and implement it on market data. We are then able to generate volatility smiles implied by the fitted parameters of the model. Using different versions of the model, with and without jumps, we are able to identify the jumps' impact on the smile. The model can then be used to price more exotic products. A first attempt at Bermudan default options pricing for example is in Ben Ameur et al (2006), where the basic SSRD model is used. The jump-extended SSRJD introduced here could be used as an improved version.

2 The basics of intensity modeling

We denote the market filtration by $\mathbb{G} = (\mathcal{G}_t)_{(t \geq 0)}$ and let $\mathbb{Q}$ be a risk-neutral probability measure. We follow the intensity based approach to default risk modeling and introduce the default time as a totally inaccessible $\mathbb{G}-$stopping time τ. We further assume the usual structure for $\mathbb{G}$, namely that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{(t \geq 0)}$ is the usual brownian filtration and $\mathbb{H} = (\mathcal{H}_t)_{(t \geq 0)}$ is the filtration generated by the default process: $\mathcal{H}_t = \sigma(\{\tau < u\}, u \leq t)$. It is also assumed that there exists a strictly positive $\mathbb{F}-$adapted process λ such that the process M given by

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^t \mathbf{1}_{\{\tau > s\}} \lambda_s ds = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s ds \qquad (1)$$

is a uniformly integrable $\mathbb{G}-$martingale under $\mathbb{Q}$. The process λ is referred to as the $\mathbb{G}-$marginal intensity of the stopping time τ under $\mathbb{Q}$ or risk-neutral default intensity.

Let $\widehat{P}(t,T)$ denote the price at time t of a hypothetical zero recovery defaultable unit face value zero-coupon maturing at time T. As in the case of default-free bonds, the value $\overline{P}(t,T)$ of the defaultable bond before default is given by an expectation of the discounted payoff, with the only difference that the discounting is adjusted for default-risk using the intensity process as proved by Duffie et al (1996):

$$\widehat{P}(t,T) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\overline{D}(t,T)\middle|\,\mathcal{F}_t\right] =: \mathbf{1}_{\{\tau>t\}}\ \overline{P}(t,T) \tag{2}$$

where $\overline{D}(t,T)$ denotes the default risk-adjusted discount factor:

$$\overline{D}(t,T) = \exp\left(-\int_t^T (r_u+\lambda_u)du\right) \tag{3}$$

Finally, let $\widehat{\mathbb{Q}}(t,T)$ denote the risk-neutral survival probability to time T as seen from time t. This risk neutral survival probability satisfies the following equation:

$$\widehat{\mathbb{Q}}(t,T) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_u du\right)\middle|\,\mathcal{F}_t\right] =: \mathbf{1}_{\{\tau>t\}}\ \overline{\mathbb{Q}}(t,T) \tag{4}$$

3 Credit Default swaps

In this section, we briefly review default swaps pricing and refer to Brigo and Alfonsi (2005) for further details. A (credit) default swap is a financial instrument used by two counterparties to buy or sell protection against the default risk of a reference credit name. In a default swap signed at time t starting at time T_a with maturity T_b, the protection buyer pays a periodic fee or spread $R^{a,b}(t)$ at the payment dates $T_{a+1},\ldots,T_b$ (typically quarterly) as long as the reference entity does not default. In case of a default occurring at time τ with $T_a<\tau\leq T_b$, the protection seller compensates the protection buyer for his loss given default that we assume to be a known constant L. In addition, the protection seller receives from the protection buyer the spread accrued since the last payment date before default. In the case where $t<T_a$, the contract is a forward default swap, while if $t=T_a$ we are dealing with a spot default swap.

Accordingly, the value $CDS(t,\Upsilon,R,L)$ of a default swap with a payment schedule $\Upsilon = \{T_{a+1},\ldots,T_b\}$, a spread R and a loss given default L at time t is given by the following expression:

$$CDS(t,\Upsilon,R,L) = \mathbf{1}_{\{\tau>t\}}\left[R\overline{C}_{a,b}(t) - L\int_{T_a}^{T_b}\mathbb{E}_{\mathbb{Q}}[\overline{D}(t,u)\lambda_u|\mathcal{F}_t]du\right] \tag{5}$$

where

$$\overline{C}_{a,b}(t) = \left[\sum_{i=a+1}^{b} \alpha_i \overline{P}(t, T_i) + \int_{T_a}^{T_b} (u - T_{(\beta(u)-1)}) \mathbb{E}_{\mathbb{Q}}[\overline{D}(t, u)\lambda_u | \mathcal{F}_t] du \right] \tag{6}$$

and $T_{\beta(t)}$ is the first date in the set $\{T_a, \ldots, T_b\}$ that follows t and $\alpha_i = T_i - T_{i-1}$ is the year fraction between T_{i-1} and T_i.

Hence, the fair spread $R_{a,b}(t)$ as long as default has not occurred can be computed as the value of R that equates the default swap value to zero:

$$\mathbf{1}_{\{\tau > t\}} R_{a,b}(t) = \mathbf{1}_{\{\tau > t\}} \frac{L \int_{T_a}^{T_b} \mathbb{E}_{\mathbb{Q}}[\overline{D}(t, u)\lambda_u | \mathcal{F}_t] du}{\overline{C}_{a,b}(t)} \tag{7}$$

Many stochastic models with correlated intensity and short rate do not result in analytical formulas for the inner expectations that are to be integrated in (7). However, default swaps have been shown in Brigo and Alfonsi (2006) to be relatively insensitive to the correlation between brownians driving the intensity and interest rate processes, hence we will assume from now on independence between interest rates and intensities when computing default swaps values.

4 Credit Default swaptions

A default swaption is an option written on a default swap. It references a specific underlying default swap and specifies a maturity, and a strike. The most common default swaptions are European style options. A payer default swaption entitles its holder the right but not the obligation to become a protection buyer in the underlying default swap at the expiration of the option, paying a protection fee equal to the strike spread. A receiver entitles to its holder the right to become a protection seller. Normally, a single name default swaption is canceled (or knocked out) by a default of the underlying reference name if this occurs before the option's maturity.

In the sequel, T will denote the default swaption's expiry date. The starting date of the underlying default swap T_a will typically be equal to T, its expiry date will be denoted by T_b, with fee payments schedule $T_{a+1}, \ldots, T_b$. That is, the default swaption holder enters a spot default swap if she chooses to exercise the option at maturity. For the pricing of a default swaption at a valuation date t, the underlying reference is thus the T maturity forward default swap with payment dates $T_{a+1}, \ldots, T_b$. The strike K specified in the contract is the periodic fixed rate that is to be paid in exchange for the default protection of the default swap in case of exercise, instead of the fair market spread $R_{a,b}(T)$ that will be available at time only at time T. A $T-$defaultable payoff can be valued at time t by taking the expectation of its properly discounted value, where the discounting is done using the default adjusted stochastic discount factor $\overline{D}(t,T)$ as shown in Duffie et al (1996). Hence, the payer default swaption can be valued as in Brigo and Alfonsi (2005) and Brigo and Cousot (2006):

$$PSO(t,T,\Upsilon,K) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\overline{D}(t,T)\overline{C}_{a,b}(T)\left(R_{a,b}(T)-K\right)^{+}|\mathcal{F}_t\right] \tag{8}$$

A single dynamics for $R_{a,b}$ leading to a market formula analogous to the one for interest rate swaptions is derived, under different assumptions, in Schönbucher (2004) and Jamshidian (2004). Brigo (2005) derives the same formula under different assumptions and sketches the construction of a whole market model for a joint family of default swap rates. Assuming that the default swap rate $R_{a,b}$ follows a geometric brownian motion with volatility $\sigma_{a,b}$, the above approaches allow to price the default swaptions using Black-style formulas. Here, we recall the formula for a payer default swaption, with self-evident notation:

$$\begin{aligned} PSO(t,T,\Upsilon,K) &= \mathbf{1}_{\{\tau>t\}}\overline{C}_{a,b}(t)\left[R_{a,b}(t)\Phi\left(d_1\right)-K\Phi\left(d_2\right)\right] \qquad (9)\\ d_{1,2} &= \left(\log(R_{a,b}(t)/K)\pm(T-t)\sigma_{a,b}^2/2\right)/\left(\sigma_{a,b}\sqrt{T-t}\right)\end{aligned}$$

When faced with the requirement of marking a default swaption position to market, the need for a different model than the market model becomes apparent. Indeed the market model requires one to input a volatility parameter. If the model could be trusted as providing an appropriate description of the world, this parameter (constant across maturities and strikes) could be implied from currently traded options. Recognizing that the model is a rather primitive approximation, one would expect to observe different volatility parameters for different strikes and maturities resulting in a volatility smile (or skew or smirk). However, for a given underlying reference name, there are often only very few different default swaptions traded, and quite often the market is limited to the At-The-Money (ATM) options. Deducing patterns in a market model context can then be difficult. On the other hand, a stochastic default intensity model can be calibrated to a default swap rates term structure and very few default swaptions, and the fitted values of the parameters can be used to value different default swaptions consistently, under the condition that the model implies meaningful patterns of implied volatilities.

5 Stochastic modeling of the intensity process

Default swap rates time series can hardly be reconciled with a geometric brownian motion. Furthermore, forward default swaps underlying default swaptions are not traded as such, and hence delta-hedging can only be done approximately with spot default swap term structures. For risk management and control purposes, it is important to recognize the relation between different default swaps and swaptions referencing the same credit name. Jumps and stochastic volatility could potentially be introduced in the market model by postulating more appropriate dynamics $dR_{a,b}$ for the default swap rate. However, this would quickly destroy the main feature of the model: its simplicity. Also, in order to value more exotic options, it becomes important to incorporate the whole term structure of the default swap rates as well as postulating dynamics that can yield appropriate deformations of this term structure in the future. A relatively simple candidate for these tasks is the stochastic default intensity approach we adopt in this paper. It models the default intensity process instead of the default swap rate, providing a plausible approach to consistently modeling the default swap rates of different maturities. Following this approach, Brigo and Alfonsi (2005) proposed a two-factor shifted square root diffusion model, where both the short rate and the default intensity are assumed to follow possibly correlated shifted square root diffusions. The processes are modeled as a sum of a deterministic function and a square root diffusion. Comparing numerical examples in Brigo and Cousot (2006) and Brigo (2005) we see that it is difficult to produce large enough implied volatilities compared to what is implied from default swaptions market data or historical volatilities of default swap spreads. Hereafter we present an extension to this model, by allowing for positive jumps in the process driving the default intensity.

5.1 A Shifted jump-diffusion square root process for the intensity: JCIR++

We write the intensity λ as the sum of a positive deterministic function ψ and of a positive stochastic process y^β:

$$\lambda_t = y_t^\beta + \psi(t;\beta),\ t \geq 0, \tag{10}$$

where ψ is a deterministic function, depending on the parameter vector β (which includes y_0^β), that is integrable on closed intervals. We denote by Ψ the integrated function of ψ, $\Psi(t,\beta) = \int_0^t \psi(s,\beta)ds$. For pricing default swaps or calibrating the model parameters to default swap spreads, we assume independence between risk-free interest rates and the process driving the default intensity. Hence this approach calibrates first the risk-free part to interest rates instruments and then conditional on the short rate parameter values calibrates the default intensity parameters. This conditionally independent calibration has been shown in Brigo and Alfonsi (2005) to be robust to correlation assumptions. In numerical experiments, they found that default swap valuation sensitivity to correlation between brownians driving short rate and default intensity is negligible.

We extend the model from Brigo and Alfonsi (2005) by allowing for jumps in the square root process modeling y_t^β, leading to a model we call JCIR++. The resulting process is a special case of the class of Affine Jump Diffusions (AJD) (see Duffie et al (2000), Duffie et al (2003)). The dynamics of y^β would then satisfy

$$dy_t^\beta = \kappa(\mu - y_t^\beta)dt + \nu\sqrt{y_t^\beta}dZ_t + dJ_t, \tag{11}$$

with the following condition

$$2\kappa\mu > \nu^2 \tag{12}$$

and J is a pure jump process with jumps arrival rate $\alpha > 0$ and exponentially distributed jump sizes with parameter $\gamma > 0$ preserving the attractive feature of positive default intensity. In other terms

$$J_t = \sum_{i=1}^{M_t} Y_i$$

where M is a Poisson process with intensity α, the Ys are exponentially distributed with parameter γ. Since this model belongs to the tractable AJD class of models, the survival probability has the typical "log-affine" shape

$$\begin{aligned}
\widehat{\mathbb{Q}}(t,T) &= \mathbf{1}_{\{\tau>t\}}\overline{\mathbb{Q}}(t,T) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_s ds\right)|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\Psi(t,\beta) - \Psi(T,\beta) - \int_t^T y_s^\beta ds\right)|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}A(t,T)\exp\left(\Psi(t,\beta) - \Psi(T,\beta) - B(t,T)y_t^\beta\right) \qquad (13)
\end{aligned}$$

where the functional forms of the terms A and B with respect to the parameters $\kappa, \mu, \nu, \alpha, \gamma$ are given in the appendix of Duffie and Garleanu (2001) and are obtained by solving the usual Riccati equations. These expressions for A and B can be recast in a form that is similar to the classical terms α and β in the bond price formula for the Cox, Ingersoll and Ross (1985) model (CIR):

$$\begin{aligned}
A(t,T) &= \alpha(t,T)\zeta(t,T) \qquad (14)\\
B(t,T) &= \beta(t,T) \qquad (15)
\end{aligned}$$

where

$$\zeta(t,T) = \left(\frac{2h\exp\left(\frac{h+\kappa+2\gamma}{2}(T-t)\right)}{2h + (\kappa+h+2\gamma)(\exp^{h(T-t)} - 1)}\right)^{\frac{2\alpha\gamma}{\nu^2-2\kappa\gamma-2\gamma^2}} \qquad (16)$$

and where $\alpha(t,T)$, $\beta(t,T)$ are the terms from the CIR model:

$$\begin{aligned}
\alpha(t,T) &= \left(\frac{2h\exp\left(\frac{h+\kappa}{2}(T-t)\right)}{2h + (\kappa+h)(\exp^{h(T-t)} - 1)}\right)^{\frac{2\kappa\mu}{\nu^2}} \qquad (17)\\
\beta(t,T) &= \frac{2(\exp^{h(T-t)} - 1)}{2h + (\kappa+h)(\exp^{h(T-t)} - 1)} \qquad (18)
\end{aligned}$$

with $h = \sqrt{\kappa^2 + 2\nu^2}$.

Given the expression for $\zeta(t,T)$, we introduce another restriction on the parameters to avoid the case where the denominator of the exponent $\nu^2 - 2\kappa\gamma - 2\gamma^2$ goes to zero:

$$\gamma \neq \frac{h-\kappa}{2}$$

Theoretically, this constraint is not necessary, since when there is equality, $\zeta(t,T)$ goes to 1. However, in numerical computations that would lead to divisions by zero with potential numerical instabilities.

Using the fact that default swaps are only marginally sensitive to the correlation between the brownians driving short rate and default intensity, we assume this correlation to be null when computing default swap rates or values. For default swap computations it is enough to be able to write the expression for $\mathbb{E}_{\mathbb{Q}}[\overline{D}(t,T)\lambda_T|\mathcal{F}_t]$:

$$\mathbb{E}_{\mathbb{Q}}[\overline{D}(t,T)\lambda_T|\mathcal{F}_t] = -P(t,T)\frac{d}{dT}\overline{\mathbb{Q}}(t,T) \tag{19}$$

where $P(t,T)$ is the value at time t of a default-free zero-coupon bond for maturity T:

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[D(t,T)|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right]$$

5.2 A Shifted square root diffusion for the short rate: CIR++

For the process driving the risk-free interest rates, we follow Brigo and Alfonsi (2005) and Brigo and Cousot (2006) and assume the short rate to follow a shifted square root diffusion, without jumps. The short rate r_t is written as a CIR++ process, i.e. as the sum of a deterministic function φ and of a square root diffusion process x_t^π:

$$r_t = x_t^\pi + \varphi(t;\pi),\ \ t \geq 0 \tag{20}$$

where φ depends on the parameter vector π including x_0^π and is integrable on closed intervals. We also recall the dynamics of the square root diffusion process x^π:

$$dx_t^\pi = k(\theta - x_t^\pi)dt + \sigma\sqrt{x_t^\pi}dW_t \tag{21}$$

where all parameters in the vector $\pi = (k,\theta,\sigma,x_0^\pi)$ are positive constants, with the usual condition

$$2k\theta > \sigma^2 \tag{22}$$

ensuring the positivity of the process x^π. This process is equivalent to the process assumed for y^β above when the jump parameters α and γ are set to zero. We also denote $\Phi(t,\pi) := \int_0^t \varphi(s,\pi)ds$. This shifted diffusion process can be calibrated to the term structure of interest rates and caps as in Brigo and Alfonsi (2005). The brownian motions driving the short rate and the default intensity can be correlated

$$d < W, Z >_t = \rho dt, \tag{23}$$

and we recall from the numerical results in Brigo and Alfonsi (2005) that the calibration can be greatly simplified by first assuming ρ to be zero, allowing a separate calibration to the interest rates instruments and the credit instruments, since default swaps' sensitivity to this correlation is shown to be negligible.

6 Model calibration

Given a set of parameters π for x^{π} and the deterministic shifts φ's obtained by calibrating a risk-free term structure and interest rates derivatives, we discuss a two-steps calibration procedure for the intensity model parameters. We calibrate the parameters β by minimizing errors in pricing default swaps with different maturities, while ensuring that the values of the deterministic shifts required to exactly fit the default swap values are positive. In case one is not interested in the deterministic shifts and prefers the homogenous version of the model, we propose a slightly different calibration procedure. It consists in calibrating the parameters β of the stochastic process so as to recover exactly default swaps that would be used for hedging the option, while minimizing errors in pricing the rest of the default swaps term structure. This approach has the benefit that it ensures we are able to price exactly our hedging instruments without being too sensitive to the others, hence more robust to stale prices (see Lvov (2006)). In addition, the errors to be minimized are the sums of relative or percentage errors as opposed to absolute errors, to avoid the possibility that the optimization be mainly driven by the long maturities default swaps, as the spreads of these will naturally be larger than those of the short maturities default swaps. Below, we describe more formally both calibration procedures.

Given a term structure of traded default swap rates, i.e. a vector of market values of default swap rates:

$$\mathbf{R}^{\mathbf{Mkt}}_{\mathbf{a},\mathbf{b}} = \begin{pmatrix} R^{Mkt}_{a^1,b^1} \\ \vdots \\ R^{Mkt}_{a^m,b^m} \end{pmatrix},$$

we denote by $\mathfrak{H}$ the set containing the indices referring to the default swaps that are considered as hedging instruments. We are interested in fitting the term structure of the shifts, that we assume to be piecewise linear with the following knots:

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_m \end{pmatrix}$$

The constraint that the parameters driving the stochastic part of the process $\beta = (\kappa, \mu, \nu, \alpha, \gamma, y_0^{\beta})$ should be enough to recover exactly the hedging instruments translate into the following condition on the shifts:

$$\psi_j = 0, \forall j \in \mathfrak{H} \tag{24}$$

Denote the model (as opposed to market values) default swap term structure as:

$$\mathbf{R}_{\mathbf{a},\mathbf{b}}(\psi, \beta) = \begin{pmatrix} R_{a^1,b^1}(\psi, \beta) \\ \vdots \\ R_{a^m,b^m}(\psi, \beta) \end{pmatrix}$$

We can now define the vector of relative pricing errors:

$$\varepsilon_i := \frac{\mathbf{R}^{\mathbf{Mkt}}_{\mathbf{a^i},\mathbf{b^i}} - \mathbf{R}_{\mathbf{a^i},\mathbf{b^i}}(\psi,\beta)}{\mathbf{R}^{\mathbf{Mkt}}_{\mathbf{a^i},\mathbf{b^i}}} \tag{25}$$

For the complete model with deterministic shifts the calibration amounts to solving the following problem:

$$\begin{aligned} \widehat{\psi} \text{ solves } \mathbf{R}^{\mathbf{Mkt}}_{\mathbf{a},\mathbf{b}} - \mathbf{R}_{\mathbf{a},\mathbf{b}}(\psi,\widehat{\beta}) = 0 \quad \text{in} \quad \psi \\ \widehat{\beta} = \arg\min_{\beta} \varepsilon'(\psi)\ \varepsilon(\psi) \\ \text{s.t. } \beta > 0,\ 2\kappa\mu > \nu^2,\ \gamma \neq \frac{h-\kappa}{2} \\ \text{and } \psi \geq 0 \end{aligned} \tag{26}$$

An example of a calibration along these lines for a diffusion-only restricted model for British Airways PLC (BA) on the date of 10 November 2004 is presented in figure (1).

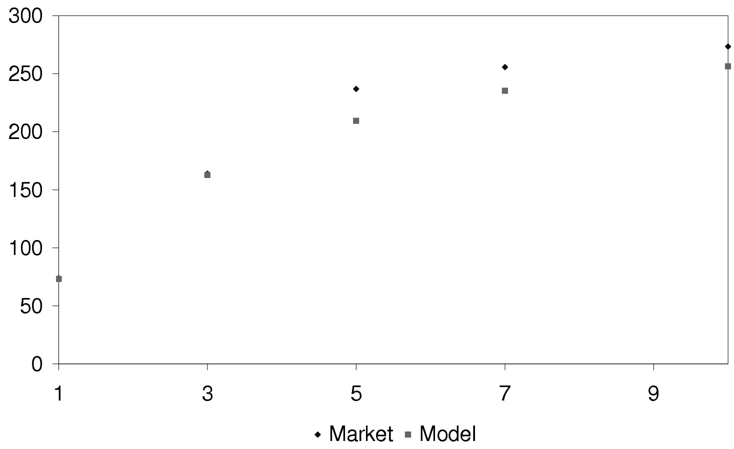

Figure 1: *Calibration of the shifted model without jumps*

We plot the market CDS term structure, along with a term structure resulting from the model where the shifts have been set to zero after the calibration, to give an idea of the effect of the shifts. Notice that the constraint that the shifts be positive forces the model generated spreads to be lower than the market spreads if the shifts are set to zero and the other parameters are kept constant. We also observe that the level of shifting required to fit exactly the data is not too dramatic.

In case we restrict the model to the homogenous version ($\psi = 0$) while fitting exactly the hedging instruments, the calibration then reduces to the following non-linear minimization problem under constraints:

$$\begin{aligned} \widehat{\beta} &= \arg\min_{\beta} \varepsilon' \varepsilon \\ \text{s.t. } R_{a^j,b^j}(0,\beta) &= R^{Mkt}_{a^j,b^j} \forall j \in \mathfrak{H}, \\ \beta > 0,\ 2\kappa\mu > \nu^2,\ \gamma &\neq \frac{h-\kappa}{2} \end{aligned} \tag{27}$$

An example of a calibration along these lines for a diffusion-only restricted model for BA is presented in figure (2).

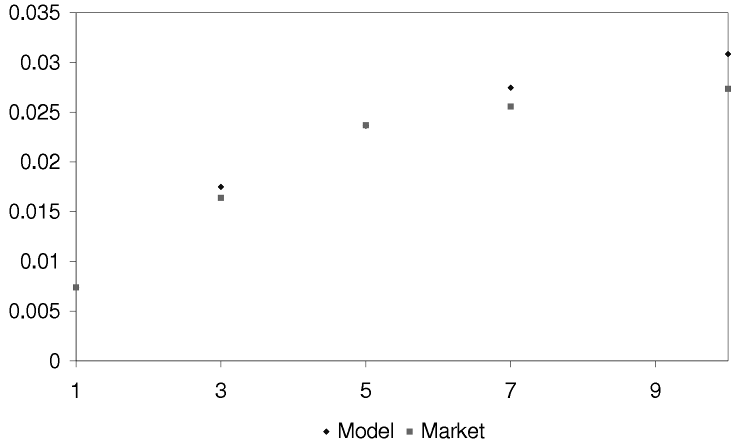

FIGURE 2: *Calibration of the homogenous model without jumps*

We plot the market CDS term structure, along with a term structure resulting from the model for which the 5−year CDS has been assumed to be the hedging instrument. This constraint coupled with the relaxation of the constraint of positive shifts results in the model spreads going higher than the market spreads for all other maturities except for the 1−year. The parameter values for both calibrations are collected in table (1).

7 Monte Carlo pricing of default swaptions

As in Brigo and Cousot (2006), default swaptions can be easily valued by simulations using the formulation:

$$PSO(t,T,K) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}\left[\overline{D}(t,T)[-CDS(T,K,L)]^{+} | \mathcal{F}_t\right] \tag{28}$$

Hence we only need to simulate the processes x^{π} and y^{β} until the swaption maturity T, and then compute the underlying default swap value at maturity for each path given

Calibration with Shifts	y_0	κ	μ	ν	Tot Abs Err without shift
Piecewise constant shifts	1.68E-06	0.4745	0.0514	0.2208	0.0059
	Maturities in years				
	1	3	5	7	10
	9.51E-05	0.0002	0.0106	0.0000	0.00096
Homogenous calibration	y_0	κ	μ	ν	Tot Abs Err
	0.0001	0.3870	0.0613	0.2172	0.0028

TABLE 1: *Numerical values of calibrated parameters*

the parameters and the value of r_T and λ_T realized on the path considered. Simulating the jump diffusion square root process is no more difficult than simulating the diffusion only correspondent. Indeed, since the brownian Z and the compound Poisson process J are independent, Mikulevicius and Platen (1988) propose to generate jump times and jump amplitudes, then proceed with the diffusion simulation schemes adding the jumps at the times when they occur. To apply this method here, one only needs to be able to generate Poisson jump times and exponentially distributed jump sizes, in addition to a simulation procedure for the diffusion square root process. More specifically, for each path one can generate a Poisson random variable with parameter $\alpha(T-t)$ giving the number of jumps to be simulated in the period of length $[t,T]$. Then simulate all the jump times as independently distributed uniforms on $[t,T]$, and generate the jump magnitude at each jump time as an independently distributed exponential random variable. At this stage, one has all the jump times and the corresponding jump magnitudes for the path under simulation, and needs to proceed with the diffusion part. For that purpose, it is enough to augment the discretization $t_0 = 0 < t_1 < \cdots < t_n = T$ by all the simulated jump times, and proceed with a suitable square root diffusion scheme like the implicit positivity preserving discretization scheme in Brigo and Alfonsi (2005) or other alternatives as in for example Alfonsi (2005) or simulating from the exact distribution (scaled non-central chi-square) in the uncorrelated case, except that for a jump time t_j, we will need to add the jump amplitude:

$$\widetilde{y}^{\beta}_{t_j} = g(\widetilde{y}^{\beta}_{t_{j-1}}) + \widetilde{Y}_{t_j} \tag{29}$$

where g is the functional form used to simulate $\widetilde{y}^{\beta}_{t_j}$ given $\widetilde{y}^{\beta}_{t_{j-1}}$ for the scheme used, and $\widetilde{Y}_{t_j}$ is the size of the simulated jump at time t_j.

8 Consistency with volatility smile

In this section, we present some numerical results concerning the behavior of the model for some parameter values. Our main focus is on the implied volatility smile that can be generated by the model. The model potentially allows one to mark-to-market (or rather mark-to-model) non-ATM default swaptions that may be present on a trading book. This task cannot be fulfilled with the market model unless we use ATM implied volatility to value all options, which should not be acceptable from a risk management perspective. On the other hand, our intensity model can be calibrated to the default swap term structure and traded ATM default swaptions to price other default swaptions more consistently.

8.1 Implied volatility patterns

To visualize the different implied volatility patterns that can be generated by the model, we present the numerical results obtained with four different values of the vector of parameters. The parameters values are collected in table (2).

Reference	y_0	κ	μ	ν	α	γ
Model1	1.7E-06	0.4745	0.0514	0.2208	0.0	0.0
Model2	0.0007	0.4066	0.0515	0.1507	0.5009	0.0050
Model3	1.3E-06	0.4851	0.0457	0.2000	0.5009	0.0050
Model4	0.005	0.2281	0.0134	0.0782	1.5000	0.0067

Table 2: *Parameter values for 4 simulated models*

We plot in figure (3) the CDS term structures generated by the different models.

Simulated prices of payer default swaptions with various strikes for these models are in figure (4), while the corresponding implied volatility smiles can be found in figure (5).

Note that the model is able to generate both a decreasing and an increasing volatility smile. It seems that an increasing smile is due to the presence of a strong jump component as in model 4.

8.2 On the impact of a strong jump component

In this section, we conduct a simple experiment to further understand the effect of the jump component. We generated a CDS term structure with a jump-diffusion model with strong jump component, and then we set the jump parameters to zero and calibrate a

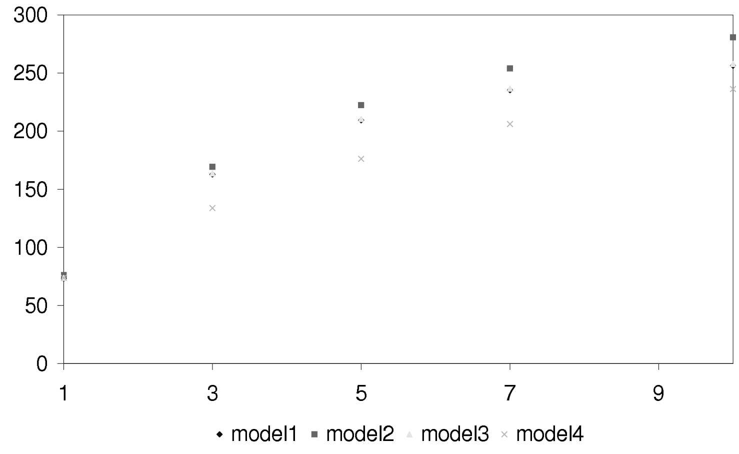

FIGURE 3: *CDS term structures for 4 models*

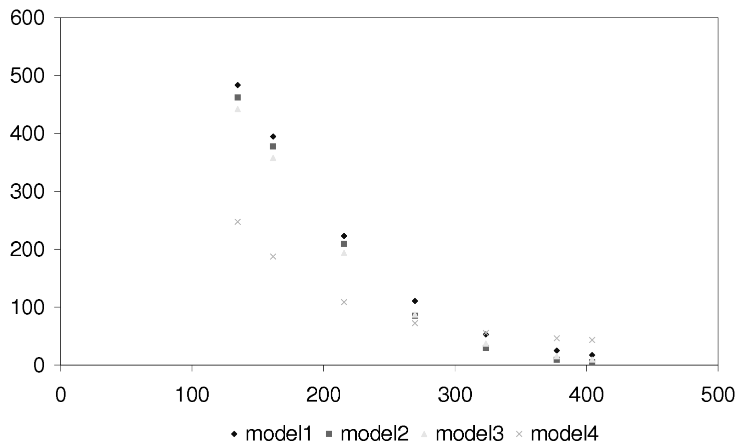

FIGURE 4: *Payer default swaption prices for 4 simulated models*

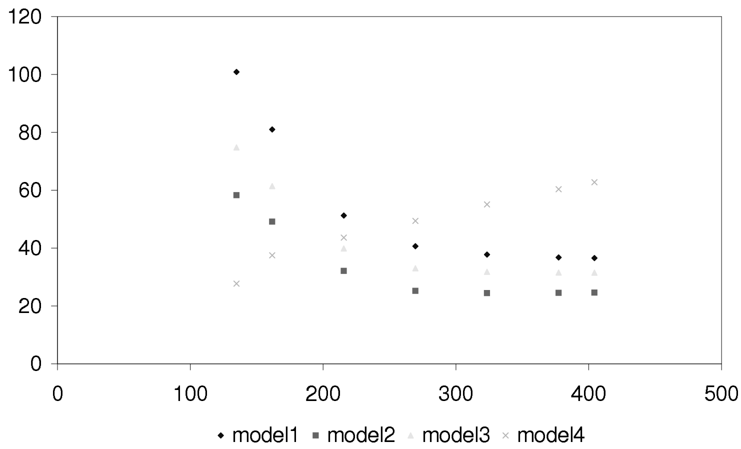

FIGURE 5: *Generated implied volatility smiles for 4 simulated models*

diffusion only model to the generated term structure. The parameters of both models are collected in table (3) and the term structures in figure (6).

Reference	y_0	κ	μ	ν	α	γ
With Jumps	0.0050	0.2281	0.0134	0.0782	1.5000	0.006705
Without Jumps	0.0050	0.1956	0.0650	0.1595	0.0000	0.0000

TABLE 3: *Parameter values for 2 cross-fitted models with and without jumps*

Note that we are able to obtain a perfect fit as the term structures match exactly.

Plots of the simulated default swaption prices with these models are in figure (7), while the implied volatility smiles are in figure (8).

From the values of the parameters, we note that the long term mean μ and the volatility ν parameters are quite significantly higher in the version without jumps to compensate for the absence of jumps. This implies a very strong volatility of the CDS spread, hence the higher prices for low strikes in the diffusion-only model. On the other hand, the model with jumps builds a significant portion of its total variance from the strong jump component, and since the jumps can only be positive, this increases the probability of reaching higher strike values, and hence higher default swaption values for higher strikes.

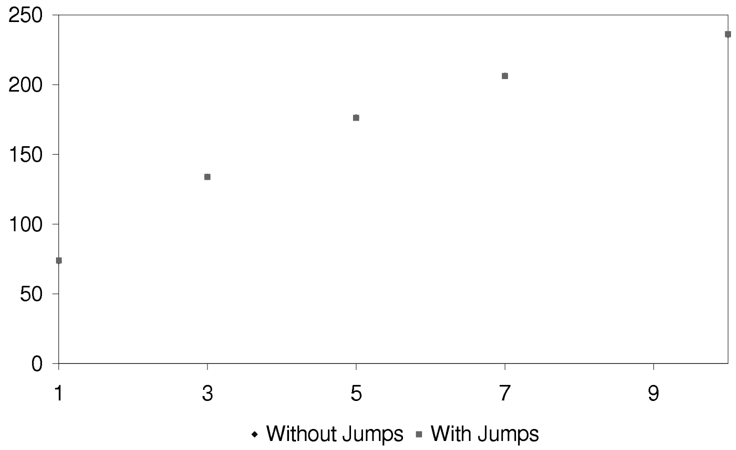

FIGURE 6: *CDS term structures given by 2 cross-fitted models with and without jumps*

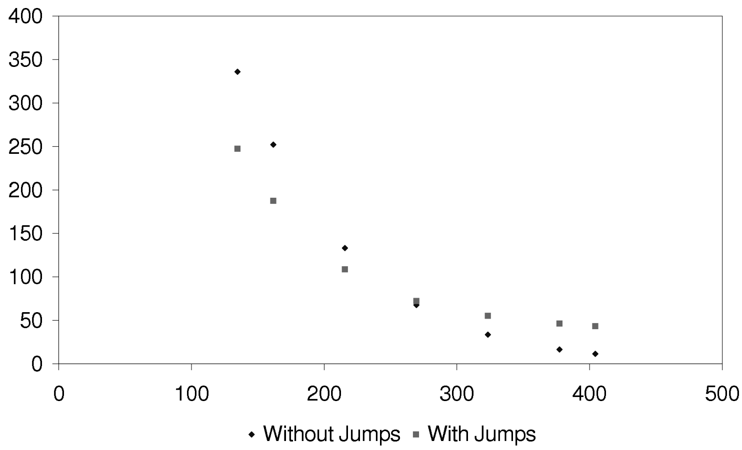

FIGURE 7: *Simulated payer default swaption prices for the 2 cross-fitted models*

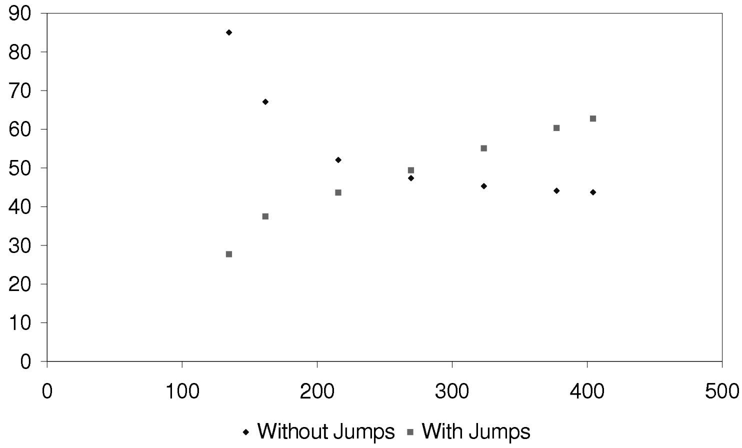

FIGURE 8: *Generated implied volatility smiles by the 2 cross-fitted models*

9 Conclusions and further research

The proposed two-factor stochastic default intensity and interest rate model for single-name credit payoffs allows for large CDS implied volatilities and realistic jump features in the intensity dynamics. Our numerical tests support the choice of this model as an improvement of the tractable analogous model without jumps in Brigo and Alfonsi (2005). We also examine volatility smile implications of the chosen dynamics. A Jamshidian decomposition extending the one in the extended version of Brigo and Alfonsi (2005) under deterministic default-free interest rates gives a closed form formula for CDS options that can be useful in calibration.

The model can be considered for valuation of credit derivatives payoffs and hybrid interest-rate/ credit payoffs. For example, results in Brigo and Pallavicini (2006) on counterparty risk with the CIR++ model could be extended to the more realistic jump diffusion JCIR++ version considered here. We also plan to investigate the application of a similar model to index options, that are much more liquid in comparison to single name options.

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