Preface

"Professor Brigo, will there be any new quotes in the second edition?"
"Yes... for example this one!"

A student at a London training course, following a similar question by a Hong Kong student to Massimo Morini, 2003.

"I would have written you a shorter letter, but I didn't have the time"
Benjamin Franklin

MOTIVATION.... five years later.

...I'm sure he's got a perfectly good reason... for taking so long...
Emily, "Corpse Bride", Tim Burton (2005).

Welcome onboard the second edition of this book on interest rate models, to all old and new readers. We immediately say this second edition is actually almost a new book, with four hundred fifty and more new pages on smile modeling, calibration, inflation, credit derivatives and counterparty risk.

As explained in the preface of the first edition, the idea of writing this book on interest-rate modeling crossed our minds in early summer 1999. We both thought of different versions before, but it was in Banca IMI that this challenging project began materially, if not spiritually (more details are given in the trivia Appendix G). At the time we were given the task of studying and developing financial models for the pricing and hedging of a broad range of derivatives, and we were involved in medium/long-term projects.

The first years in Banca IMI saw us writing a lot of reports and material on our activity in the bank, to the point that much of those studies ended up in the first edition of the book, printed in 2001.

In the first edition preface we described motivation, explained what kind of theory and practice we were going to address, illustrated the aim and readership of the book, together with its structure and other considerations. We do so again now, clearly updating what we wrote in 2001.

Why a book on interest rate models, and why this new edition?

"Sorry I took so long to respond, Plastic Man. I'd like to formally declare my return to active duty, my friends... This is J'onn J'onzz activating full telepathic link. Counter offensive has begun". JLA 38, DC Comics (2000).

In years where every month a new book on financial modeling or on mathematical finance comes out, one of the first questions inevitably is: why one more, and why one on interest-rate modeling in particular?

The answer springs directly from our job experience as mathematicians working as quantitative analysts in financial institutions. Indeed, one of the major challenges any financial engineer has to cope with is the practical implementation of mathematical models for pricing derivative securities.

When pricing market financial products, one has to address a number of theoretical and practical issues that are often neglected in the classical, general basic theory: the choice of a satisfactory model, the derivation of specific analytical formulas and approximations, the calibration of the selected model to a set of market data, the implementation of efficient routines for speeding up the whole calibration procedure, and so on. In other words, the general understanding of the theoretical paradigms in which specific models operate does not lead to their complete understanding and immediate implementation and use for concrete pricing. This is an area that is rarely covered by books on mathematical finance.

Undoubtedly, there exist excellent books covering the basic theoretical paradigms, but they do not provide enough instructions and insights for tackling concrete pricing problems. We therefore thought of writing this book in order to cover this gap between theory and practice.

The first version of the book achieved this task in several respects. However, the market is rapidly evolving. New areas such as smile modeling, inflation, hybrid products, counterparty risk and credit derivatives have become fundamental in recent years. New bridges are required to cross the gap between theory and practice in these recent areas.

The Gap between Theory and Practice

> *But Lo! Siddârtha turned/ Eyes gleaming with divine tears to the sky,/ Eyes lit with heavenly pity to the earth;/ From sky to earth he looked, from earth to sky,/ As if his spirit sought in lonely flight/ Some far-off vision, linking this and that,/ Lost - past - but searchable, but seen, but known.*
> From "The Light of Asia", Sir Edwin Arnold (1879).

A gap, indeed. And a fundamental one. The interplay between theory and practice has proved to be an extremely fruitful ingredient in the progress of science and modeling in particular. We believe that practice can help to appreciate theory, thus generating a feedback that is one of the most important and intriguing aspects of modeling and more generally of scientific investigation.

If theory becomes deaf to the feedback of practice or vice versa, great opportunities can be missed. It may be a pity to restrict one's interest only to extremely abstract problems that have little relevance for those scientists or quantitative analysts working in "real life".

Now, it is obvious that everyone working in the field owes a lot to the basic fundamental theory from which such extremely abstract problems stem.

It would be foolish to deny the importance of a well developed and consistent theory as a fundamental support for any practical work involving mathematical models. Indeed, practice that is deaf to theory or that employs a sloppy mathematical apparatus is quite dangerous.

However, besides the extremely abstract refinement of the basic paradigms, which are certainly worth studying but that interest mostly an academic audience, there are other fundamental and more specific aspects of the theory that are often neglected in books and in the literature, and that interest a larger audience.

Is This Book about Theory? What kind of Theory?

> *"Our paper became a monograph. When we had completed the details, we rewrote everything so that no one could tell how we came upon our ideas or why. This is the standard in mathematics."*
>
> David Berlinski, "Black Mischief" (1988).

In the book, we are not dealing with the fundamental no-arbitrage paradigms with great detail. We resume and adopt the basic well-established theory of Harrison and Pliska, and avoid the debate on the several possible definitions of no-arbitrage and on their mutual relationships. Indeed, we will raise problems that can be faced in the basic framework above. Insisting on the subtle aspects and developments of no-arbitrage theory more than is necessary would take space from the other theory we need to address in the book and that is more important for our purposes.

Besides, there already exist several books dealing with the most abstract theory of no-arbitrage. On the theory that we deal with, on the contrary, there exist only few books, although in recent years the trend has been improving. What is this theory? For a flavor of it, let us select a few questions at random:

- How can the market interest-rate curves be defined in mathematical terms?
- What kind of interest rates does one select when writing the dynamics? Instantaneous spot rates? Forward rates? Forward swap rates?
- What is a sufficiently general framework for expressing no-arbitrage in interest-rate modeling?
- Are there payoffs that do not require the interest-rate curve dynamics to be valued? If so, what are these payoffs?
- Is there a definition of volatility (and of its term structures) in terms of interest-rate dynamics that is consistent with market practice?
- What kinds of diffusion coefficients in the rate dynamics are compatible with different qualitative evolutions of the term structure of volatilities over time?
- How is "humped volatility shape" translated in mathematical terms and what kind of mathematical models allow for it?

- What is the most convenient probability measure under which one can price a specific product, and how can one derive concretely the related interest-rate dynamics?
- Are different market models of interest-rate dynamics compatible?
- What does it mean to calibrate a model to the market in terms of the chosen mathematical model? Is this always possible? Or is there a degree of approximation involved?
- Does terminal correlation among rates depend on instantaneous volatilities or only on instantaneous correlations? Can we analyze this dependence?
- What is the volatility smile, how can it be expressed in terms of mathematical models and of forward-rate dynamics in particular?
- Is there a diffusion dynamics consistent with the quoting mechanism of the swaptions volatility smile in the market?
- What is the link between dynamics of rates and their distributions?
- What kind of model is more apt to model correlated interest-rate curves of different currencies, and how does one compute the related dynamics under the relevant probability measures?
- When does a model imply the Markov property for the short rate and why is this important?
- What is inflation and what is its link with classical interest-rate modeling?
- How does one calibrate an inflation model?
- Is the time of default of a counterparty predictable or not?
- Is it possible to value payoffs under an equivalent pricing measure in presence of default?
- Why are Poisson and Cox processes so suited to default modeling?
- What are the mathematical analogies between interest-rate models and credit-derivatives models? For what kind of mathematical models do these analogies stand?
- Does counterparty risk render a payoff dynamics-dependent even if without counterparty risk the payoff valuation is model-independent?
- What kind of mathematical models may account for possible jump features in the stochastic processes needed in credit spread modeling?
- Is there a general way to model dependence across default times, and across market variables more generally, going beyond linear correlation? What are the limits of these generalizations, in case?
- ......

We could go on for a while with questions of this kind. Our point is, however, that the theory dealt with in a book on interest-rate models should consider this kind of question.

We sympathize with anyone who has gone to a bookstore (or perhaps to a library) looking for answers to some of the above questions with little success. We have done the same, several times, and we were able to find only limited material and few reference works, although in the last few years the situation has improved. We hope the second edition of this book will cement the steps forward taken with the first edition.

We also sympathize with the reader who has just finished his studies or with the academic who is trying a life-change to work in industry or who is considering some close cooperation with market participants. Being used to precise statements and rigorous theory, this person might find answers to the above questions expressed in contradictory or unclear mathematical language. This is something else we too have been through, and we are trying not to disappoint in this respect either.

Is This Book about Practice? What kind of Practice?

> *If we don't do the work, the words don't mean anything. Reading a book or listening to a talk isn't enough by itself.*
> Charlotte Joko Beck, "Nothing Special: Living Zen", Harper Collins, 1995.

We try to answer some questions on practice that are again overlooked in most of the existing books in mathematical finance, and on interest-rate models in particular. Again, here are some typical questions selected at random:

- What are accrual conventions and how do they impact on the definition of rates?
- Can you give a few examples of how time is measured in connection with some aspects of contracts? What are "day-count conventions"?
- What is the interpretation of most liquid market contracts such as caps and swaptions? What is their main purpose?
- What kind of data structures are observed in the market? Are all data equally significant?
- How is a specific model calibrated to market data in practice? Is a joint calibration to different market structures always possible or even desirable?
- What are the dangers of calibrating a model to data that are not equally important, or reliable, or updated with poor frequency?
- What are the requirements of a trader as far as a calibration results are concerned?
- How can one handle path-dependent or early-exercise products numerically? And products with both features simultaneously?
- What numerical methods can be used for implementing a model that is not analytically tractable? How are trees built for specific models? Can instantaneous correlation be a problem when building a tree in practice?
- What kind of products are suited to evaluation through Monte Carlo simulation? How can Monte Carlo simulation be applied in practice? Under which probability measure is it convenient to simulate? How can we reduce the variance of the simulation, especially in presence of default indicators?

- Is there a model flexible enough to be calibrated to the market smile for caps?
- How is the swaptions smile quoted? Is it possible to "arbitrage" the swaption smile against the cap smile?
- What typical qualitative shapes of the volatility term structure are observed in the market?
- What is the impact of the parameters of a chosen model on the market volatility structures that are relevant to the trader?
- What is the accuracy of analytical approximations derived for swaptions volatilities and terminal correlations?
- Is it possible to relate CMS convexity adjustments to swaption smiles?
- Does there exist an interest-rate model that can be considered "central" nowadays, in practice? What do traders think about it?
- How can we express mathematically the payoffs of some typical market products?
- How do you handle in practice products depending on more than one interest-rate curve at the same time?
- How do you calibrate an inflation model in practice, and to what quotes?
- What is the importance of stochastic volatility in inflation modelling?
- How can we handle hybrid structures? What are the key aspects to take into account?
- What are typical volatility sizes in the credit market? Are these sizes motivating different models?
- What's the impact of interest-rate credit-spread correlation on the valuation of credit derivatives?
- Is counterparty risk impacting interest-rate payoffs in a relevant way?
- Are models with jumps easy to calibrate to credit spread data?
- Is there a way to imply correlation across default times of different names from market quotes? What models are more apt at doing so?
- ......

Again, we could go on for a while, and it is hard to find a single book answering these questions with a rigorous theoretical background. Also, answering some of these questions (and others that are similar in spirit) motivates new theoretical developments, maintaining the fundamental feedback between theory and practice we hinted at above.

AIMS, READERSHIP AND BOOK STRUCTURE

> *"And these people are sitting up there seriously discussing intelligent stars and trips through time to years that sound like telephone numbers. Why am I here?"* Huntress/Helena Bertinelli, DC One Million (1999).

Contrary to what happens in other derivatives areas, interest-rate modeling is a branch of mathematical finance where no general model has been yet accepted as "standard" for the whole sector, although the LIBOR market model is emerging as a possible candidate for this role. Indeed, there exist market standard models for both main interest-rate derivatives "submarkets", namely the caps and swaptions markets. However, such models are theoretically incompatible and cannot be used jointly to price other interest-rate derivatives.

Because of this lack of a standard, the choice of a model for pricing and hedging interest-rate derivatives has to be dealt with carefully. In this book, therefore, we do not just concentrate on a specific model leaving all implementation issues aside. We instead develop several types of models and show how to use them in practice for pricing a number of specific products.

The main models are illustrated in different aspects ranging from theoretical formulation to a possible implementation on a computer, always keeping in mind the concrete questions one has to cope with. We also stress that different models are suited to different situations and products, pointing out that there does not exist a single model that is uniformly better than all the others.

Thus our aim in writing this book is two-fold. First, we would like to help quantitative analysts and advanced traders handle interest-rate derivatives with a sound theoretical apparatus. We try explicitly to explain which models can be used in practice for some major concrete problems. Secondly, we would also like to help academics develop a feeling for the practical problems in the market that can be solved with the use of relatively advanced tools of mathematics and stochastic calculus in particular. Advanced undergraduate students, graduate students and researchers should benefit as well, from seeing how some sophisticated mathematics can be used in concrete financial problems.

The Prerequisites

The prerequisites are some basic knowledge of stochastic calculus and the theory of stochastic differential equations and Poisson processes in particular. The main tools from stochastic calculus are Ito's formula, Girsanov's theorem, and a few basic facts on Poisson processes, which are, however, briefly reviewed in Appendix C.

The Book is Structured in Eight Parts

The first part of the book reviews some basic concepts and definitions and briefly explains the fundamental theory of no-arbitrage and its implications as far as pricing derivatives is concerned.

In the second part the first models appear. We review some of the basic short-rate models, both one- and two-dimensional, and then hint at forward-rate models, introducing the so called Heath-Jarrow-Morton framework.

In the third part we introduce the "modern" models, the so-called market models, describing their distributional properties, discussing their analytical tractability and proposing numerical procedures for approximating the interest-rate dynamics and for testing analytical approximations. We will make extensive use of the "change-of-numeraire" technique, which is explained in detail in a initial section. This third part contains a lot of new material with respect to the earlier 2001 edition. In particular, the correlation study and the cascade calibration of the LIBOR market model have been considerably enriched, including the work leading to the Master's and PhD theses of Massimo Morini.

The fourth part is largely new, and is entirely devoted to smile modeling, with a parade of models that are studied in detail and applied to the caps and swaptions markets.

The fifth part is devoted to concrete applications. We in fact list a series of market financial products that are usually traded over the counter and for which there exists no uniquely consolidated pricing model. We consider some typical interest-rate derivatives dividing them into two classes: i) derivatives depending on a single interest-rate curve; ii) derivatives depending on two interest-rate curves.

Part Six is new and we introduce and study inflation derivatives and related models to price them.

Part Seven is new as well and concerns credit derivatives and counterparty risk, and besides introducing the payoffs and the models we explain the analogies between credit models and interest-rate models.

Appendices

Part Eight regroups our appendices, where we have also moved the "other interest rate models" and the "equity payoffs under stochastic rates" sections, which were separate chapters in the first edition. We updated the appendix on stochastic calculus with Poisson processes and updated the "Talking to the Traders" appendix with conversations on the new parts of the book.

We also added an appendix with trivia and frequently asked questions such as "who's who of the two authors", "what does the cover represent", "what about all these quotes" etc.

It is sometimes said that no one ever reads appendices. This book ends with eight appendices, and the last one is an interview with a quantitative trader, which should be interesting enough to convince the reader to have a look at the appendices, for a change.

FINAL WORD AND ACKNOWLEDGMENTS

Whether our treatment of the theory fulfills the targets we have set ourselves, is for the reader to judge. A disclaimer is necessary though. Assembling a book in the middle of the "battlefield" that is any trading room, while quite stimulating, leaves little space for planned organization. Indeed, the book is not homogeneous, some topics are more developed than others.

We have tried to follow a logical path in assembling the final manuscript, but we are aware that the book is not optimal in respect of homogeneity and linearity of exposition. Hopefully, the explicit contribution of our work will emerge over these inevitable little misalignments.

Acknowledgments

A book is always the product not only of its authors, but also of their colleagues, of the environment where the authors work, of the encouragements and critique gathered from conferences, referee reports for journal publications, conversations after seminars, university lectures, training courses, summer and winter schools, e-mail correspondence, and many analogous events. While we cannot do justice to all the above, we thank explicitly our recently acquired colleagues Andrea "Fifty levels of backtrack and I'm not from Vulcan" Pallavicini, who joined us in the last year with both analytical and numerical impressive skills, and Roberto "market-and-modeling-super-speed" Torresetti, one of the founders of the financial engineering department, who came back after a tour through Chicago and London, enhancing our activity with market understanding and immediate and eclectic grasp of modeling issues.

Some of the most important contributions, physically included in this book, especially this second edition, come from the "next generation" of quants and PhD students. Here is a roll call:

- Aurélien Alfonsi (PhD in Paris and Banca IMI trainee, Credit Derivatives with Damiano);
- Cristina Capitani (Banca IMI trainee, LIBOR model calibration, with Damiano);
- Laurent Cousot (PhD student in NY and Banca IMI trainee, Credit Derivatives with Damiano);
- Naoufel El-Bachir (PhD student in Reading and Banca IMI trainee, Credit Derivatives with Damiano);
- Eymen Errais (PhD student at Stanford and Banca IMI trainee, Credit Derivatives with Damiano and Smile Modeling with Fabio);
- Jan Liinev (PhD in Ghent, LIBOR / Swap models distance with differential geometric methods, with Damiano);
- Dmitri Lvov (PhD in Reading and Banca IMI trainee, Bermudan Swaption Pricing and Hedging with the LFM, with Fabio);
- Massimo Masetti (PhD in Bergamo, currently working for a major bank in London, Counterparty Risk and Credit Derivatives with Damiano);
- Nicola Moreni (PhD in Paris, and Banca IMI trainee, currently our colleague, Inflation Modeling with Fabio);

- Giulio Sartorelli (PhD in Pisa, Banca IMI trainee and currently our colleague, Short-Rate and Smile Modeling with Fabio);
- Marco Tarenghi (Banca IMI trainee and our former colleague, Credit derivatives and counterparty risk with Damiano);

Special mention is due to Massimo Morini, Damiano's PhD student in Milan, who almost learned the first edition by heart. His copy is the most battered and travel-worn we have ever seen; Massimo is virtually a co-author of this second edition, having contributed largely to the new parts of Chapters 6 and 7, and having developed recent and promising results on smile calibration and credit derivatives market models that we have not been in time to include here. Massimo also taught lectures and training courses based on the book all around the world, helping us whenever we were too busy to travel abroad.

As before we are grateful to our colleagues Gianvittorio "Tree and Optimization Master" Mauri and Francesco "Monte Carlo" Rapisarda, for their help and continuous interaction concerning both modeling and concrete implementations on computers. Francesco also helped by proofreading the manuscript of the first edition and by suggesting modifications.

The feedback from the trading desks (interest-rate-derivatives and credit derivatives) has been fundamental, first in the figures of Antonio Castagna and Luca Mengoni (first edition) and then Andrea Curotti (now back in London), Stefano De Nuccio (now with our competitors), Luca Dominici, Roberto Paolelli, and Federico Veronesi. They have stimulated many developments with their objections, requirements and discussions. Their feeling for market behavior has guided us in cases where mere mathematics and textbook finance could not help us that much. Antonio has also helped us with stimulating discussions on inflation modeling and general pricing and hedging issues. As before, this book has been made possible also by the farsightedness of our head Aleardo Adotti, who allowed us to work on the frontiers of mathematical finance inside a bank.

Hundreds of e-mails in the last years have reached us, suggesting improvements, asking questions, and pointing out errors. Again, it would be impossible to thank all single readers who contacted us, so we say here a big collective "Thank-you" to all our past readers. All mistakes that are left are again, needless to say, ours.

It has been tough to remain mentally sane in these last years, especially when completing this almost-one-thousand pages book. So the list of "external" acknowledgments has lengthened since the last time.

Damiano is grateful to the next generation above, in particular to the ones he is working/has worked with most: his modeling colleagues Andrea and Roberto, and then Aurélien, Eymen, Jan and Massimo Masetti (who held the "fortress" all alone in a difficult moment); Marco and Massimo Morini are gratefully mentioned also for the Boston MIT -Miami-Key West-Cape Canaveral "tournee" of late 2004. Umberto Cherubini has been lots

of fun with the "Japanese experiences" of 1999-2004 and many professional suggestions. Gratitude goes also to Suzuki "Freccetta" SV650, to the Lake of Como (Lario) and the Dolomites (Dolomiti), to Venice, Damiano's birthplace, a dream still going after all these years, to the Venice carnival for the tons of fun with the "Difensori della Terra" costume players, including Fabrizio "Spidey", Roberto "Cap", Roberto "Ben" and Graziano "Thor" among many others; to Diego and Bojana, they know why, to Chiara and Marco Salcoacci (and the newly arrived Carlo!), possibly the nicest persons on Earth, to Lucia and Massimo (Hayao Miyazaki is the greatest!), and to the many on-line young friends at ComicUS and DCForum. Damiano's gratitude goes finally to his young fiancée, who in the best tradition of comic-books and being quite shy asked to maintain a secret identity here, and especially to his whole family past and present, for continued affection, support and encouragement, in particular to Annamaria, Francesco, Paolo, Dina and Mino.

Fabio is grateful to his colleagues Gianvittorio Mauri, Andrea Pallavicini, Francesco Rapisarda and Giulio Sartorelli for their invaluable contribution in the modeling, pricing and hedging of the bank's derivatives. Their skilled efficiency has allowed (and still allows) him to devote himself also to more speculative matters. Special thanks then go to his friends, and especially to the "ammiragliato" (admiralty) group, for the fun they have planning their missions around a table in "trattorie" near Treviso, to Antonio, Jacopo and Raffaele for the great time they spend together in Milan and travelling all over the world, to Chiara and Eleonora for their precious advices and sincere affection, to his pastoral friends for their spiritual support, and last, but not least, to his family for continued affection and support.

Finally, our ultimate gratitude is towards transcendence and is always impossible to express with words. We just say that we are grateful for the Word of the Gospel and the Silence of Zen.

A Special Final Word for Young Readers and Beginners

It looked insanely complicated, and this was one of the reasons why the snug plastic cover it fitted into had the words "Don't Panic" printed on it in large friendly letters.
Douglas Adams (1952 - 2001).

We close this long preface with a particular thought and encouragement for young readers. Clearly, if you are a professional or academic experienced in interest-rate modeling, we believe you will not be scared by a first quick look at the table of contents and at the chapters.

However, even at a first glance when flipping through the book, some young readers might feel discouraged by the variety of models, by the difference in approaches, by the book size, and might indeed acquire the impression of a chaotic sequence of models that arose in mathematical finance without a particular order or purpose. Yet, we assure you that this subject is interesting, relevant, and that it can (and should) be fun, however "clichéd" this may sound to you. We have tried at times to be colloquial in the book, in an attempt to avoid writing a book on formal mathematical finance from A to Zzzzzzzzzz... (where have you heard this one before?).

We are trying to avoid the two apparent extremes of either scaring or boring our readers. Thus you will find at times opinions from market participants, guided tours, intuition and discussion on things as they are seen in the market. We would like you to give it at least a try. So, if you are one of the above young readers, and be you a student or a practitioner, we suggest you take it easy. This book might be able to help you a little in entering this exciting field of research. This is why we close this preface with the by-now classic recommendations...

> *.. a brief hiss of air as the green plasma seals around him and begins to photosynthesize oxygen, and then the dead silence of space. A silence as big as everything. [...] Cool green plasma flows over his skin, maintaining his temperature, siphoning off sweat, monitoring muscle tone, repelling micro-meteorites. He thinks green thoughts. And his thoughts become things. Working the ring is like giving up cigarettes.*
>
> *He feels like a "sixty-a-day" man.*
>
> Grant Morrison on Green Lantern (Kyle Rayner)'s ring, *JLA*, 1997

> *"May fear and dread not conquer me".* Majjhima Nikaya VIII.6

> *"Do not let your hearts be troubled and do not be afraid".* St. John XIV.27

> Martian manhunter: *"...All is lost...."*
> Batman: *"I don't believe that for a second. What should I expect to feel?"*
> M: *"Despair. Cosmic despair. Telepathic contact with Superman is only possible through the Mageddon mind-field that holds him in thrall. It broadcasts on the lowest psychic frequencies...horror...shame...fear...anger..."*
> B: *"Okay, okay. Despair is fine. I can handle despair and so can you."*
> Grant Morrison, JLA: World War Three, 2000, DC Comics.

> *"Non abbiate paura!" [Don't be afraid!].* Karol Wojtyla (1920- 2005)

> *"For a moment I was afraid."* *"For no reason".*
> Irma [Kati Outinen] and M [Markku Peltola], "The Man without a Past", Aki Kaurismaki (2002).

Venice and Milan, May 4, 2006

Damiano Brigo and Fabio Mercurio

DESCRIPTION OF CONTENTS BY CHAPTER

We herewith provide a detailed description of the contents of each chapter, highlighting the updates for the new edition.

Part I: BASIC DEFINITIONS AND NO ARBITRAGE

Chapter 1: Definitions and Notation. The chapter is devoted to standard definitions and concepts in the interest-rate world, mainly from a static point of view. We define several interest-rate curves, such as the LIBOR, swap, forward-LIBOR and forward-swap curves, and the zero-coupon curve.

We explain the different possible choices of rates in the market. Some fundamental products, whose evaluation depends only on the initially given curves and not on volatilities, such as bonds and interest-rate swaps, are introduced. A quick and informal account of fundamental derivatives depending on volatility such as caps and swaptions is also presented, mainly for motivating the following developments.

Chapter 2: No-Arbitrage Pricing and Numeraire Change. The chapter introduces the theoretical issues a model should deal with, namely the no-arbitrage condition and the change of numeraire technique. The change of numeraire is reviewed as a general and powerful theoretical tool that can be used in several situations, and indeed will often be used in the book.

We remark how the standard Black models for either the cap or swaption markets, the two main markets of interest-rate derivatives, can be given a rigorous interpretation via suitable numeraires, as we will do later on in Chapter 6.

We finally hint at products involving more than one interest-rate curve at the same time, typically quanto-like products, and illustrate the no-arbitrage condition in this case.

Part II: FROM SHORT RATE MODELS TO HJM

Chapter 3: One-Factor Short-Rate Models. In this chapter, we begin to consider the dynamics of interest rates. The chapter is devoted to the short-rate world. In this context, one models the instantaneous spot interest rate via a possibly multi-dimensional driving diffusion process depending on some parameters. The whole yield-curve evolution is then characterized by the driving diffusion.

If the diffusion is one-dimensional, with this approach one is directly modeling the short rate, and the model is said to be “one-factor”. In this chapter, we focus on such models, leaving the development of the multi-dimensional (two-dimensional in particular) case to the next chapter.

As far as the dynamics of one-factor models is concerned, we observe the following. Since the short rate represents at each instant the initial point of the yield curve, one-factor short-rate models assume the evolution of the whole yield curve to be completely determined by the evolution of its initial point. This is clearly a dangerous assumption, especially when pricing products depending on the correlation between different rates of the yield curve at a certain time (this limitation is explicitly pointed out in the guided tour of the subsequent chapter).

We then illustrate the no-arbitrage condition for one-factor models and the fundamental notion of market price of risk connecting the objective world, where rates are observed, and the risk-neutral world, where expectations leading to prices occur. We also show how choosing particular forms for the market price of risk can lead to models to which one can apply both econometric techniques (in the objective world) and calibration to market prices (risk-neutral world). We briefly hint at this kind of approach and subsequently leave the econometric part, focusing on the market calibration.

A short-rate model is usually calibrated to some initial structures in the market, typically the initial yield curve, the caps volatility surface, the swaptions volatility surface, and possibly other products, thus determining the model parameters. We introduce the historical one-factor time-homogeneous models of Vasicek, Cox Ingersoll Ross (CIR), Dothan, and the Exponential Vasicek (EV) model. We hint at the fact that such models used to be calibrated only to the initial yield curve, without taking into account market volatility structures, and that the calibration can be very poor in many situations.

We then move to extensions of the above one-factor models to models including "time-varying coefficients", or described by inhomogeneous diffusions. In such a case, calibration to the initial yield curve can be made perfect, and the remaining model parameters can be used to calibrate the volatility structures. We examine classic one-factor extensions of this kind such as Hull and White's extended Vasicek (HW) model, classic extensions of the CIR model, Black and Karasinski's (BK) extended EV model and a few more.

We discuss the volatility structures that are relevant in the market and explain how they are related to short-rate models. We discuss the issue of a humped volatility structure for short-rate models and give the relevant definitions. We also present the Mercurio-Moraleda short-rate model, which allows for a parametric humped-volatility structure while exactly calibrating the initial yield curve, and briefly hint at the Moraleda-Vorst model.

We then present a method of ours for extending pre-existing time-homogeneous models to models that perfectly calibrate the initial yield curve while keeping free parameters for calibrating volatility structures. Our method preserves the possible analytical tractability of the basic model. Our extension is shown to be equivalent to HW for the Vasicek model, whereas it is original in case of the CIR model. We call CIR++ the CIR model being extended through our procedure. This model will play an important role in the final part of the book devoted to credit derivatives, in the light of the

Brigo-Alfonsi SSRD stochastic intensity and interest rate model, with the Brigo-El Bachir jump diffusion extensions (JCIR++) playing a fundamental role to attain high levels of implied volatility in CDS options. The JCIR++ model, although not studied in this chapter and delayed to the credit chapters, retains an interest of its own also for interest rate modelling, possibly also in relationship with the volatility smile problem. The reader, however, will have to adapt the model from intensity to interest rates on her own.

We then show how to extend the Dothan and EV models, as possible alternatives to the use of the popular BK model.

We explain how to price coupon-bearing bond options and swaptions with models that satisfy a specific tractability assumption, and give general comments and a few specific instructions on Monte Carlo pricing with short-rate models.

We finally analyze how the market volatility structures implied by some of the presented models change when varying the models parameters. We conclude with an example of calibration of different models to market data.

Chapter 4: Two-Factor Short-Rate Models. If the short rate is obtained as a function of all the driving diffusion components (typically a summation, leading to an additive multi-factor model), the model is said to be "multi-factor".

We start by explaining the importance of the multi-factor setting as far as more realistic correlation and volatility structures in the evolution of the interest-rate curve are concerned.

We then move to analyze two specific two-factor models.

First,we apply our above deterministic-shift method for extending pre-existing time-homogeneous models to the two-factor additive Gaussian case (G2). In doing so, we calibrate perfectly the initial yield curve while keeping five free parameters for calibrating volatility structures. As usual, our method preserves the analytical tractability of the basic model. Our extension G2++ is shown to be equivalent to the classic two-factor Hull and White model. We develop several formulas for the G2++ model and also explain how both a binomial and a trinomial tree for the two-dimensional dynamics can be obtained. We discuss the implications of the chosen dynamics as far as volatility and correlation structures are concerned, and finally present an example of calibration to market data.

The second two-factor model we consider is a deterministic-shift extension of the classic two-factor CIR (CIR2) model, which is essentially the same as extending the Longstaff and Schwartz (LS) models. Indeed, we show that CIR2 and LS are essentially the same model, as is well known. We call CIR2++ the CIR2/LS model being extended through our deterministic-shift procedure, and provide a few analytical formulas. We do not consider this model with the level of detail devoted to the G2++ model, because of the fact that its volatility structures are less flexible than the G2++'s, at least if one wishes to preserve analytical tractability. However, following some new developments coming from using this kind of model for credit derivatives, such as the Brigo-Alfonsi SSRD stochastic intensity model, we point out some further extensions and approximations that can render the CIR2++ model both flexible and tractable, and reserve their examination for further work.

Chapter 5: The Heath-Jarrow-Morton Framework. In this chapter we consider the Heath-Jarrow-Morton (HJM) framework. We introduce the general framework and point out how it can be considered the right theoretical framework for developing interest-rate theory and especially no-arbitrage. However, we also point out that the most significant models coming out concretely from such a framework are the same models we met in the short-rate approach.

We report conditions on volatilities leading to a Markovian process for the short rate. This is important for implementation of lattices, since one then obtains (linearly-growing) recombining trees, instead of exponentially-growing ones. We show that in the one-factor case, a general condition leading to Markovianity of the short rate yields the Hull-White model with all time-varying coefficients, thus confirming that, in practice, short-rate models already contained some of the most interesting and tractable cases.

We then introduce the Ritchken and Sankarasubramanian framework, which allows for Markovianity of an enlarged process, of which the short rate is a component. The related tree (Li, Ritchken and Sankarasubramanian) is presented. Finally, we present a different version of the Mercurio-Moraleda model obtained through a specification of the HJM volatility structure, pointing out its advantages for realistic volatility behavior and its analytical formula for bond options.

Part III: MARKET MODELS

Chapter 6: The LIBOR and Swap Market Models (LFM and LSM). This chapter presents one of the most popular families of interest-rate models: the market models. A fact of paramount importance is that the lognormal forward-LIBOR model (LFM) prices caps with Black's cap formula, which is the standard formula employed in the cap market. Moreover, the lognormal forward-swap model (LSM) prices swaptions with Black's swaption formula, which is the standard formula employed in the swaption market. Now, the cap and swaption markets are the two main markets in the interest-rate-derivatives world, so compatibility with the related market formulas is a very desirable property. However, even with rigorous separate compatibility with the caps and swaptions classic formulas, the LFM and LSM are not compatible with each other. Still, the separate compatibility above is so important that these models, and especially the LFM, are nowadays seen as the most promising area in interest-rate modeling.

We start the chapter with a guided tour presenting intuitively the main issues concerning the LFM and the LSM, and giving motivation for the developments to come.

We then introduce the LFM, the "natural" model for caps, modeling forward-LIBOR rates. We give several possible instantaneous-volatility structures for this model, and derive its dynamics under different measures. We explain how the model can be calibrated to the cap market, examining the impact of the different structures of instantaneous volatility on the calibration. We introduce rigorously the term structure of volatility, and again check the impact of the different parameterizations of instantaneous volatilities on its evolution in time. We point out the difference between instantaneous and terminal correlation, the latter depending also on instantaneous volatilities.

We then introduce the LSM, the "natural" model for swaptions, modeling forward-swap rates. We show that the LSM is distributionally incompatible with the LFM. We discuss possible parametric forms for instantaneous correlations in the LFM, enriching the treatment given in the first edition. We introduce several new parametric forms for instantaneous correlations, and we deal both with full rank and reduced rank matrices. We consider their impact on swaptions prices, and how, in general, Monte Carlo simulation should be used to price swaptions with the LFM instead of the LSM. Again enriching the treatment given in the first edition, we analyze the standard error of the Monte Carlo method in detail and suggest some variance reduction techniques for simulation in the LIBOR model, based on the control variate techniques. We derive several approximated analytical formulas for swaption prices in the LFM (Brace's, Rebonato's and Hull-White's). We point out that terminal correlation depends on the particular measure chosen for the joint dynamics in the LFM. We derive two analytical formulas based on "freezing the drift" for terminal correlation. These formulas clarify the relationship between instantaneous correlations and volatilities on one side and terminal correlations on the other side.

Expanding on the first edition, we introduce the problem of swaptions calibration, and illustrate the important choice concerning instantaneous correlations: should they be fixed exogenously through some historical estimation, or implied by swaptions cross-sectional data? With this new part of the book based on Massimo Morini's work we go into some detail concerning the historical instantaneous correlation matrix and some ways of smoothing it via parametric or "pivot" forms. This work is useful later on when actually calibrating the LIBOR model.

We develop a formula for transforming volatility data of semi-annual or quarterly forward rates in volatility data of annual forward rates, and test it against Monte Carlo simulation of the true quantities. This is useful for joint calibration to caps and swaptions, allowing one to consider only annual data.

We present two methods for obtaining forward LIBOR rates in the LFM over non-standard periods, i.e. over expiry/maturity pairs that are not in the family of rates modeled in the chosen LFM.

Chapter 7: Cases of Calibration of the LIBOR Market Model. In this chapter, we start from a set of market data including zero-coupon curve, caps volatilities and swaptions volatilities, and calibrate the LFM by resorting to several parameterizations of instantaneous volatilities and by several constraints on instantaneous correlations. Swaptions are evaluated through the analytical approximations derived in the previous chapter. We examine the evolution of the term structure of volatilities and the ten-year terminal correlation coming out from each calibration session, in order to assess advantages and drawbacks of every parameterization.

We finally present a particular parameterization establishing a one-to-one correspondence between LFM parameters and swaption volatilities, such that the calibration is immediate by solving a cascade of algebraic second-order equations, leading to Brigo's basic cascade calibration algorithm. No optimization is necessary in general and the calibration is instantaneous. However, if the initial swaptions data are misaligned because of illiquidity or other reasons, the calibration can lead to negative or imaginary volatilities. We show that smoothing the initial data leads again to positive real volatilities.

The first edition stopped at this point, but now we largely expanded the cascade calibration with the new work of Massimo Morini. The impact of different exogenous instantaneous correlation matrices on the swaption calibration is considered, with several numerical experiments. The interpolation of missing quotes in the original input swaption matrix seems to heavily affect the subsequent calibration of the LIBOR model. Instead of smoothing the swaption matrix, we now develop a new algorithm that makes interpolated swaptions volatilities consistent with the LIBOR model by construction, leading to Morini and Brigo's extended cascade calibration algorithm. We test this new method and see that practically all anomalies present in earlier cascade calibration experiments are surpassed. We conclude with some further remarks on joint caps/swaptions calibration and with Monte Carlo tests establishing that the swaption volatility drift freezing approximation on which the cascade calibration is based holds for the LIBOR volatilities parameterizations used in this chapter.

Chapter 8: Monte Carlo Tests for LFM Analytical Approximations. In this chapter we test Rebonato's and Hull-White's analytical formulas for swaptions prices in the LFM, presented earlier in Chapter 6, by means of a Monte Carlo simulation of the true LFM dynamics. Partial tests had already been performed at the end of Chapter 7. The new tests are done under different parametric assumptions for instantaneous volatilities and under different instantaneous correlations. We conclude that the above formulas are accurate in non-pathological situations.

We also plot the real swap-rate distribution obtained by simulation against the lognormal distribution with variance obtained by the analytical approximation. The two distributions are close in most cases, showing that the previously remarked theoretical incompatibility between LFM and

LSM (where swap rates are lognormal) does not transfer to practice in most cases.

We also test our approximated formulas for terminal correlations, and see that these too are accurate in non-pathological situations.

With respect to the first edition, based on the tests of Brigo and Capitani, we added an initial part in this chapter computing rigorously the distance between the swap rate in the LIBOR model and the lognormal family of densities, under the swap measure, resorting to Brigo and Liinev's Kullback-Leibler calculations. The distance results to be small, confirming once again the goodness of the approximation.

Part IV: THE VOLATILITY SMILE

The old section on smile modeling in the LFM has now become a whole new part of the book, consisting of four chapters.

Chapter 9: Including the Smile in the LFM. This first new smile introductory chapter introduces the smile problem with a guided tour, providing a little history and a few references. We then identify the classes of models that can be used to extend the LFM and briefly describe them; some of them are examined in detail in the following three smile chapters.

Chapter 10: Local-Volatility Models. Local-volatility models are based on asset dynamics whose absolute volatility is a deterministic transformation of time and the asset itself. Their main advantages are tractability and ease of implementation. We start by introducing the forward-LIBOR model that can be obtained by displacing a given lognormal diffusion, and also describe the constant-elasticity-of-variance model by Andersen and Andreasen. We then illustrate the class of density-mixture models proposed by Brigo and Mercurio and Brigo, Mercurio and Sartorelli, providing also an example of calibration to real market data. A seemingly paradoxical result on the correlation between the underlying and the volatility, also in relation with later uncertain parameter models, is pointed out. In this chapter mixtures resort to the eralier lognormal mixture diffusions of the first edition but also to Mercurio's Hyperbolic-Sine mixture processes. We conclude the Chapter by describing Mercurio's second general class, which combines analytical tractability with flexibility in the cap calibration.

The local-volatility models in this chapter are meant to be calibrated to the caps market, and to be only used for the pricing of LIBOR dependent derivatives. The task of a joint calibration to the cap and swaption markets and the pricing of swap-rates dependent derivatives under smile effects, is, in this book, left to stochastic-volatility models and to uncertain-parameters models, the subject of the last two smile chapters.

Chapter 11: Stochastic-Volatility Models. We then move on to describe LIBOR models with stochastic volatility. They are extensions of the LFM where the instantaneous volatility of forward rates evolves according to a diffusion process driven by a Brownian motion that is possibly instantaneously correlated with those governing the rates' evolution. When the (instantaneous) correlation between a forward rate and its volatility is zero, the existence itself of a stochastic volatility leads to smile-shaped implied volatility curves. Skew-shaped volatilities, instead, can be produced as soon as we i) introduce a non-zero (instantaneous) correlation between rate and volatility or ii) assume a displaced-diffusion dynamics or iii) assume that the rate's diffusion coefficient is a non-linear function of the rate itself. Explicit formulas for both caplets and swaptions are usually derived by calculating the characteristic function of the underlying rate under its canonical measure.

In this chapter, we will describe some of the best known extensions of the LFM allowing for stochastic volatility, namely the models of i) Andersen and Brotherton-Ratcliff, ii) Wu and Zhang, iii) Hagan, Kumar, Lesniewski and Woodward, iv) Piterbarg and v) Joshi and Rebonato.

Chapter 12: Uncertain Parameters Models. We finally consider extensions of the LFM based on parameter uncertainty. Uncertain-volatility models are an easy-to-implement alternative to stochastic-volatility models. They are based on the assumption that the asset's volatility is stochastic in the simplest possible way, modelled by a random variable rather than a diffusion process. The volatility, therefore, is not constant and one assumes several possible scenarios for its value, which is to be drawn immediately after time zero. As a consequence, option prices are mixtures of Black's option prices and implied volatilities are smile shaped with a minimum at the at-the-money level. To account for skews in implied volatilities, uncertain-volatility models are usually extended by introducing (uncertain) shift parameters.

Besides their intuitive meaning, uncertain-parameters models have a number of advantages that strongly support their use in practice. In fact, they enjoy a great deal of analytical tractability, are relatively easy to implement and are flexible enough to accommodate general implied volatility surfaces in the caps and swaptions markets. As a drawback, future implied volatilities lose the initial smile shape almost immediately. However, our empirical analysis will show that the forward implied volatilities induced by the models do not differ much from the current ones. This can further support their use in the pricing and hedging of interest rate derivatives.

In this chapter, we will describe the shifted-lognormal model with uncertain parameters, namely the extension of Gatarek's one-factor uncertain-parameters model to the general multi-factor case as considered by Errais, Mauri and Mercurio. We will derive caps and (approximated) swaptions prices in closed form. We will then consider examples of calibration to caps and swaptions data. A curious relationship between one of the simple models in this framework and the earlier lognormal-mixture local volatility dynamics, related also to underlying rates and volatility decorrelation, is pointed out as from Brigo's earlier work.

Part V: EXAMPLES OF MARKET PAYOFFS

> *We thought that by making your world more violent, we would make it more "realistic", more "adult". God help us if that's what it means. Maybe, for once we could try to be kind.*
>
> Grant Morrison, Animal Man 26, 1990, DC Comics.

Chapter 13: Pricing Derivatives on a Single Interest-Rate Curve. This chapter deals with pricing specific derivatives on a single interest-rate curve. Most of these are products that are found in the market and for which no standard pricing technique is available. The model choice is made on a case-by-case basis, since different products motivate different models. The differences are based on realistic behaviour, ease of implementation, analytical tractability and so on. For each product we present at least one model based on a compromise between the above features, and in some cases we present more models and compare their strong and weak points. We try to understand which model parameters affect prices with a large or small influence. The financial products we consider are: in-arrears swaps, in-arrears caps, autocaps, caps with deferred caplets, ratchet caps and floors (new for the second edition), ratchets (one-way floaters), constant-maturity swaps (introducing also the convexity-adjustment technique), average rate caps, captions and floortions, zero-coupon swaptions, Eurodollar futures, accrual swaps, trigger swaps and Bermudan-style swaptions. We add numerical examples for Bermudan swaptions. Further, in this new edition we consider target redemption notes and CMS spread options.

Chapter 14: Pricing Derivatives on Two Interest-Rate Curves. The chapter deals with pricing specific derivatives involving two interest-rate curves. Again, most of these are products that are found in the market and for which no standard pricing technique is available. As before, the model choice is made on a case-by-case basis, since different products motivate different models. The used models reduce to the LFM and the G2++ shifted two-factor Gaussian short-rate model. Under the G2++ model, we are able to model correlation between the interest rate curves of the two currencies. The financial products we consider include differential swaps, quanto caps, quanto swaptions, quanto constant-maturity swaps. A market quanto adjustment and market formulas for basic quanto derivatives are also introduced. We finally price, in a market-model setting, spread options on two-currency LIBOR rates, options on the product of two-currency LIBOR rates and trigger swaps with payments, in domestic currency, triggered by either the domestic rate or the foreign one.

Part VI: INFLATION

In this new part, we describe new derivatives, which are based on inflation rates, together with possible models to price them.

Chapter 15: Pricing of Inflation Indexed Derivatives. Inflation is defined in terms of the percentage increments of a reference index, the consumer price index, which is a representative basket of goods and services.

Floors with low strikes are the most actively traded options on inflation rates. Other extremely popular derivatives are inflation-indexed swaps, where the inflation rate is either payed on an annual basis or with a single amount at the swap maturity. All these inflation-indexed derivatives require a specific model to be valued.

Most articles on inflation modeling in the financial literature are based on the so called *foreign-currency analogy*, according to which real rates are viewed as interest rates in the real (i.e. foreign) economy, and the inflation index is interpreted as the exchange rate between the nominal (i.e. domestic) and real "currencies". In this setting, the valuation of an inflation-indexed payoff becomes equivalent to that of a cross-currency interest rate derivative.

A different approach has also been developed using the philosophy of market models. The idea is to model the evolution of forward inflation indices, so that an inflation rate can be viewed as the ratio of two consecutive "assets", and derivatives are priced accordingly.

Chapters 16, 17 and 18: Inflation-Indexed Swaps, Inflation-Indexed Caplets/Floorlets, and Calibration to Market Data. The purpose of these chapters is to define the main types of inflation-indexed swaps and caps present in the market and price them analytically and consistently with no arbitrage. To this end, we will review and use i) the Jarrow and Yildirim model, where both nominal and real rates are assumed to evolve as in a one-factor Gaussian HJM model, ii) the Mercurio application of the LFM, and iii) the market model of Kazziha, also independently developed by Belgrade, Benhamou and Koehler and by Mercurio. Examples of calibration to market data will also be presented.

Chapter 19: Introducing Stochastic Volatility. In this chapter we add stochastic volatility to the market model introduced in Chapters 16 and 17. Precisely, we describe the approach followed by Mercurio and Moreni (2006), who modelled forward CPI's with a common volatility process that evolves according to a square-root diffusion.

Modelling the stochastic volatility as in Heston (1993) has the main advantage of producing analytical formulas for options on inflation rates. In fact, we first derive an explicit expression for the characteristic function of the ratio between two consecutive forward CPI's, and then price caplets and floorlets by Carr and Madan's (1998) Fourier transform method.

Numerical examples including a calibration to market cap data are finally shown.

Chapter 20: Pricing Hybrids with an Inflation Component. In this chapter, we tackle the pricing issue of a specific hybrid payoff involving inflation features when no smile effects are taken into account. It is meant to be an important example from an increasing family of hybrid payoffs that are getting popular in the market.

Part VII: CREDIT

This new part deals with credit derivatives, counterparty risk, credit models and their analogies with interest-rate models.

Chapter 21: Introduction and Pricing under Counterparty Risk. This first chapter starts this new part of the book devoted to credit derivatives and counterparty risk. In this first chapter we introduce the financial payoffs and the families of rates we deal with in the following. We present a guided tour to give some orientation and general feeling for this credit part of the book. The guided tour also focuses on multiname credit derivatives, introducing collateralized debt obligations (CDO) and first to default (FtD) contracts as fundamental examples. The first generation pricing of these products involves copula functions, that are introduced and reviewed, including the recent family of Alfonsi and Brigo periodic copulas. The need for dynamical models of dependence is pointed out. This is the only part of the book where we mention multi-name credit derivatives. The book focuses mostly on single name credit derivatives.

Then we introduce as first credit payoffs the prototypical defaultable bonds, the Credit Default Swaps (CDS) payoffs and defaultable floaters, including a relationship between the last two. In particular, we consider some different definitions of CDS forward rates, with analogies with LIBOR vs swap rates. We explore in detail possible equivalence between CDS payoffs and rates and defaultable floaters payoffs and rates.

We then introduce CDS options payoffs, pointing out some formal analogies with the swaption payoff encountered earlier in the book. We also introduce constant maturity CDS, a product that has grown in popularity in recent times. This product presents analogies with constant maturity swaps in the default free market. Finally, we close the chapter with counterparty risk pricing in interest rate derivatives. We show how to include the event that the counterparty may default in the risk neutral valuation of the financial payoff. This is particularly important after the recent regulatory directions given by the Basel II agreement and subsequent amendments and also by the "IAS 39" (international accounting standard) system. The counterparty risk pricing formula of Brigo and Masetti for non-standard swaps and swaps under netting agreements is only hinted at.

Chapter 22: Intensity Models. In this new chapter we focus completely on intensity models, exploring in detail also the issues we have anticipated in the earlier chapter in order to be able to deal with CDS and notions of implied hazard rates and functions.

Intensity models, part of the family of reduced form models, all move from the basic idea of describing the default time as the first jump time of a

Poisson process. Default is not induced by basic market observables but has an exogenous component that is independent of all the default free market information. Monitoring the default free market (interest rates, exchange rates, etc) does not give complete information on the default process, and there is no economic rationale behind default. This family of models is particularly suited to model credit spreads and in its basic formulation is easy to calibrate to Credit Default Swap (CDS) or corporate bond data.

The basic facts from probability are essentially the theory of Poisson and Cox processes. We start from the simplest, constant intensity Poisson process and explain the interpretation of the intensity as a probability of first jumping (defaulting) per unit of time. We then move to time-inhomogeneous Poisson processes, that allow to model credit spreads without volatility. Further, we move to stochastic intensity Poisson processes, where the probability of first jumping (defaulting) is itself random and follows a stochastic process of a certain kind. This last case is referred to as "Cox process" approach, or "doubly stochastic Poisson process". This approach allows us to take into account credit spread volatility. In all three cases of constant, deterministic-time-varying and stochastic intensity we point out how the Poisson process structure allows to view survival probabilities as discount factors, the intensity as credit spread, and how this helps us in recycling the interest-rate technology for default modeling. We then analyze in detail the CDS calibration with deterministic intensity models, illustrating the notion of implied hazard function with a case study based on Parmalat CDS data. We illustrate how the only hope of inducing dependence between the default event and interest rates in a diffusion setting is through a stochastic intensity correlated with the interest rate. We explain the fundamental idea of conditioning only to the partial information of the default free market when pricing credit derivatives. This result has fundamental consequences in that it will allow us later to define the CDS market model under a measure that is equivalent to the risk neutral one. Also, our definition of forward CDS rate itself owes much to this result.

We also explain how to simulate the default time, illustrating the notion of standard error and presenting suggestions on how to keep the number of paths under control. These suggestions take into account peculiarities of default modeling that make the variance reduction more difficult than in the default free market case.

We then introduce our choice for the stochastic intensity in a diffusion setting, the Brigo-Alfonsi stochastic intensity model. We term the stochastic intensity and interest rate model SSRD: Shifted Square Root Diffusion model. It is essentially a CIR++ model for the intensity correlated with a CIR++ model for the short rate. We argue the choice is the only reasonable one in a diffusion setting for the intensity given that one wishes analytical tractability for survival probabilities (CDS calibration) and positivity of the intensity process. We show how to calibrate the SSRD model to CDS quotes and interest rate data in a separable way, and argue that the instantaneous correlation has a negligible impact on the CDS price, allowing us to maintain the separability of the calibration in practice even when correlation is not zero. We present some original numerical schemes due to Alfonsi and Brigo for the simulation of the SSRD model that preserve positivity of the discretized process and analyze the convergence of such schemes. We also introduce the Brigo-Alfonsi Gaussian mapping technique that maps the model into a two factor Gaussian model, where calculations in presence of correlation are much easier. We analyze the mapping procedure and its accuracy by means of Monte Carlo tests. We also analyze the impact of the correlation on some prototypical payoff. As an exercise we price a cancellable structure with the stochastic intensity model. We also introduce Brigo's CDS option closed form formula under deterministic interest rates and CIR++ stochastic intensity, a particular case of the SSRD model. We analyze implied CDS volatilities patterns in the full SSRD case by means of Monte Carlo simulation. Finally, we explain why the CIR++ model for the intensity cannot attain large levels (such as 50%) of implied volatilities for CDS rates, and introduce jumps in the CIR++ model, hinting at the JCIR model and at its possible calibration to both CDS and options, Brigo and El-Bachir JCIR++ model.

Chapter 23: CDS Options Market Models. In this last new chapter of the credit part we start with the payoffs and structural analogies between CDS options and callable defaultable floating rate notes (DFRN).

We then introduce the market formula for CDS options and callable DFRN, based on a rigorous change of numeraire technique as in Brigo's CDS market model, different from Schönbucher's in that it guarantees equivalence of pricing measures notwithstanding default. Numerical examples of implied volatilities from CDS option quotes are given, and are found to be rather high, in agreement with previous studies dealing with historical CDS rate volatilities (Hull and White).

We discuss possible developments towards a compete specifications of the vector dynamics of CDS forward rates under a single pricing measure, based on one-period CDS rates.

We give some hints on modeling of the volatility smile for CDS options, based on the general framework introduced earlier.

We also illustrate how to use Brigo's market model to derive an approximated formula for Constant Maturity CDS. This formula is based on a sort of convexity adjustment and bears resemblance to the formula for valuing constant maturity swaps with the LIBOR model, seen earlier in the book. The adjustment is illustrated with several numerical examples.

Part VIII: APPENDICES

Appendix A: Other Interest-Rate Models. We present a few interest-rate models that are particular in their assumptions or in the quantities they model, and that have not been treated elsewhere in the book. We do not give a detailed presentation of these models but point out their particular features, compared to the models examined earlier in the book. This was a chapter in the first edition but to simplify the layout we included it here as an appendix.

Appendix B: Pricing Equity Derivatives under Stochastic Interest Rates. The appendix treats equity-derivatives valuation under stochastic interest rates, presenting us with the challenging task of modeling stock prices and interest rates at the same time. Precisely, we consider a continuous-time economy where asset prices evolve according to a geometric Brownian motion and interest rates are either normally or lognormally distributed. Explicit formulas for European options on a given asset are provided when the instantaneous spot rate follows the Hull-White one-factor process. It is also shown how to build approximating trees for the pricing of more complex derivatives, under a more general short-rate process. This was a chapter in the first edition but to simplify the layout we included it here as an appendix.

Appendix C: a Crash Introduction to Stochastic Differential Equations and Poisson Processes.

> *There is, of course, a dearth of good mathematics teachers [...] Why subject themselves to a lifetime surrounded by a pandemonium of fresh-faced young people in uniform shouting to each other across the classroom for nine grand a year, they say, when they can do exactly the same in the trading room of any stockbrokers for ninety?*
> Robert Ainsley, "Bluff your way in Maths", Ravette Books, 1988

This appendix is devoted to a quick intuitive introduction on SDE's and Poisson processes. We start from deterministic differential equation and gradually introduce randomness. We introduce intuitively Brownian motion and explain how it can be used to model the "random noise" in the differential equation. We observe that Brownian motion is not differentiable, and explain that SDE's must be understood in integral form. We quickly introduce the related Ito and Stratonovich integrals, and introduce the fundamental Ito formula.

We then introduce the Euler and Milstein schemes for the time-discretization of an SDE. These schemes are essential when in need of Monte Carlo simulating the trajectories of an Ito process whose transition density is not explicitly known.

We include two important theorems: the Feynman-Kac theorem and the Girsanov theorem. The former connects PDE's to SDE's, while the latter permits to change the drift coefficient in an SDE by changing the basic probability measure. The Girsanov theorem in particular is used in the book to derive the change of numeraire toolkit.

Given its importance in default modeling, we also introduce the Poisson process, to some extent the purely jump analogous of Brownian motion.

Brownian motions and Poisson processes are among the most important random processes of probability.

Appendix D: a Useful Calculation. This appendix reports the calculation of a particular integral against a standard normal density, which is useful when dealing with Gaussian models.

Appendix E: a Second Useful Calculation. This appendix shows how to calculate analytically the price of an option on the spread between two assets, under the assumption that both assets evolve as (possibly correlated) geometric Brownian motions.

Appendix F: Approximating Diffusions with Trees. This appendix explains a general method to obtain a trinomial tree approximating the dynamics of a general diffusion process. This is then generalized to a two-dimensional diffusion process, which is approximated via a two-dimensional trinomial tree.

Appendix G: Trivia and Frequently Asked Questions (FAQ). In this appendix we answer a number of frequently asked questions concerning the book trivia and curiosities. It is a light appendix, meant as a relaxing moment in a book that at times can be rather tough.

Appendix H: Talking to the Traders. This is the ideal conclusion of the book, consisting of an interview with a quantitative trader. Several issues are discussed, also to put the book in a larger perspective. This version for the second edition has been enriched.