Approximated moment-matching dynamics for basket-options simulation

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Abstract

The aim of this paper is to present two moment matching procedures for basket-options pricing and to test its distributional approximations via distances on the space of probability densities, the Kullback-Leibler information (KLI) and the Hellinger distance (HD). We are interested in measuring the KLI and the HD between the real simulated basket terminal distribution and the distributions used for the approximation, both in the lognormal and shifted-lognormal families. We isolate influences of instantaneous volatilities and instantaneous correlations, in order to assess which configurations of these variables have a major impact on the KLI and HD and therefore on the quality of the approximation.

Contents

1 Introduction

In this paper we start by introducing the standard moment-matching procedure that one can apply to simulate the average price of a basket of basic assets. The basic idea is that of approximating the *actual* process of the basket value by a sufficiently simple stochastic process. The expression "sufficiently simple" should be interpreted as "simple enough to allow for analytic solutions to the pricing problem at hand".

The approximation happens on the basis of a *moment matching* principle, which can be stated as follows: set the parameters of the approximating process so that as many moments as possible of the actual basket-price process are exactly reproduced. With the usual lack of fantasy, the market choice of an approximating process seems to have fallen onto the lognormal one [9]. The distinctive parameters of such a process being only two (the return's average and standard deviation over the time horizon set by the option to price) this moment-matching procedure can only match the first two moments of the original distribution. The lengthy calculations of the parameters' values can be performed so as to keep into account the effect of dividends, either continuous or discrete (but in any case deterministic, both in payment dates and in amounts). A more compact formulation of this method is obtained by resorting to forward prices, which incorporate both interest rates and dividends.

Another similar procedure consists of matching the first three moments, through an additional shift in the approximating lognormal basket by a deterministic parameter. This new parameter allows to fit the first three moments without losing analytical tractability, in that we can immediately characterize the distributional properties of the resulting process in a trivial fashion.

Wanting to match further moments, we will show how this issue can be addressed by resorting to a mixture of lognormal densities. Such a mixture, in fact, is flexible enough for practical purposes and implies closed form formulas for option prices.

We have performed an empirical analysis of the two- and three-moments-matching approximations based on the case of a basket of two equities in the Italian stock exchange and compare results by resorting to a Monte Carlo simulation to obtain the “true” distribution and statistics of the basket within a Black-Scholes world.

We subsequently analyze specifically the implications of the three-moments-method as far as a call option pricing is concerned.

The second part of the paper addresses the problem of computing a synthetic but at the same time rigorous measure of the deviation of the approximated baskets distributions from the true basket distribution. To characterize rigorously this distributional discrepancy, we introduce both the Kullback Leibler information and the Hellinger distance in suitable spaces of densities, and explain how this can help us in our investigation. We compute the distances of the true basket from the parametric families of densities being used in the two and three moments approximations through Monte Carlo simulation. The two families are respectively the lognormal and shifted lognormal families. Finally, we try and isolate the variables and the situations causing this distance to increase drastically, thus characterizing the cases where the two and three moments approximations can fail.

2 Assumptions and notation

Let $\{T_1, \ldots, T_N\}$ be the set of dates at which the values of the basket are contributing to the contract payoff. We shall denote by $\{\tau_1, \ldots, \tau_N\}$ the corresponding time lengths, meaning that $\tau_i = T_i - T_{i-1}$, $T_0 = 0$, $i = 1, 2, \ldots, N$.

For example, if we are pricing an Asian-style option on a basket, the T's are the times at which the average is taken.

Denote by S_t^i or equivalently $S^i(t)$ the price of the i-th asset in the basket at time t. The basket value is then given by

$$A(t) = \sum_{i=1}^{n} a_i S^i(t), \quad A(0) = A_0,$$

where the a's denote deterministic and constant weights specified by the option contract, and $A_0 > 0$ is the initial basket value.

3 Matching the first two moments under different dividends formulations

3.1 Case with continuous dividends, constant rates and volatilities

The first and simplest case we consider is the case where all interest rates are equal to r, and where each asset S^i in the basket pays a continuous dividend yield q_i and has constant instantaneous volatility σ_i. In other terms, we denote by $W^1, \dots, W^n$ n correlated Brownian motions and assume that, under the risk-neutral measure,

$$\begin{aligned} dS_t^i &= (r - q_i)S_t^i \, dt + \sigma_i S_t^i \, dW_t^i, \quad i = 1, \dots, n \\ \mathrm{Corr}(d\ln(S_t^i), d\ln(S_t^j))\, dt &= dW_t^i dW_t^j = \rho_{i,j}\, dt. \end{aligned} \tag{1}$$

We therefore assume the different asset returns to be instantaneously correlated according to the matrix ρ.

Our purpose is to approximate the basket value A with a (lognormal) geometric Brownian motion (GBM). We take as proxy of A the process $\bar{A}$ defined by

$$d\bar{A}(t) = (r - \bar{q})\bar{A}(t)\, dt + \bar{\sigma}\bar{A}(t)\, dW_t, \quad \bar{A}(0) = A_0, \tag{2}$$

where W is a Brownian motion, and $\bar{q}$ and $\bar{\sigma}$ are the basket dividend yield and volatility to be determined in terms of the single assets' dividends q_i, volatilities σ_i, and correlations $\rho_{i,j}$, $i, j = 1, \dots, n$.

We aim at finding the value of the basket volatility $\bar{\sigma}$ and dividend yield $\bar{q}$ that are consistent, in some sense, with the true basket dynamics. We may reason as follows. By Itô's formula applied to (2) we have

$$d\ln \bar{A}(t) = (r - \bar{q} - \tfrac{1}{2}\bar{\sigma}^2)\, dt + \bar{\sigma}\, dW_t,$$

so that, by integrating, we immediately obtain

$$\bar{A}(t) = A_0 \exp[(r - \bar{q} - \tfrac{1}{2}\bar{\sigma}^2)t + \bar{\sigma} W_t].$$

It is easy to compute the first and second moments of the approximated basket $\bar{A}$. We obtain, by using the Gaussian distribution $W_t \sim \mathcal{N}(0, t)$ and the moment generating function of Gaussian variables,

$$E(\bar{A}(t)) = A_0 \exp[(r - \bar{q})t],$$

and

$$E(\bar{A}(t)^2) = A_0^2 \exp[(2r - 2\bar{q} + \bar{\sigma}^2)t].$$

We now compute the first two moments of the true basket A. To do this, we need to know $E(S_t^i S_t^j)$. This can be computed as follows. Consider the differential

$$d(S^i(t)S^j(t)) = S^i(t)dS^j(t) + S^j(t)dS^i(t) + dS^i(t)dS^j(t)\ ,$$

and substitute for dS^i and dS^j from (1). Once done this, take expectation on both sides, and recall that $E(dW) = 0$, thus concluding that

$$dE(S^i(t)S^j(t)) = (2r - q_i - q_j + \rho_{i,j}\sigma_i\sigma_j)E(S^i(t)S^j(t))\,dt,$$

which leads, through integration, to

$$E(S^i(t)S^j(t)) = S_0^i S_0^j \exp[(2r - q_i - q_j + \rho_{i,j}\sigma_i\sigma_j)t].$$

Going back to the first two moments of A, we obtain easily

$$\begin{aligned} E(A(t)) &= \sum_{i=1}^n a_i E(S^i(t)) = \sum_{i=1}^n a_i S_0^i \exp((r - q_i)t) \\ E(A(t)^2) &= \sum_{i,j=1}^n a_i a_j E(S^i(t)S^j(t)) = \sum_{i,j=1}^n a_i a_j S_0^i S_0^j \exp[(2r - q_i - q_j + \rho_{i,j}\sigma_i\sigma_j)t]. \end{aligned}$$

The *moment matching procedure* consists in imposing, at a chosen time $t = T$, the equality between the first two moments of the true basket and of its approximation,

$$E(A(T)) = E(\bar{A}(T)), \quad E(A(T)^2) = E(\bar{A}(T)^2)$$

through the expressions found above. By doing this, we can solve in $\bar{\sigma}$ and $\bar{q}$, thus obtaining, after straightforward computations,

$$\boxed{\begin{aligned} \bar{q} &= -\tfrac{1}{T}\ln\left(\frac{\sum_{i=1}^n a_i S_0^i e^{-q_i T}}{\sum_{i=1}^n a_i S_0^i}\right), \\ \bar{\sigma}^2 &= \tfrac{1}{T}\ln\left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j \exp[(-q_i - q_j + \rho_{i,j}\sigma_i\sigma_j)T]}{e^{-2\bar{q}T}(\sum_{i=1}^n a_i S_0^i)^2}\right) \\ &= \tfrac{1}{T}\ln\left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j \exp[(-q_i - q_j + \rho_{i,j}\sigma_i\sigma_j)T]}{(\sum_{i=1}^n a_i S_0^i e^{-q_i T})^2}\right) \end{aligned}} \tag{3}$$

Therefore, in this case the problem is solved.

Remark 3.1. (Choice of T) *Here T is a single time instant upon which the equivalence is based. In case of a European-style option, it can be set to the maturity of the option. However, for more complex situations it is better to break down the above analysis by considering all the instants T that contribute to the final payoff.*

3.2 Case with continuous dividends, time-varying rates and volatilities

When wishing to take into account the initial term structures of interest rates and volatilities, one has to include time-varying rates and volatilities in the model. This can be accomplished by taking time-varying r and σ:

$$\begin{aligned} dS_t^i &= (r(t) - q_i)S_t^i\,dt + \sigma_i(t)S_t^i\,dW_t^i, \quad i = 1, \ldots, n \\ \mathrm{Corr}(d\ln(S_t^i), d\ln(S_t^j))\,dt &= dW_t^i dW_t^j = \rho_{i,j}\,dt, \end{aligned} \tag{4}$$

where we stick to the hypothesis of continuous dividends.

What is commonly involved in pricing payoffs depending on the basket at times T's is the average of the just introduced time-varying quantities:

$$R(U,T) = \tfrac{1}{T-U}\int_U^T r(s)ds, \quad V^i(U,T) = \sqrt{\tfrac{1}{T-U}\int_U^T \sigma_i(u)^2 du}\ .$$

Here $R(U,T)$ is the continuously-compounded (interest) rate at time U for the maturity T, and $V^i(U,T)$ is the volatility needed to price at time U a plain-vanilla option on S^i with maturity T. These quantities are usually available in the market at time $U = 0$, and in-between volatilities can be stripped from the volatility curve at $U = 0$.

As before, we approximate the basket value A with a GBM. We thus take as proxy of A the process $\bar{A}$, now defined by

$$d\bar{A}(t) = (r(t) - \bar{q})\bar{A}(t)dt + \bar{\sigma}(t)\bar{A}(t)dW_t, \quad \bar{A}(0) = A_0,$$

where $\bar{q}$ and $\bar{\sigma}(\cdot)$ are the basket dividend yield and *instantaneous* volatility. What we need this time are the integrals of the instantaneous volatility. These integrals express volatilities corresponding to options with times and maturities in the set of chosen extremes of the integrals. To this end, denote by $V_A(0,T)$ the basket volatility at time 0 for maturity T, that is

$$V_A(0,T) := \sqrt{\tfrac{1}{T}\int_0^T \bar{\sigma}^2(t)dt}.$$

Straightforward generalizations of the above case lead to

$$V_A(0,T)^2 = \tfrac{1}{T}\ln\left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j \exp[(-q_i - q_j)T + \rho_{i,j}\int_0^T \sigma_i(t)\sigma_j(t)dt]}{e^{-2\bar{q}T}(\sum_{i=1}^n a_i S_0^i)^2}\right), \tag{5}$$

whereas the formula for $\bar{q}$ remains the same as in (3).

The only problematic term in Eq. (5) is

$$\int_0^T \sigma_i(t)\sigma_j(t)dt.$$

We can easily compute this integral if we take each σ_i to be piecewise constant in intervals $[T_{k-1}, T_k]$ and set to the average volatilities $V^i(T_{k-1}, T_k) := \sqrt{\int_{T_{k-1}}^{T_k} \sigma_i(t)^2 dt/\tau_k}$. Indeed, in such a case we write

$$\int_0^{T_k} \sigma_i(t)\sigma_j(t)dt \approx \sum_{h=1}^{k} V^i(T_{h-1}, T_h)V^j(T_{h-1}, T_h)\tau_h\ .$$

To obtain the average volatilities

$$V_A(T_{k-1},T_k) = \sqrt{\frac{1}{\tau_k}\int_{T_{k-1}}^{T_k} \bar{\sigma}^2(t)dt}$$

in-between the time instants T's that are relevant for the payoff, compute first, applying the above formula,

$$\boxed{\begin{aligned} V_A(0,T_k)^2 &= \tfrac{1}{T_k}\ln\left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j \exp[(-q_i-q_j)T_k + \rho_{i,j}\sum_{h=1}^k V^i(T_{h-1},T_h)V^j(T_{h-1},T_h)\tau_h]}{e^{-2\bar{q}_k T_k}(\sum_{i=1}^n a_i S_0^i)^2}\right) \\ &= \tfrac{1}{T_k}\ln\left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j \exp[(-q_i-q_j)T_k + \rho_{i,j}\sum_{h=1}^k V^i(T_{h-1},T_h)V^j(T_{h-1},T_h)\tau_h]}{\left(\sum_{i=1}^n a_i S_0^i e^{-q_i T_k}\right)^2}\right) \end{aligned}} \tag{6}$$

where, since also $\bar{q}$ depends on T in general, we have set

$$\bar{q}_k := -\tfrac{1}{T_k}\ln\left(\frac{\sum_{i=1}^n a_i S_0^i e^{-q_i T_k}}{\sum_{i=1}^n a_i S_0^i}\right).$$

Now, since

$$\int_{T_{k-1}}^{T_k} \bar{\sigma}(t)^2 dt = \int_0^{T_k} \bar{\sigma}(t)^2 dt - \int_0^{T_{k-1}} \bar{\sigma}(t)^2 dt,$$

we obtain

$$V_A(T_{k-1},T_k)^2 = \frac{T_k V_A(0,T_k)^2 - T_{k-1}V_A(0,T_{k-1})^2}{T_k - T_{k-1}}.$$

The values of $\bar{q}$ in $[T_{k-1},T_k]$ are stripped similarly: we set

$$\bar{q}_{k-1,k} := \frac{\bar{q}_k T_k - \bar{q}_{k-1}T_{k-1}}{T_k - T_{k-1}}.$$

A Monte Carlo simulation of the basket price will be based on integration of the GBM dynamics between times T_{k-1} and T_k, leading to

$$\bar{A}(T_k) = \bar{A}(T_{k-1})\exp\{[F(0;T_{k-1},T_k) - \bar{q}_{k-1,k} - \tfrac{1}{2}V_A(T_{k-1},T_k)^2]\tau_k + V_A(T_{k-1},T_k)\sqrt{\tau_k}\mathcal{N}(0,1)\}, \tag{7}$$

where all realizations of $\mathcal{N}(0,1)$ involved can be taken to be independent, and where $F(0;T_{k-1},T_k)$ is the continuously compounded forward rate at time 0 for the period from T_{k-1} to T_k.[1]

[1]Notice that $F(0;T_{k-1},T_k) = R(T_{k-1},T_k)$ since rates are deterministic.

3.3 Case with discrete dividends, time-varying rates and volatilities

In this case dividends are no longer given by continuous dividend yields, so that we can set $q_i = \bar{q} = 0$ in the previous cases. Let us denote by $t_1, \dots, t_m$ the times at which at least one asset in the basket pays a known discrete dividend, and assume that the i-th asset in the basket pays a known dividend K_p^i at time t_p, $p = 1, \dots, m$. This quantity K is simply set to 0 for those assets that pay no dividends at t_p.

We denote by $D^i(t)$ the present value of all *future* dividend payments for the i-th asset at time t, i.e. the sum of all dividend payments occurring after time t, each discounted from its payment date to t. We model each asset price C^i in the basket as

$$C^i(t) = S^i(t) + D^i(t). \tag{8}$$

Notice that $D^i(t)$ is a deterministic process with jumps (it jumps each time a dividend payment occurs for the i-th asset), whereas S^i now models the *continuous* part of our price process. The whole price of the i-th asset is now C^i, and this is the asset price observed in the market. We assume the continuous part S^i of every such asset price C^i to follow again a GBM analogous to (4) with all the q's (and hence $\bar{q}$) set to zero:

$$\begin{aligned} dS_t^i &= r(t)S_t^i\,dt + \ \sigma_i(t)S_t^i\,dW_t^i, \quad i = 1, \dots, n \\ \mathrm{Corr}(d\ln(S_t^i), d\ln(S_t^j))\,dt &= dW_t^i dW_t^j := \rho_{i,j}\,dt. \end{aligned} \tag{9}$$

According to this formulation, the basket value is at time t

$$B(t) = \sum_i a_i C^i(t) = \sum_i a_i S^i(t) + \sum_i a_i D^i(t). \tag{10}$$

Set

$$D(t) := \sum_i a_i D^i(t), \quad A(t) := \sum_i a_i S_t^i.$$

It then follows that the basket value is

$$B(t) = A(t) + D(t). \tag{11}$$

We now approximate the continuous part $A(t)$ of the basket value through the same process $\bar{A}$ as in the previous section with all q's (and therefore $\bar{q}$) set to 0,

$$d\bar{A}(t) = r(t)\bar{A}(t)dt + \bar{\sigma}(t)\bar{A}(t)dW_t, \quad \bar{A}(0) = A_0\ ,$$

so that we obtain again a method to simulate the basket. We have however to be careful: *The volatilities we are using are not the volatilities of the assets C, but merely the volatilities of their continuous parts S.* We need therefore to correct for this difference, since our simulation requires the S volatilities while the market provides us with the C volatilities.

3.3.1 From the C volatilities to the A volatilities

We reason as follows. Due to the deterministic nature of D's, the time t-conditional variance of the "instantaneous increment" dS_t is

$$\mathrm{Var}_t(dS_t^i) = \mathrm{Var}_t[d(C^i(t) - D^i(t))] = \mathrm{Var}_t(dC^i(t)).$$

By assuming a lognormal-like dynamics for C^i, so that $\mathrm{var}_t(dC^i(t)) = \sigma_i^C(t)^2 C^i(t)^2 dt$, we immediately have

$$\sigma_i(t)^2 S^i(t)^2 dt = \sigma_i^C(t)^2 C^i(t)^2 dt,$$

leading to

$$\sigma_i(t)^2 = \frac{\sigma_i^C(t)^2 \; C^i(t)^2}{(C^i(t) - D^i(t))^2} \; .$$

Integrating $\sigma_i(t)^2$ between T_{k-1} and T_k, and defining $V_C^i(T_{k-1}, T_k)^2 = \int_{T_{k-1}}^{T_k} \sigma_i^C(t)^2 dt/\tau_k$, we obtain, by freezing stochastic processes at time 0:

$$V^i(T_{k-1}, T_k)^2 \approx \frac{V_C^i(T_{k-1}, T_k)^2 C^i(0)^2}{(C^i(0) - D^i(0))^2} \; .$$

A different approximation consists in replacing random variables with their expected values at time T_{k-1}:

$$\boxed{V^i(T_{k-1}, T_k)^2 \approx V_C^i(T_{k-1}, T_k)^2 \frac{e^{R(0,T_{k-1})T_{k-1}}(C^i(0) - D^i(0)) + D^i(T_{k-1})}{e^{2R(0,T_{k-1})T_{k-1}}(C^i(0) - D^i(0))^2}} \; .$$

This last formula is used to obtain the V's from the market observed V_C's. Indeed, it is the C asset volatilities V_C that are observed in the market, and the above formula provides us with the volatilities of their continuous parts S.

We can now obtain the integrated volatility of the continuous part $\bar{A}$ of the basket from the above V's via formula (6) with all q's and $\bar{q}$ set to 0:

$$\boxed{V_A(0, T_k)^2 = \frac{1}{T_k} \ln \left(\frac{\sum_{i,j} a_i a_j S_0^i S_0^j \exp[\rho_{i,j} \sum_{h=1}^k V^i(T_{h-1}, T_h) V^j(T_{h-1}, T_h) \tau_h]}{(\sum_{i=1}^n a_i S_0^i)^2} \right)} \; . \quad (12)$$

In-between average volatilities are obtained again as

$$V_A(T_{k-1}, T_k)^2 = \frac{T_k V_A(0, T_k)^2 - T_{k-1} V_A(0, T_{k-1})^2}{T_k - T_{k-1}} \; .$$

Now we can simulate $\bar{A}$ as before:

$$\bar{A}(T_k) = \bar{A}(T_{k-1}) \exp[(F(0; T_{k-1}, T_k) - \tfrac{1}{2} V_A(T_{k-1}, T_k)^2)\tau_k + V_A(T_{k-1}, T_k)\sqrt{\tau_k}\mathcal{N}(0,1)]. \quad (13)$$

4 Matching further moments: formulation via forward prices

The above results can be formulated in a more compact way by resorting to forward prices. Forward prices incorporate dividends and interest rates, with a considerable ease of notation.

Given a maturity T, the T-forward price at time t for an asset S is simply the strike K that makes the forward contract payoff $(S_T - K)$ fair at time t. We solve in K the equation

$$E_t[S_T - K] = 0,$$

thus obtaining trivially the forward price for S:

$$F_S(t,T) := K = E_t(S_T).$$

The process $F_S(t,T)$, being the t-conditional expectation of a fixed random variable, follows a martingale. It follows that if we model $F_S(t,T)$ via a diffusion, this will be a driftless one. When the canonical maturity is clear from the context, we will simply write $F(t)$ or F_t instead of $F(t,T)$. We denote by $F^i(t)$ the forward price for the asset S^i,

$$F^i(t) = F^i_t = E_t(S^i_T).$$

Given the martingale property of forward prices, we have

$$dF^i(t) = \sigma_i(t) F^i(t)\, dW^i_t, \quad i = 1\ldots, n,$$

$$\mathrm{Corr}(d\ln F^i(t), d\ln F^j(t)) \; = \rho_{i,j}$$

where it is easy to check that the σ's are the same we had for the S^i's, both in the continuous dividend case (where the S^i's are the underlying assets) and in the discrete dividend case (where the S^i's are the continuous parts of the underlying asset prices).

Now we can express expectations in terms of forward prices F^i instead of spot prices S^i. In fact, under any of the formulations for dividends, we can write the basket's m-th moment as

$$E_0\{(A_T)^m\} = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1}\cdots a_{i_m} F_0^{i_1}\cdots F_0^{i_m} \exp\left\{\sum_{k=1}^{m-1}\sum_{h=k+1}^{m} \rho_{i_k,i_h}\int_0^T \sigma_{i_k}(u)\sigma_{i_k}(u)du\right\},$$

so that in particular the first three moments are, respectively,

$$\begin{cases} E_0\{A_T\} = \sum_i a_i F_0^i \\ E_0\{A_T^2\} = \sum_{ij} a_i a_j F_0^i F_0^j \exp[\rho_{i,j}\int_0^T \sigma_i(s)\sigma_j(s)ds] \\ E_0\{A_T^3\} = \sum_{ijk} a_i a_j a_k F_0^i F_t^j F_0^k \exp[\rho_{i,j}\int_0^T \sigma_i(s)\sigma_j(s)ds + \rho_{i,k}\int_0^T \sigma_i(s)\sigma_k(s)ds \\ \qquad\qquad + \rho_{j,k}\int_0^T \sigma_j(s)\sigma_k(s)ds] \end{cases} \tag{14}$$

Consider once again a generic GBM, under the risk-neutral measure,

$$d\bar{B}(t) = \mu(t)\bar{B}(t)\,dt + \sigma_B(t)\bar{B}(t)\,dW_t. \tag{15}$$

We can easily match the first two "terminal" moments of this process with the corresponding basket ones, thus obtaining a case very similar to that of Section 3.2 with forwards F's replacing spots S's, so that all r's and q's are set to 0.

On the other hand, we might decide to match more than the first two moments. Notice that the m-th moment of our benchmark GBM $\bar{B}$ is easily computed as

$$E\{(\bar{B}(T))^m\} = (\bar{B}(0))^m \exp\left[\int_0^T \left(m\mu(s) + \frac{m(m-1)}{2}\sigma_B(s)^2\right) ds\right]. \tag{16}$$

Notice, in particular, that

$$\begin{aligned}
E\{\bar{B}(T)\} &= \bar{B}(0)\exp\left[\int_0^T \mu(s)ds\right] =: \xi, & (17)\\
E\{\bar{B}(T)^2\} &= \bar{B}(0)^2\exp\left[\int_0^T (2\mu(s)+\sigma_B(s)^2)ds\right] & (18)\\
&= E\{\bar{B}(T)\}^2\exp\left[\int_0^T \sigma_B(s)^2 ds\right] =: E\{\bar{B}(T)\}^2\alpha = \xi^2\alpha, &\\
E\{\bar{B}(T)^3\} &= \bar{B}(0)^3\exp\left[3\int_0^T (\mu(s)+\sigma_B(s)^2)ds\right] = (E\{\bar{B}(T)\})^3\alpha^3 = \xi^3\alpha^3. & (19)
\end{aligned}$$

If we wish to match more than just two moments of the actual distribution, we need to leave the lognormal terminal distribution for $\bar{B}$ to obtain something more general. In doing this, we need be to careful in order to preserve analytical tractability. One of the easiest ways out is the following "shifting" technique. Set

$$B(t) = \bar{B}(t) + \gamma(t), \quad \gamma(t) := \gamma\exp\left(\int_0^t \mu(s)\,ds\right),\ t \geq 0. \tag{20}$$

The new parameter γ represents a new degree of freedom that can be exploited to match the third moment. The corresponding dynamics is immediately

$$dB(t) = \mu(t)B(t)\,dt + \sigma_B(t)(B(t)-\gamma(t))\,dW_t, \tag{21}$$

so that the risk-neutral drift-rate μ is preserved. The distribution of B has a shifted lognormal density p_B, which is related to $\bar{B}$'s lognormal density $p_{\bar{B}}$ through

$$p_{B(t)}(x) = p_{\bar{B}(t)}(x-\gamma(t)),\ t \geq 0,\ x \geq \gamma(t).$$

We are primarily interested in the shift at terminal time T, so that we set $\lambda := \gamma(T)$.

Compute

$$\begin{cases} E\{B(T)\} = E\{\bar{B}(T)\} + \lambda \\ E\{B(T)^2\} = E\{\bar{B}(T)^2\} + 2\lambda E\{\bar{B}(T)\} + \lambda^2 \\ E\{B(T)^3\} = E\{\bar{B}(T)^3\} + 3\lambda^2 E\{\bar{B}(T)\} + 3\lambda E\{\bar{B}(T)^2\} + \lambda^3 \end{cases} \tag{22}$$

Now use the shorthand notations $m_i = E_0[A(T)^i]$ to denote the generic i-th moments of the true basket $A(T)$. Then the first-three-moments-matching procedure reads, in this context, taking into account (14) and (22,17,18,19), respectively,

$$\begin{cases} m_1 = \xi + \lambda \\ m_2 = \xi\alpha + 2\lambda\xi + \lambda^2 \\ m_3 = \xi^3\alpha^3 + 3\lambda^2\xi + 3\lambda\xi\alpha + \lambda^3. \end{cases} \tag{23}$$

Lengthy but straightforward calculations lead to the following equations, which can be used to determine ξ, α and λ from m_1, m_2 and m_3 through Eq. (23)

$$\begin{cases} \lambda = m_1 - \xi \\ \xi^2 = \frac{m_2 - m_1^2}{\alpha - 1} \\ ((\alpha-1)+3)(\alpha-1)^{\frac{1}{2}}(m_2 - m_1^2)^{\frac{3}{2}} + (m_1(3m_2 - 2m_1^2) - m_3) = 0 \end{cases} \tag{24}$$

The last equation has three (generally complex) solutions for α. We seek a real solution of the kind $\alpha > 1$, which corresponds to $\int_0^t \sigma_B(s)^2 ds > 0$. The last equation when cast in the form $x^3 + 3x + \beta = 0$ has a (unique) real solution given by

$$x = \frac{\left(-4\beta + 4\sqrt{4+\beta^2}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-4\beta + 4\sqrt{4+\beta^2}\right)^{\frac{1}{3}}}\ , \tag{25}$$

which is positive only when $\beta < 0$, *i.e.* when $m_3 > m_1(3m_2 - 2m_1^2)$ (since $m_2 > m_1^2$), or still in other terms, *when the original distribution is (highly) positively skewed.* This case commonly occurs empirically when dealing with baskets.

In the following, we propose a simple method for matching an arbitrary (but finite) number of moments of the basket's density at terminal date T.

4.1 The call option value in the three-moment matching procedure

From the definition of Eq. (20) the value at maturity T of the European call option with strike K on the basket is approximated by

$$X_T = [B(T) - K]^+ = [\bar{B}(T) - (K - \lambda)]^+ = [\bar{B}(T) - \kappa]^+, \tag{26}$$

with $\kappa = K - \lambda$, whose value at time 0 is

$$X_0 = P(0,T)E\{X_T\}, \tag{27}$$

with $P(0,T)$ denoting the discount factor for maturity T.

From the assumptions made on process $\bar{B}(t)$ we have

$$\begin{cases} \bar{B}(T) = \bar{B}(0)\exp\left[\int_0^T \left(\mu(s) - \frac{\sigma_B(s)^2}{2}\right) ds + \int_0^T \sigma_B(s)\, dW_s\right] \\ \bar{B}(0) = A_0 - \gamma \end{cases} \tag{28}$$

so that

$$p_{\bar{B}(T)}(y) = \frac{1}{y\sqrt{2\pi\int_0^T \sigma_B(s)^2 ds}} \exp\left\{-\frac{1}{2\int_0^T \sigma_B(s)^2}\left[\ln\left(\frac{y}{\bar{B}(0)}\right) - \int_0^T \left(\mu(s) - \frac{\sigma_B(s)^2}{2}\right) ds\right]^2\right\}.$$

Eq. (27) therefore becomes

$$X_0 = P(0,T)\int_0^\infty [y-\kappa]^+ p_{\bar{B}(T)}(y)\, dy. \tag{29}$$

Caution must be taken for the cases (not rare, though) when $\kappa < 0$: γ can in fact assume values comparable (and, often much greater than) the initial value of the basket A_0. In such a case, the option price is trivially computed, and is equal to

$$\begin{aligned} P(0,T)\left[\bar{B}(0)\exp\left(\int_0^T \mu(s)ds\right) - \kappa\right] &= P(0,T)\left[(A_0-\gamma)\exp\left(\int_0^T \mu(s)ds\right) - K + \lambda\right] \\ &= P(0,T)\left[A_0\exp\left(\int_0^T \mu(s)ds\right) - K\right]. \end{aligned} \tag{30}$$

The contract has thus lost its optionality and has become a forward contract.

For all other cases, the pricing argument is standard, and leads to the Black and Scholes formula for an option whose underlying is the geometric Brownian motion $\bar{B}$ and with shifted strike κ:

$$\begin{aligned} X_0 &= P(0,T)\left[F_0 N(d_1) - \kappa N(d_2)\right] \\ d_{1,2} &= \frac{\ln\left(\frac{F(0)}{\kappa}\right) \pm \int_0^T \sigma_B(s)^2/2\ ds}{\sqrt{\int_0^T \sigma_B(s)^2\ ds}} \end{aligned} \tag{31}$$

where we set $F_0 = E\{\bar{B}(T)\} = \bar{B}(0)\exp[\int_0^T \mu(s)ds]$.

5 Approximating the basket density via a lognormal mixture

In this section, we show how to approximate the density function of A_T by means of a mixtures of lognormal densities. This method turns out to be useful because even though the density of A_T is not explicitly known, its associated moments are.

As in Brigo and Mercurio (2000, 2001), let us consider the following mixture of ν lognormal densities:

$$f(x) = \sum_{j=1}^{\nu} \lambda_j \frac{1}{x v_j \sqrt{2\pi}} \exp\left\{ -\frac{1}{2v_j^2} [\ln(x) - m_j]^2 \right\},$$

where λ_j's, v_j's and m_j's are real constants with $\lambda_j \geq 0$, $v_j > 0$ and $\sum_{j=1}^{\nu} \lambda_j = 1$.

Since the moment generating function of a normal random variable $Z = \mathcal{N}(\mu, \sigma^2)$ is

$$\psi_Z(t) = E\left(e^{tZ}\right) = e^{t\mu + \frac{1}{2}t^2\sigma^2},$$

the moments of the density f are explicitly given by

$$\mathcal{M}_k = \int_0^{+\infty} x^k f(x) dx = \sum_{j=1}^{\nu} \lambda_j \, e^{k m_j + \frac{1}{2}k^2 v_j^2},$$

for any positive integer k.

Our method is based on matching the moments $\mathcal{M}_k$ and $M_k := E\{(A_t)^m\}$ for each $k \leq r$ (the r-th is the last moment we want to match). To this end, let us define:

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,\nu} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,\nu} \end{pmatrix} \qquad \Lambda := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_\nu \end{pmatrix} \qquad M := \begin{pmatrix} M_0 \\ \vdots \\ M_r \end{pmatrix}$$

where

$$a_{k,j} := e^{(k-1)m_j + \frac{1}{2}(k-1)^2 v_j^2}.$$

We then want to find λ_j's, m_j's and v_j's solving the following (constrained) system:

$$\begin{aligned} A\Lambda &= M \\ \lambda_j &\geq 0 \ \forall j = 1, \ldots, \nu \\ v_j &> 0 \ \forall j = 1, \ldots, \nu. \end{aligned} \tag{32}$$

Note that the constraint $\sum_{j=1}^{\nu} \lambda_j = 1$ has been inserted as first equation in the system (0-th moment matching).

The system (32) does not necessarily have a (unique) solution. Think for instance of the situation where we want to match a large number of moments just using few lognormal densities (large r and small ν). However, we should not forget we are free to suitably choose the number ν of lognormal densities.

A natural question arises now: how many lognormal densities shall we introduce? Of course, there is no definite answer, given also the degree of approximation one wants to achieve. One can start with a low value for ν (e.g. 2 or 3) and minimize the squared Euclidean norm $||A\Lambda - M||^2$ over all λ_j's, m_j and v_j's. If the resulting optimization error is not satisfactory enough (hopefully zero), one can then increase ν accordingly.

Remark 5.1. *Let us suppose that all m_j's and v_j's are fixed, so that (32) becomes a linear system in the variables λ_j's. A version of the Farkas lemma states that, for $\nu > 1$ and $r > 0$, the following are equivalent:*

1. *There exists some $\Lambda^* \in \mathbb{R}^n$, $\Lambda^* \geq 0$, such that $A\Lambda^* = M$;*
2. *There exists some $X \in \mathbb{R}^{r+1}$ such that $A^T X \leq 0$ and $M^T X > 0$ (upperscript T denotes transposition).*

Therefore, to prove 1., and hence to solve our system (32), it is enough to prove 2. Condition 2. however is not so straightforward to verify, so that other methods, like the above minimization, are normally required in practice.

A major advantage of a mixture of lognormal densities is that it immediately leads to closed form formulas for option prices. In fact, the price of a European call (put), with strike K and maturity T, implied by such a mixture is simply the mixture of the corresponding Black-Scholes call (put) prices:

$$P(0,T)\sum_{j=1}^{\nu}\lambda_j\omega\left[e^{m_j+\frac{1}{2}v_j^2}\Phi\left(\omega\frac{m_j-\ln(K)+v_j^2}{v_j}\right)-K\Phi\left(\omega\frac{m_j-\ln(K)}{v_j}\right)\right],$$

where $\omega = 1$ for a call and $\omega = -1$ for a put.

6 An empirical analysis of the two- and three-moment matching approximations

We have performed a thorough analysis of the approximations for the particular case of a basket comprised of two stocks in the Italian market (a realistic case): Fiat and Generali. The analysis consisted of *i)* plotting the "terminal" probability distribution of the basket price and *ii)* pricing a European call option on the basket. Maturity is approximately equal to five years (from Dec. 29^{th}, 2000 to Dec. 7^{th}, 2005). Today's date is Dec. 29^{th}, 2000 in the calculations. The Monte Carlo simulations are all based on 100,000 paths. The invariant quantities for the pricing are reported in Table 6. The option payoff formula is

$$X_T = N\left[\frac{\sum_{i=1}^n a_i S_T^i}{\sum_{i=1}^n a_i S_0^i} - 1\right]^+ \tag{33}$$

with N the nominal (conventionally set equal to 100).

We have systematically studied prices and probability distributions for a set of different values of *i)* individual stock price volatilities and *ii)* correlations among stock price returns. We can devise three correlation regimes: the highly positive, zero and highly negative correlation. Also for individual volatilities the high-high, high-low and low-low regimes are of interest.

Nominal	100	
Today	29-Dec-00	
Payoff Date	7-Dec-05	
Discount Factor At Payoff	0.783895779	
Strike	32.661	
Stock	FIAT	GENERALI
Weights	0.5956	0.4044
Spot Prices	26.3	42.03
Maturity Date	7-Dec-05	
Forward Price	31.68017702	51.57651408

Table 1: The inputs for the two-stock basket option

The basic conclusions as far as option pricing is concerned are enclosed in Table 2. They can be generally summed up as follows: for symmetric systems (*i.e.* when volatilities are roughly equal) and high correlation, the two approximations give the same price with good accuracy with respect to the "real" price. Matching three moments instead of two leads to generally better results especially in the case of highly asymmetric and/or negatively correlated systems (when the correlation equals one, both approximations converge to the same result for equal volatilities, due to the fact that the system is really following a geometric Brownian motion). The effect of matching an additional moment beyond the first two only is the more sizable, the more negative the correlation.

The quality of the approximations can be justified by examining Figs. 1–3 which show the probability distributions for the real basket (obtained through a Monte Carlo simulation, solid line), the two-moment matching (analytic distribution, dash-dotted line) and the three-moment matching one (analytic distribution, dashed line) for a few possible choices of the individual volatilities and of the correlation between stock price returns. Normally, negative correlations give rise to a highly peaked basket terminal distribution, which can only be approximated by the shifted lognormal distribution, the ordinary lognormal one being too smooth to adapt for the task. However, there are situations where matching up to three moments can still be insufficient for a reasonable approximation (see Fig. 1).[2]

Positive correlations instead allow for a good degree of approximation of the basket terminal distribution, as can be seen from Fig. 3.

The same conclusions roughly apply to the case of baskets of more than just two securities (see Figs. 4– 5).

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[2]In such a case, resorting to mixtures of lognormal densities for matching also higher moments can be quite helpful.

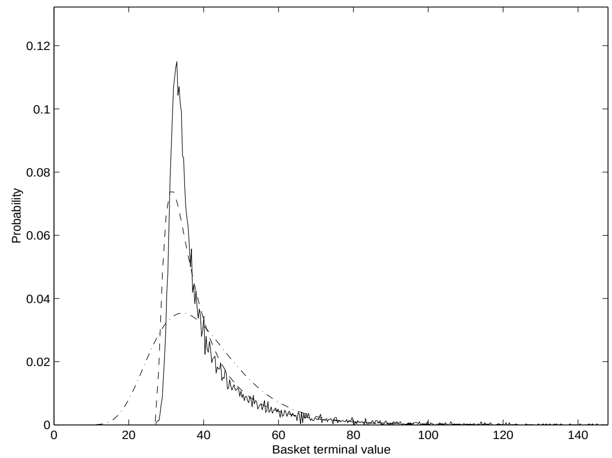

Figure 1: The probability distributions for the actual basket (solid), the two-moment matching procedure (dash-dotted), the three-moment matching procedure (dashed) when $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $\rho = -0.99$.

7 The Kullback Leibler information and the Hellinger distance

In this section we introduce briefly the Kullback-Leibler information and the Hellinger distance, and we explain their importance for our problem, see also Brigo and Hanzon (1998), Brigo (1999). Suppose we are given the space $\mathcal{D}$ of all the densities of probability measures on the real line equipped with its Borel field, which are absolutely continuous with respect to the Lebesgue measure. Functions in $\mathcal{D}$ belong to L^1, so that their square roots belong to L^2. Then define

$$K(p_1, p_2) := E_{p_1}\{\log p_1 - \log p_2\} \geq 0, \quad p_1, p_2 \in \mathcal{D}, \tag{34}$$

$$H(p_1, p_2) := \|\sqrt{p_1} - \sqrt{p_2}\|_2 \tag{35}$$

in $\mathcal{D}$, where $\|\cdot\|_2$ denotes the L_2 norm, and where in general

$$E_p\{\phi\} = \int \phi(x)p(x)\,dx.$$

The above quantities are respectively the well-known Kullback-Leibler information (KLI) and the Hellinger distance (HD). The KLI non-negativity follows from the Jensen inequality. The KLI gives a measure of how much the density p_2 is displaced with respect to the density p_1. We remark the important fact that K is not a distance: in order to be a metric, it should be symmetric and satisfy the triangular inequality, which is not the case. Instead, the HD is a real metric. However, the KLI features many properties of a distance in a generalized geometric setting (see for instance Amari (1985)). Notice finally that if p_2 vanishes in a measurable set of positive measure where p_1 does not vanish, the

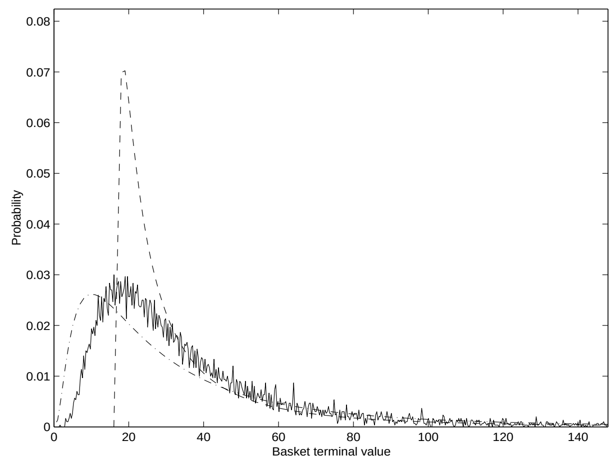

Figure 2: Same as in Fig. 1 for $\sigma_1 = 0.3$, $\sigma_2 = 0.6$, $\rho = 0$.

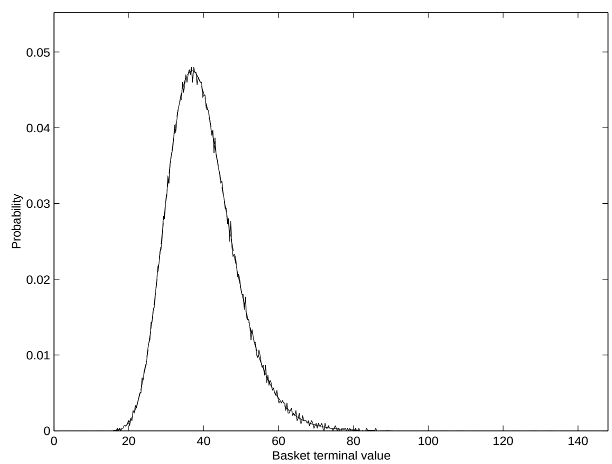

Figure 3: Same as in Fig. 1 for $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\rho = 0.99$.

KLI becomes infinite. This means that the KLI assigns infinite distance to densities with different support, contrary to the HD. For the Hellinger case, instead,

$$H(p_1, p_2) = \sqrt{2 - 2\int \sqrt{p_1(x)p_2(x)}\, dx}, \tag{36}$$

from which we see that the HD takes values in $[0, \sqrt{2}]$. It can be shown that this distance, when defined directly on measures rather than on densities, is independent of the particular basic measure with respect to which densities are expressed, as long as both measures whose distance is considered are absolutely continuous with respect to the basic measure. Since one can always find a basic measure with respect to which two given measures μ_1 and μ_2 are absolutely continuous (take for example $(\mu_1 + \mu_2)/2$), the distance is well defined on the set of all finite and positive measures on a given space $(\Omega, \mathcal{F})$, independently of the basic measure chosen.

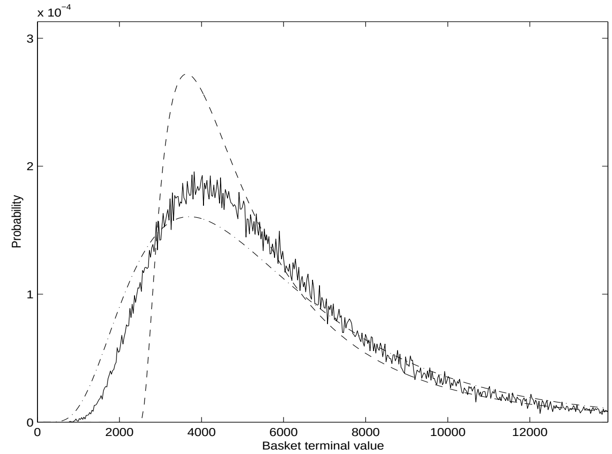

Figure 4: The probability distributions for the actual basket (solid), the two-moment matching procedure (dash-dotted), the three-moment matching procedure (dashed) for a three component basket when $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $\sigma_3 = 0.2$ and the stock-stock correlation is $\rho = -0.14$.

Consider now a finite dimensional manifold of exponential probability densities such as

$$\begin{aligned} EM(c) &= \{p(\cdot,\theta) : \theta \in \Theta \subset I\!R^m\}, \quad \Theta \text{ open in } I\!R^m, \\ p(\cdot,\theta) &= \exp[\theta_1 c_1(\cdot) + \ldots + \theta_m c_m(\cdot) - \psi(\theta)], \end{aligned} \tag{37}$$

expressed w.r.t. the expectation parameters η defined by

$$\eta_i(\theta) = E_{p(\cdot,\theta)}\{c_i\} = \partial_{\theta_i}\psi(\theta), \quad i = 1,\ldots,m \tag{38}$$

with ∂_z denoting partial derivative with respect to z (see for example Brigo (1999) or Brigo, Hanzon and Le Gland (1999) for more details). We define $p(x;\eta(\theta)) := p(x,\theta)$ (the semicolon/colon notation identifies the parameterization).

Now suppose we are given a density $p \in \mathcal{D}$, and we want to approximate it by a density of the finite dimensional manifold $EM(c)$.

It seems then reasonable to find a density $p(\cdot,\theta)$ in $EM(c)$ which minimizes the Kullback Leibler information $K(p,.)$. Compute

$$\begin{aligned} \min_\theta K(p,p(\cdot,\theta)) &= \min_\theta\{E_p[\log p - \log p(\cdot,\theta)]\} \\ &= E_p \log p - \max_\theta\{\theta_1 E_p c_1 + ... + \theta_m E_p c_m - \psi(\theta)\} \\ &= E_p \log p - \max_\theta V(\theta), \\ V(\theta) &:= \theta_1 E_p c_1 + \ldots + \theta_m E_p c_m - \psi(\theta). \end{aligned}$$

It follows immediately that a necessary condition for the minimum to be attained at θ^* is $\partial_{\theta_i} V(\theta^*) = 0, \quad i = 1,...,m$ which yields

$$E_p c_i - \partial_{\theta_i}\psi(\theta^*) = E_p c_i - E_{p(\cdot,\theta^*)} c_i = 0, \qquad i = 1,\ldots,m$$

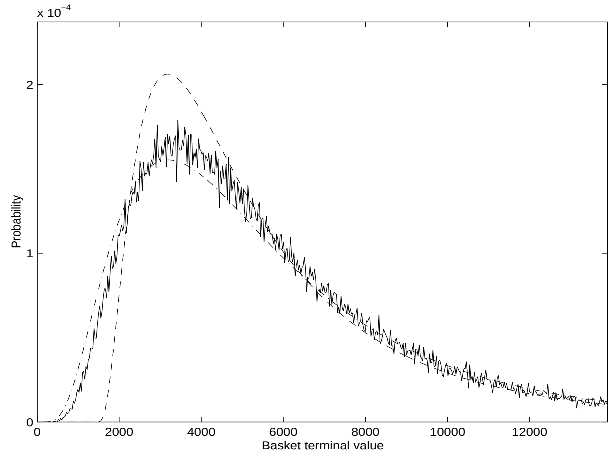

Figure 5: Same as in Fig. 4 for $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $\sigma_3 = 0.2$ and the stock-stock correlation is $\rho = 0.14$.

i.e. $E_p c_i = \eta_i(\theta^*), \quad i = 1, \ldots, m$. This last result indicates that according to the Kullback Leibler information, the best approximation of p in the manifold $EM(c)$ is given by the density of $EM(c)$ which shares the same c_i expectations (c_i-moments) as the given density p. This means that in order to approximate p we only need its c_i moments, $i = 1, 2, .., m$.

The above discussion provides also a way to compute the KLI distance of the density p from the exponential family $EM(c)$ as the distance between p and its projection $p(\cdot, \theta^*)$ onto $EM(c)$ in the KL sense. We have

$$\begin{aligned} K(p, EM(c)) &= E_p \log p \ - (\theta_1^* E_p c_1 + ... + \theta_m^* E_p c_m - \psi(\theta^*)) \\ &= E_p \log p \ - (\theta_1^* \eta_1(\theta^*) + ... + \theta_m^* \eta_m(\theta^*) - \psi(\theta^*)). \end{aligned} \tag{39}$$

One can look at the problem from the opposite point of view. Suppose we decide to approximate the density p by taking in account only its m c_i-moments. It can be proved (see Kagan et al. (1973), Theorem 13.2.1) that the maximum entropy distribution which shares the c-moments with the given p belongs to the family $EM(c)$.

Summarizing, if we decide to approximate by using c-moments, then entropy analysis supplies arguments to use the family $EM(c)$; and if we decide to use the approximating family $EM(c)$, Kullback-Leibler says that the "closest" approximating density in $EM(c)$ shares the c-moments with the given density. These nice characterizations are not shared by the HD, whose advantage over KLI is that of being a real metric and of giving finite distances for densities with different supports.

Finally, it is well-known that the KLI is infinitesimally equivalent to the Fisher information metric around every point of a finite-dimensional manifold of densities such as $EM(c)$ defined above. For this reason one refers to the KLI as to a "distance" even if it is not a metric. Indeed, consider the two densities $p(\cdot, \theta)$ and $p(\cdot, \theta + d\theta)$ of $EM(c)$. By expanding in Taylor series, we obtain easily

$$\begin{aligned} K(p(\cdot,\theta),p(\cdot,\theta+d\theta)) &= -\sum_{i=1}^{m} E_{p(\cdot,\theta)}\left\{\frac{\partial \log p(\cdot,\theta)}{\partial\theta_i}\right\} d\theta_i \\ &\qquad - \sum_{i,j=1}^{m} E_{p(\cdot,\theta)}\left\{\frac{\partial^2 \log p(\cdot,\theta)}{\partial\theta_i\partial\theta_j}\right\} d\theta_i\, d\theta_j + O(|d\theta|^3) \end{aligned}$$

which is the same expression we obtain by expanding the Hellinger distance

$$H(p(\cdot,\theta),p(\cdot,\theta+d\theta)).$$

8 Distance of the true basket distribution from the lognormal family of distributions and other numerical tests

Consider again the true basket terminal distribution at time T, coming from direct simulation of the single underlying assets. If we approximate the true basket by a terminal lognormal density, corresponding for example to the basket dynamics (15) (or to any other leading to a terminal lognormal distribution), then the question of computing the KLI distance between the true terminal distribution of $A(T)$ and the lognormal family becomes all that is relevant. Indeed, this distance expresses the best we can do by remaining in the lognormal family.

Just to recall the lognormal distribution in exponential class form, notice that the approximate basket dynamics (15) leads to

$$\bar{B}(t) = A_0 \exp\left[\int_0^t (\mu(u) - \tfrac{1}{2}\sigma_B^2(u))du + \int_0^t \sigma_B(u)dW_u\right],$$

so that

$$\log \bar{B}(t) \sim \mathcal{N}\left(\log A_0 + \int_0^t (\mu(u) - \tfrac{1}{2}\sigma_B^2(u))du, \int_0^t \sigma_B^2(u)du\right) \ . \tag{40}$$

The probability density $p_{\bar{B}(t)}$ of $\bar{B}(t)$, at any time t, is therefore given by

$$\begin{aligned} p_{\bar{B}(t)}(x) &= p(x,\theta(t)) = \exp\left\{\theta_1(t)\ln\frac{x}{A_0} + \theta_2(t)\ln^2\frac{x}{A_0} - \psi(\theta_1(t),\theta_2(t))\right\}, \\ \theta_1(t) &= \frac{\int_0^t \mu(u)\ du}{\int_0^t \sigma_B^2(u)du} - \frac{3}{2}, \quad \theta_2(t) = -\frac{1}{2\int_0^t \sigma_B^2(u)du}, \\ \psi(\theta_1(t),\theta_2(t)) &= -\frac{(\theta_1(t)+1)^2}{4\theta_2(t)} + \tfrac{1}{2}\ln\left(\frac{-\pi A_0^2}{\theta_2(t)}\right), \end{aligned}$$

where $x > 0$, and is clearly in the exponential class, with $c_1(x) = \ln(x/A_0)$, $c_2(x) = (c_1(x))^2$. We will denote by $\mathcal{L}$ the related exponential family $EM(c)$. As concerns the expectation parameters for this family, they are readily computed as follows:

$$\begin{aligned}
\eta_1 &= E_\theta \ln(x/A_0) = \partial_{\theta_1}\psi(\theta_1,\theta_2) = -\frac{\theta_1+1}{2\theta_2} \\
\eta_2 &= E_\theta \ln^2(x/A_0) = \partial_{\theta_2}\psi(\theta_1,\theta_2) = \left(\frac{\theta_1+1}{2\theta_2}\right)^2 - \frac{1}{2\theta_2}\ .
\end{aligned}$$

As for the Gaussian family, in this particular family the θ parameters can be computed back from the η parameters by inverting the above formulas:

$$\begin{aligned}
\theta_1 &= \frac{\eta_1}{\eta_2-\eta_1^2} - 1\ , \qquad (41)\\
\theta_2 &= -\frac{1}{2(\eta_2-\eta_1^2)}\ ,\\
\psi(\theta_1,\theta_2) &= \tfrac{1}{2}\left[\frac{\eta_1^2}{\eta_2-\eta_1^2} + \ln(2\pi(\eta_2-\eta_1^2)A_0^2)\right]\ .
\end{aligned}$$

We can now compute the distance of a density p from the lognormal family $\mathcal{L}$ by applying formula (39):

$$K(p,\mathcal{L}) = E_p \ln p\ - (\theta_1^*\eta_1(\theta^*) + \theta_2^*\eta_2(\theta^*) - \psi(\theta^*)),$$

where, as previously seen, minimizing the distance implies finding the parameters θ^* such that

$$\eta_1(\theta^*) = E_p \ln(x/A_0), \quad \eta_2(\theta^*) = E_p \ln^2(x/A_0)\ .$$

By substituting (41), omitting the argument θ^* and simplifying, we obtain

$$K(p,\mathcal{L}) = E_p \ln p\ + \tfrac{1}{2} + \eta_1 + \tfrac{1}{2}\ln(2\pi(\eta_2-\eta_1^2)A_0^2)\ . \qquad (42)$$

This distance is readily computed with no need of optimization procedures, once one has an estimate of the true basket density and of its first two *log*-moments. Notice, indeed, an important point.

Remark 8.1 (Moments versus log-moments). *The best moment-matching technique in the KLI sense is obtained by matching the first two log-moments of the true distribution, instead of the first two usual moments. However, we do not have an analytical formula for the log-moments of the true basket, as opposed to the usual moments, so that one prefers to match the usual moments instead of what KLI would suggest.*

It is interesting to measure numerically the loss in terms of KLI induced by this “wrong moments” choice. This is easily feasible, as measuring the distance between two lognormal densities, which has an easy closed form expression. The only problem is that the basket log moments have to be computed through simulation. Then, for consistency, we may compute the usual moments by the same simulations and compare the distance between the two related lognormal densities to the distance of the basket from the family of lognormal densities. In this way, we can understand whether the wrong moments choice worsens considerably the approximation or not.

Given that the KLI is not a real metric as opposed to the HD, and since, once one leaves exponential families, its appeal diminishes considerably, we reason as follows. As far as the shifted lognormal dynamics (21) is concerned, its shifted lognormal distribution is not in an exponential class if we let the shifting parameter be free. It follows that interpreting the KLI projection as (suitable) moments matching no longer applies. Moreover, the shifting technique changes the support of the distribution, and this aspect renders the Hellinger distance again preferable, as discussed in Section 7.

We then start with a non-shifted lognormal approximating basket dynamics, by simulating the basic assets as correlated geometric Brownian motions, and present several plots of the KLI distance between the two moments approximation and the family of lognormal densities, as a function of volatility and correlation of the single assets forming the basket. We try and isolate what are the possible causes of large distances from the lognormal family. We answer questions such as: i) is asymmetry in the single assets volatilities and/or initial values good or bad for the KLI? ii) are negative correlations better or worse than positive or mixed ones? iii) are large correlations better or worse than small ones?

We already answered these questions for the specific pricing problem. We are now trying to justify what already seen by analyzing the distributional properties, instead.

Then we check the impact of the wrong moments choice on our approximation.

We do some of the same measures numerically with the HD and compare results with the KLI. Given that the two distances are infinitesimally equivalent, we expect them not to differ too much when densities are close.

Then, we move to the shifted lognormal dynamics. In doing this we leave the KLI and fully switch to the HD. We measure the distances between the true basket, two moments and three moments approximations as functions of the underlying assets parameters. We try and characterize situations where the two moments matching procedure does better or worse than the three moments one when compared to the true distribution, and also compute the distance between the two approximations themselves.

We first need to assess a number of facts affecting the accuracy of our calculation of distances. The question we need to find an answer to is: is the number of Monte Carlo paths sufficient for the accuracy of the calculation?

A brief description of the procedure follows. We sampled the "real" terminal distribution at time T of the basket values through a simulation consisting of one million paths for each configuration of individual volatilities and correlations. By "real" we mean the distribution of the basket under the Black-Scholes assumptions we made at the very beginning, but where *no* use of approximations of any sort was ever made. The terminal distribution at time T of the basket values was then histogrammed in bins, each of unitary width. A similar procedure was then applied to sample both the two-moment and the three-moment matching distributions.

Given the three histograms, the two distances of interest (KLD, HD) were calculated through a numerical integration for each couple (real basket versus two-moments matched, real basket versus three-moments matched, two-moments matched versus three-moments matched). Due to the fact that by construction, the support for the three-moment matching distribution does not equal $[0, \infty)$ as for the two-moment matching one, the convention we used is the following: the three-moment matching distribution is taken as a reference for the computation of the distances between the two (in other words, it plays the role of distribution p_1 in Eq. (34)).

The Kullback-Leibler distances from the "real" basket distribution were calculated using the latter as a reference. To prevent numerical divergences in the calculation of the Kullback-Leibler distance when the two distributions do not have the same support, a conventional value of 10^{-8} replaces zero in the non reference distribution.

Distances obtained for two- versus three-moment matching distributions were compared with a numerical integration of the analytic distributions, performed using Simpson's rule [10]. This allowed us to check the accuracy of calculating distances through Monte Carlo histograms. Some results, covering a significant range of configurations of correlation and individual volatilities, are reported in Table 3. The accuracy in the calculation of distances is generally acceptable, being typically of the order of less than one percent. This is judged to be sufficient for a qualitative information to be gathered from the data. Normally, low accuracy results from basket terminal distributions displaying a high degree of kurtosis.

Tables 4–6 show the actual numbers computed for the three values of the correlation among stock price returns, $\rho = -0.99, 0, 0.99$, for different values of individual volatilities. Some more insight can be gained from inspection of Figs. 6–14, in which the two distances under examination are viewed as functions of the individual stock price volatilities for extreme degrees of correlation. One thing is immediately clear: the two distances have qualitatively the same behaviour as functions of the stock volatilities. In fact, the shape of both surfaces is typically that of a valley running along the diagonal in the (σ_1, σ_2) plane. The broadness of the valley increases when passing from the fully anticorrelated (Figs. 6–8) to the fully correlated case (Figs. 12–14). This valley is also monotonically increasing along the diagonal.

However, some more features can be noticed: in the anticorrelated case the distance between the two different moment matching distributions (where, as stated above, the three-moment matching one is the reference) can be seen to be increasing with asymmetry (*i.e.*, when moving away from the diagonal) up to a limiting value, and then decreasing again. This is due to the fact that, when the two volatilities become very different, the basket closely resembles a one-dimensional system affected mostly by the dominant volatility, and both approximations converge to the same terminal distribution. This feature gradually disappears when correlation increases to 0 and 0.99: the valley becomes broad and flat (in other words, both approximations are distributionally close to the real one for not-too-asymmetric systems) and then steeply increases away from the diagonal.

The distances of the two moment matching distributions from the "real" basket terminal distribution are instead monotonically increasing functions of the asymmetry of the system, and only in the anticorrelated case suggest a plateau for high asymmetry (see Fig. 7). The three-moment to real distance is instead always quite structured, with a clear maximum (in the range spanned by our simulations) for asymmetric volatilities of the order of (30%,60%).

Whenever the system gets either more symmetric or more asymmetric, distances clearly decrease.

As a general result, we could state that the two-moment matching procedure performs quite well for nearly symmetric systems, especially when the correlation is not maximally negative; switching to a shifted lognormal distribution that matches the first three (instead of two) moments gives best results for highly asymmetric cases, when either of the two volatilities is greater than the other. This result is confirmed by a close inspection of the contour plots (Figs. 15–23).

9 Conclusions

Through a numerical study of a sample case (a basket composed of two stocks), we have tested the quality of two different approximations under various conditions. The study has been developed on two levels: the comparison of option prices and the analysis of probability distributions. The latter analysis has been based on the calculation of two "distances", the Kullback-Leibler information and the Hellinger distance, which, although of different nature, give similar results as far as qualitative information is concerned.

The answers to the questions posed in the preceding section are well represented by the graphs drawn: the approximation of reproducing a basket by a family of lognormal distributions, in terms of terminal distributions, breaks down in general when the system becomes asymmetric. A crucial quantity for this is the correlation among basket components: negative correlations worsen considerably the quality of both moment-matching approximations. To the decrease in quality there corresponds a significant change in the behaviour of the approximations, though: for negative correlations the performance of the three-moment matching approximation is better than the other, and generally this applies also to the case of highly asymmetric volatilities.

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σ_{FIAT}	σ_{GEN}	ρ	MC price	three-mom price	two-mom price	three-mom index vol.	three-mom Fwd.Index	Index Shift
0.1	0.1	-0.99	18.19(2)	23.872	18.504	0.179368697	37.27655382	2.449701908
0.1	0.1	-0.6	18.254(2)	18.249	18.568	0.056535864	32.75296506	6.973290669
0.1	0.1	-0.2	18.572(5)	18.567	18.861	0.065772043	38.68295302	1.043302709
0.1	0.1	0	18.78(7)	19.012	19.059	0.419742958	39.30651277	0.419742958
0.1	0.1	0.2	19.005(8)	19.003	19.275	0.078092443	39.56436475	0.161890976
0.1	0.1	0.6	19.48(10)	19.479	19.735	0.089558901	39.71135159	0.014904135
0.1	0.1	0.99	19.95(1)	20.198	20.198	0.099750353	39.72625301	2.72056E-06
0.1	0.3	-0.99	18.566(3)	21.589	22.235	0.364860528	12.87454865	26.85170708
0.1	0.3	-0.6	20.793(14)	20.717	23.208	0.320613608	17.15123272	22.57502301
0.1	0.3	-0.2	22.321(19)	22.151	24.177	0.291209254	21.41426579	18.31198993
0.1	0.3	0	22.99(2)	22.827	24.650	0.281297023	23.38643352	16.33982221
0.1	0.3	0.2	23.61(2)	23.473	25.118	0.273932173	25.21356364	14.51269209
0.1	0.3	0.6	24.76(2)	24.685	26.035	0.265220441	28.38539664	11.34085909
0.1	0.3	0.99	25.834(18)	25.781	26.909	0.262403165	30.86533776	8.86091797
0.1	0.6	-0.99	27.052(14)	29.699	40.121	0.612769846	19.05988101	20.66637472
0.1	0.6	-0.6	28.69(3)	29.521	40.457	0.607291631	19.79134873	19.934907
0.1	0.6	-0.2	30.30(4)	30.318	40.831	0.601430756	20.61876782	19.1074879
0.1	0.6	0	31.05(4)	31.312	41.030	0.59842992	21.06275597	18.66349976
0.1	0.6	0.2	31.77(4)	31.188	41.236	0.595396983	21.52721347	18.19904226
0.1	0.6	0.6	33.14(3)	32.132	41.671	0.589290058	22.51742826	17.20882747
0.1	0.6	0.99	34.45(2)	33.371	42.127	0.583388957	23.55947295	16.16678278
0.3	0.1	-0.99	18.206(1)	21.474	21.177	0.377708169	10.55236604	29.17388969
0.3	0.1	-0.6	20.092(12)	19.859	22.240	0.319236587	15.35999423	24.3662615
0.3	0.1	-0.2	21.600(17)	21.348	23.294	0.283904276	20.1392719	19.58698383
0.3	0.1	0	22.269(19)	22.051	23.807	0.272945917	22.30634833	17.4199074
0.3	0.1	0.2	22.90(2)	22.720	24.311	0.265279588	24.27803033	15.44822539
0.3	0.1	0.6	24.10(2)	23.969	25.296	0.257197224	27.60629099	12.11996474
0.3	0.1	0.99	25.189(17)	25.092	26.228	0.255639835	30.12248874	9.603766989
0.3	0.3	-0.99	18.732(4)	21.650	22.447	0.372353145	12.86049241	26.86576332
0.3	0.3	-0.6	23.35(2)	22.973	24.591	0.268745727	24.561214	15.16504173
0.3	0.3	-0.2	26.20(3)	26.064	26.743	0.237001481	34.28681779	5.439437939
0.3	0.3	0	27.44(3)	27.375	27.806	0.238226287	36.9960551	2.730200625
0.3	0.3	0.2	28.60(4)	28.584	28.861	0.245733145	38.53433843	1.191917299
0.3	0.3	0.6	30.78(4)	30.811	30.952	0.270522262	39.61697148	0.109284249
0.3	0.3	0.99	32.83(5)	32.968	32.968	0.299251811	39.72625075	4.97996E-06
0.3	0.6	-0.99	27.36(2)	28.101	39.737	0.61767253	18.34476252	21.38149321
0.3	0.6	-0.6	31.17(4)	29.502	40.384	0.606049429	19.80184924	19.92440649
0.3	0.6	-0.2	34.36(5)	31.443	41.257	0.591544263	21.84904686	17.87720887
0.3	0.6	0	35.79(5)	32.931	41.785	0.583592113	23.11631831	16.60993742
0.3	0.6	0.2	37.15(5)	33.974	42.379	0.575528206	24.55104967	15.17520606
0.3	0.6	0.6	39.77(5)	37.079	43.780	0.560962997	27.84712848	11.87912724
0.3	0.6	0.99	42.37(4)	40.487	45.421	0.552874339	31.28074401	8.445511719
0.6	0.1	-0.99	24.467(15)	27.716	38.098	0.615307514	16.93041852	22.79583721
0.6	0.1	-0.6	26.48(3)	27.211	38.508	0.608519153	17.73960256	21.98665317
0.6	0.1	-0.2	28.29(3)	28.085	38.962	0.601319285	18.65499651	21.07125922
0.6	0.1	0	29.11(3)	29.424	39.201	0.597660061	19.14599994	20.58025579
0.6	0.1	0.2	29.88(3)	29.039	39.449	0.593981906	19.65935482	20.06690091
0.6	0.1	0.6	31.35(3)	30.071	39.970	0.586643391	20.7523157	18.97394002
0.6	0.1	0.99	32.67(2)	31.151	40.511	0.579650096	21.8992758	17.82697992
0.6	0.3	-0.99	26.12(2)	26.118	37.869	0.616911806	16.62363281	23.10262291
0.6	0.3	-0.6	30.20(4)	27.656	38.644	0.602715533	18.24538251	21.48087322
0.6	0.3	-0.2	33.43(5)	29.780	39.676	0.585341296	20.52205794	19.20419779
0.6	0.3	0	34.86(5)	31.074	40.294	0.576029241	21.9253794	17.80087632
0.6	0.3	0.2	36.22(5)	32.524	40.985	0.566797952	23.50394931	16.22230642
0.6	0.3	0.6	38.85(5)	35.829	42.591	0.551009385	27.06642078	12.65983495
0.6	0.3	0.99	41.40(4)	39.369	44.440	0.543725695	30.64441364	9.081842089
0.6	0.6	-0.99	36.96(4)	37.217	44.838	0.583032219	27.26094977	12.46530596
0.6	0.6	-0.6	39.80(6)	38.494	45.219	0.575437673	28.58147336	11.14478237
0.6	0.6	-0.2	43.06(7)	40.756	45.960	0.562767476	31.10305642	8.62319931
0.6	0.6	0	44.63(7)	42.162	46.534	0.555250316	32.94721857	6.779037159
0.6	0.6	0.2	46.18(7)	44.460	47.298	0.548988389	35.10230883	4.623946893
0.6	0.6	0.6	49.27(8)	48.914	49.500	0.554653025	38.8624482	0.863807527
0.6	0.6	0.99	52.57(8)	52.585	52.585	0.598508754	39.72623077	2.49572E-05

Table 2: Prices of call options on the two–stock basket outlined in Table 6. The first three columns give the individual volatilities of the stocks and their instantaneous correlation, the fourth gives the Monte Carlo price in a Black–Scholes framework for the single assets (statistical uncertainty on the estimate is given in parentheses). The two subsequent columns give the price of the option calculated with the three– and the two–moment matching procedures, respectively.

rho	v1	v2	KLD23	H23	aKLD23	aH23	rel.err. KLD23	rel.err. H23
-0.99	0.1	0.1	0.218838	0.131945	0.241505	0.142837	-0.094	-0.076
-0.99	0.1	0.2	0.222958	0.15232	0.225789	0.154711	-0.013	-0.015
-0.99	0.1	0.3	0.263829	0.194942	0.263915	0.196539	-0.00033	-0.0083
-0.99	0.2	0.1	0.278072	0.185579	0.285108	0.189641	-0.025	-0.021
-0.99	0.2	0.2	0.269175	0.182059	0.274568	0.185373	-0.020	-0.018
-0.99	0.3	0.1	0.313993	0.226037	0.315621	0.22846	-0.0052	-0.011
-0.99	0.3	0.2	0.354847	0.250804	0.357976	0.253975	-0.0087	-0.012
-0.99	0.3	0.3	0.27737	0.205057	0.277309	0.20661	0.00022	-0.0075
0	0.1	0.1	2.03548e-005	9.54271e-006	2.49846e-006	1.3e-006	7.1	6.6
0	0.1	0.2	0.0125227	0.00744924	0.01208	0.00728964	0.037	0.022
0	0.1	0.3	0.0758187	0.0563726	0.0750839	0.0564204	0.0098	-0.00085
0	0.2	0.1	0.00986339	0.0057038	0.00944828	0.00556002	0.044	0.026
0	0.2	0.2	0.000447922	0.000213649	0.000154143	7.92155e-005	1.9	1.7
0	0.2	0.3	0.0208195	0.0138063	0.0200611	0.0134852	0.038	0.024
0	0.2	0.4	0.131393	0.105765	0.130439	0.106207	0.0073	-0.0042
0	0.3	0.1	0.0765328	0.0563849	0.0759533	0.0564474	0.0076	-0.0011
0	0.3	0.2	0.0135796	0.00850388	0.0128723	0.00823717	0.055	0.032
0	0.3	0.3	0.00324979	0.00175223	0.00248637	0.00142438	0.31	0.23
0	0.3	0.4	0.052081	0.03988	0.0508833	0.0395785	0.024	0.0076
0	0.4	0.1	0.214642	0.170028	0.213671	0.170692	0.0045	-0.0039
0	0.4	0.2	0.121511	0.0968438	0.120635	0.0972102	0.0073	-0.0038
0	0.4	0.3	0.0360245	0.0263782	0.03479	0.0259586	0.035	0.016
0	0.4	0.4	0.0200428	0.013843	0.0185312	0.01323	0.082	0.046
0.99	0.1	0.2	0.00247715	0.00131638	0.00204039	0.00112665	0.21	0.17
0.99	0.1	0.3	0.0261437	0.0179323	0.0252194	0.0175598	0.037	0.021
0.99	0.1	0.4	0.105989	0.0852947	0.104825	0.0855886	0.011	-0.0034
0.99	0.2	0.1	0.00248854	0.00133487	0.00211696	0.00116846	0.18	0.14
0.99	0.2	0.3	0.00296912	0.00158455	0.00207561	0.00118835	0.43	0.33
0.99	0.2	0.4	0.0341034	0.025326	0.0326595	0.0248244	0.044	0.020
0.99	0.3	0.1	0.0273298	0.0187098	0.0265303	0.0184368	0.030	0.015
0.99	0.3	0.2	0.00288854	0.00154994	0.00207492	0.00118583	0.39	0.31
0.99	0.3	0.3	3.91215e-006	1.17163e-006	1.7e-10	3.2e-010	23011.6	3700.56
0.99	0.3	0.4	0.00569767	0.00314874	0.00372715	0.00232535	0.53	0.35
0.99	0.4	0.1	0.112083	0.0899206	0.111135	0.0902053	0.0085	-0.0032
0.99	0.4	0.2	0.034608	0.0256035	0.0330881	0.02504	0.046	0.023
0.99	0.4	0.3	0.00546638	0.0030425	0.00366671	0.00227812	0.49	0.34

Table 3: A check of the numerical procedure used to calculate the distances among distributions through Monte Carlo: for different correlations and individual volatilities, we report in columns four and five the Monte-Carlo-calculated distances, in columns six and seven the same quantities calculated through numerical integration, in the last two columns the percentage errors in the calculation.

rho	v1	v2	KLD23	H23	KLD2b	H2b	KLD3b	H3b
-0.99	0.1	0.1	0.218838	0.131945	0.205638	0.103334	0.0466358	0.0122691
-0.99	0.1	0.2	0.222958	0.15232	0.378935	0.237993	0.0815244	0.0498708
-0.99	0.1	0.3	0.263829	0.194942	0.488671	0.319677	0.150626	0.103711
-0.99	0.1	0.4	0.364353	0.276284	0.651024	0.423278	0.226744	0.161889
-0.99	0.1	0.5	0.52253	0.389967	0.859131	0.543759	0.310822	0.223349
-0.99	0.1	0.6	0.747401	0.534888	1.10166	0.674223	0.39278	0.282593
-0.99	0.2	0.1	0.278072	0.185579	0.413961	0.249738	0.0630203	0.0349268
-0.99	0.2	0.2	0.269175	0.182059	0.345933	0.207495	0.0369139	0.0176367
-0.99	0.2	0.3	0.344408	0.247488	0.475467	0.303794	0.074178	0.0414038
-0.99	0.2	0.4	0.440804	0.324966	0.620168	0.405165	0.135779	0.0832207
-0.99	0.2	0.5	0.5845	0.425442	0.780567	0.510204	0.190715	0.12302
-0.99	0.2	0.6	0.79381	0.555048	0.95729	0.620269	0.224679	0.146796
-0.99	0.3	0.1	0.313993	0.226037	0.543327	0.344283	0.151778	0.0994899
-0.99	0.3	0.2	0.354847	0.250804	0.442019	0.279974	0.0520929	0.0250491
-0.99	0.3	0.3	0.27737	0.205057	0.413571	0.269918	0.0749936	0.0427038
-0.99	0.3	0.4	0.38355	0.288381	0.49624	0.333911	0.072609	0.0402728
-0.99	0.3	0.5	0.558177	0.410284	0.62032	0.424071	0.0671154	0.0325614
-0.99	0.3	0.6	0.779854	0.548832	0.76553	0.524921	0.11671	0.0286924
-0.99	0.4	0.1	0.415552	0.30643	0.707351	0.447946	0.232912	0.161783
-0.99	0.4	0.2	0.479655	0.34425	0.593016	0.386193	0.0839096	0.0466189
-0.99	0.4	0.3	0.367315	0.275285	0.466945	0.313785	0.0604056	0.0323824
-0.99	0.4	0.4	0.298797	0.234079	0.460432	0.31628	0.108052	0.0663061
-0.99	0.4	0.5	0.439902	0.339211	0.522838	0.366877	0.0596868	0.0317828
-0.99	0.4	0.6	0.697456	0.50669	0.625387	0.44507	0.353023	0.0328962
-0.99	0.5	0.1	0.579246	0.421093	0.912306	0.566172	0.314723	0.222994
-0.99	0.5	0.2	0.638226	0.45162	0.757402	0.495773	0.123541	0.0726772
-0.99	0.5	0.3	0.571969	0.416126	0.579043	0.397233	0.0911699	0.0182084
-0.99	0.5	0.4	0.412474	0.320184	0.498426	0.349886	0.0574602	0.0309544
-0.99	0.5	0.5	0.345918	0.279226	0.496176	0.354114	0.113617	0.0713113
-0.99	0.5	0.6	0.541518	0.417485	0.546168	0.396514	0.0932755	0.0157356
-0.99	0.6	0.1	0.811753	0.562102	1.15016	0.69391	0.389074	0.277442
-0.99	0.6	0.2	0.86123	0.588191	0.937644	0.609372	0.155307	0.0880039
-0.99	0.6	0.3	0.829726	0.572337	0.721927	0.498392	0.377364	0.0439781
-0.99	0.6	0.4	0.700628	0.505369	0.58638	0.418798	0.673677	0.0590863
-0.99	0.6	0.5	0.506447	0.394764	0.525409	0.381653	0.0774958	0.0155701
-0.99	0.6	0.6	0.427892	0.345799	0.521773	0.383757	0.0775218	0.04483

Table 4: Distances between distributions for correlation equal to -0.99, for different configurations of the individual volatilities: prefixes "KLD" and "H" denote the Kullback-Leibler and Hellinger distances, respectively; suffixes "23", "2b" and "3b" denote distances between two- and three-moment matching distributions, two-moment matching and simulated basket distribution, three-moment and simulated basket distributions, respectively.

rho	v1	v2	KLD23	H23	KLD2b	H2b	KLD3b	H3b
0	0.1	0.1	2.03548e-005	9.54271e-006	0.000140322	6.86961e-005	0.000146713	7.11875e-005
0	0.1	0.2	0.0125227	0.00744924	0.0071684	0.00358477	0.00385037	0.00133796
0	0.1	0.3	0.0758187	0.0563726	0.0408127	0.0239167	0.0572321	0.0104558
0	0.1	0.4	0.200258	0.160586	0.111815	0.0732579	0.266938	0.0331652
0	0.1	0.5	0.390454	0.307333	0.227022	0.158684	0.686602	0.0706529
0	0.1	0.6	0.647891	0.481056	0.387918	0.277853	1.27488	0.125002
0	0.2	0.1	0.00986339	0.0057038	0.00537269	0.00262088	0.00327326	0.00123624
0	0.2	0.2	0.000447922	0.000213649	0.000397248	0.000178155	0.000452666	0.000201092
0	0.2	0.3	0.0208195	0.0138063	0.00554336	0.00269605	0.0243588	0.0060096
0	0.2	0.4	0.131393	0.105765	0.0330267	0.0184525	0.392242	0.0477799
0	0.2	0.5	0.337531	0.270964	0.0977425	0.0606575	1.24246	0.125243
0	0.2	0.6	0.614648	0.459699	0.205713	0.137447	2.18084	0.213401
0	0.3	0.1	0.0765328	0.0563849	0.0373632	0.0212218	0.0688288	0.0126113
0	0.3	0.2	0.0135796	0.00850388	0.00352887	0.00163265	0.0130374	0.0040226
0	0.3	0.3	0.00324979	0.00175223	0.00139718	0.000598234	0.00263712	0.0010667
0	0.3	0.4	0.052081	0.03988	0.00714095	0.00351836	0.156625	0.0239217
0	0.3	0.5	0.246209	0.20385	0.0338496	0.0188801	1.25359	0.126827
0	0.3	0.6	0.55169	0.423695	0.097609	0.0597938	2.66157	0.261888
0	0.4	0.1	0.214642	0.170028	0.1084	0.0691868	0.370033	0.0421094
0	0.4	0.2	0.121511	0.0968438	0.0257942	0.0139214	0.386015	0.0482085
0	0.4	0.3	0.0360245	0.0263782	0.00475006	0.00224611	0.0906869	0.0164131
0	0.4	0.4	0.0200428	0.013843	0.00337574	0.00154235	0.0396564	0.00870625
0	0.4	0.5	0.133688	0.112584	0.0110779	0.00568262	0.749945	0.0792556
0	0.4	0.6	0.443227	0.354811	0.0409504	0.0231643	2.66987	0.256253
0	0.5	0.1	0.421459	0.325621	0.225561	0.154442	0.885229	0.0878191
0	0.5	0.2	0.347493	0.2761	0.0851904	0.0516173	1.42164	0.140675
0	0.5	0.3	0.22827	0.188721	0.0253467	0.0137065	1.24387	0.124694
0	0.5	0.4	0.103791	0.0864284	0.00811102	0.00403361	0.542399	0.0620074
0	0.5	0.5	0.0779295	0.0647728	0.00729799	0.00356486	0.39218	0.0454111
0	0.5	0.6	0.292278	0.246853	0.0181572	0.00961736	2.02903	0.18953
0	0.6	0.1	0.699886	0.506027	0.390641	0.275477	1.48501	0.149327
0	0.6	0.2	0.651246	0.480124	0.191427	0.125504	2.53976	0.245299
0	0.6	0.3	0.563175	0.427771	0.0819161	0.0491097	2.89708	0.281373
0	0.6	0.4	0.422827	0.338669	0.0311138	0.0171874	2.66881	0.254568
0	0.6	0.5	0.249081	0.212021	0.0142666	0.00737331	1.77329	0.164523
0	0.6	0.6	0.203063	0.176558	0.0138126	0.00708588	1.49339	0.133764

Table 5: Distances between distributions for zero correlation, for different configurations of the individual volatilities: prefixes "KLD" and "H" denote the Kullback-Leibler and Hellinger distances, respectively; suffixes "23", "2b" and "3b" denote distances between two- and three-moment matching distributions, two-moment matching and simulated basket distribution, three-moment and simulated basket distributions, respectively.

rho	v1	v2	KLD23	H23	KLD2b	H2b	KLD3b	H3b
0.99	0.1	0.1	0	0	0.000107971	5.11245e-005	0.000107971	5.11245e-005
0.99	0.1	0.2	0.00247715	0.00131638	0.00180259	0.000887342	0.000506053	0.000225129
0.99	0.1	0.3	0.0261437	0.0179323	0.0151956	0.00878737	0.00870935	0.00240444
0.99	0.1	0.4	0.105989	0.0852947	0.0554089	0.0367852	0.1053	0.0165266
0.99	0.1	0.5	0.273531	0.225125	0.136948	0.0993227	0.581213	0.0615275
0.99	0.1	0.6	0.53764	0.416669	0.268315	0.202482	1.67602	0.146105
0.99	0.2	0.1	0.00248854	0.00133487	0.00173617	0.000876315	0.000566874	0.000248296
0.99	0.2	0.2	1.34346e-010	6.71732e-011	0.000401436	0.00018961	0.000401448	0.000189616
0.99	0.2	0.3	0.00296912	0.00158455	0.00160406	0.000755162	0.00164385	0.000646363
0.99	0.2	0.4	0.0341034	0.025326	0.0111926	0.00610658	0.0452897	0.0086098
0.99	0.2	0.5	0.151168	0.128071	0.0433732	0.0269546	0.489087	0.0538192
0.99	0.2	0.6	0.396959	0.323941	0.112782	0.0771913	1.7243	0.162367
0.99	0.3	0.1	0.0273298	0.0187098	0.0156094	0.00898144	0.00856241	0.00258375
0.99	0.3	0.2	0.00288854	0.00154994	0.00161802	0.000753736	0.00159446	0.000661345
0.99	0.3	0.3	3.91215e-006	1.17163e-006	0.00108095	0.000490822	0.00106388	0.000487976
0.99	0.3	0.4	0.00569767	0.00314874	0.00242609	0.00111609	0.00500561	0.00187435
0.99	0.3	0.5	0.0604346	0.0495344	0.0109638	0.00575735	0.22298	0.0272083
0.99	0.3	0.6	0.251215	0.214938	0.040486	0.0241799	1.405	0.130163
0.99	0.4	0.1	0.112083	0.0899206	0.0570993	0.0374854	0.129105	0.0184497
0.99	0.4	0.2	0.034608	0.0256035	0.0111801	0.00604018	0.0471548	0.00881703
0.99	0.4	0.3	0.00546638	0.0030425	0.00230223	0.00104461	0.00468877	0.00178471
0.99	0.4	0.4	3.57694e-009	1.78847e-009	0.00216022	0.000953194	0.00216026	0.000953218
0.99	0.4	0.5	0.0120589	0.00768187	0.00351894	0.00159972	0.0233744	0.00554219
0.99	0.4	0.6	0.119706	0.104229	0.0118259	0.00608048	0.781695	0.0736051
0.99	0.5	0.1	0.291476	0.236958	0.140415	0.100492	0.655374	0.0686964
0.99	0.5	0.2	0.155735	0.131282	0.0430273	0.0265361	0.531289	0.0564551
0.99	0.5	0.3	0.060377	0.0490669	0.010562	0.00555887	0.201945	0.0270741
0.99	0.5	0.4	0.0117624	0.00747111	0.00332857	0.0015208	0.0221119	0.00545388
0.99	0.5	0.5	1.51803e-008	7.58959e-009	0.00364798	0.00158672	0.0036477	0.00158659
0.99	0.5	0.6	0.028717	0.022096	0.00510558	0.00232141	0.159045	0.0183311
0.99	0.6	0.1	0.573514	0.434252	0.274789	0.204741	1.75092	0.158506
0.99	0.6	0.2	0.414716	0.334451	0.112145	0.0759121	1.88212	0.171978
0.99	0.6	0.3	0.256498	0.219246	0.0393561	0.023384	1.42439	0.134745
0.99	0.6	0.4	0.119867	0.104229	0.0114463	0.00584327	0.737733	0.0739362
0.99	0.6	0.5	0.0280795	0.0213199	0.00507705	0.00227889	0.147963	0.0177646
0.99	0.6	0.6	3.98853e-006	1.20982e-006	0.0054301	0.00240345	0.00542962	0.0024032

Table 6: Distances between distributions for correlation equal to 0.99, for different configurations of the individual volatilities: prefixes "KLD" and "H" denote the Kullback-Leibler and Hellinger distances, respectively; suffixes "23", "2b" and "3b" denote distances between two- and three-moment matching distributions, two-moment matching and simulated basket distribution, three-moment and simulated basket distributions, respectively.

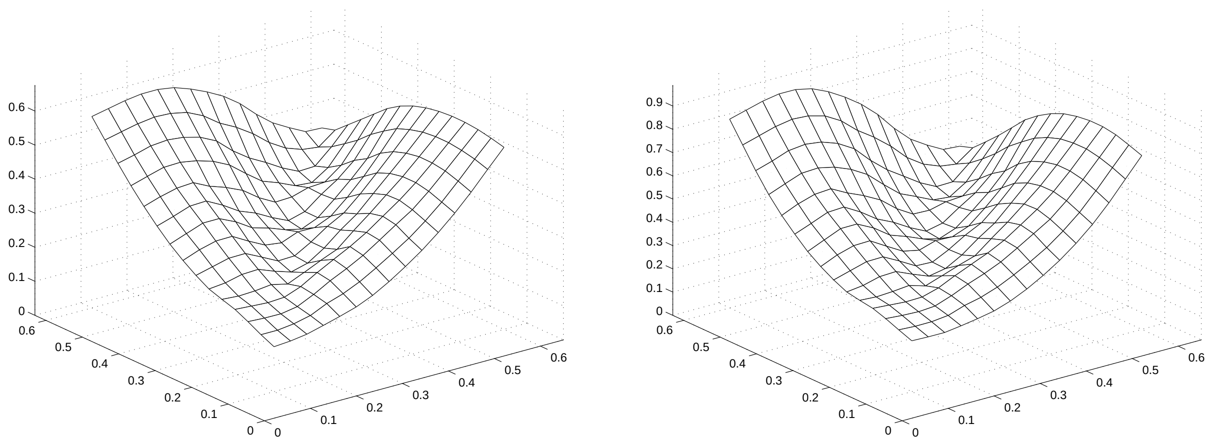

Figure 6: The Hellinger (left) and Kullback-Leibler (right) distances between the two- and the three-moment matching distributions, for fully negative correlation ($\rho = -0.99$), as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 15.

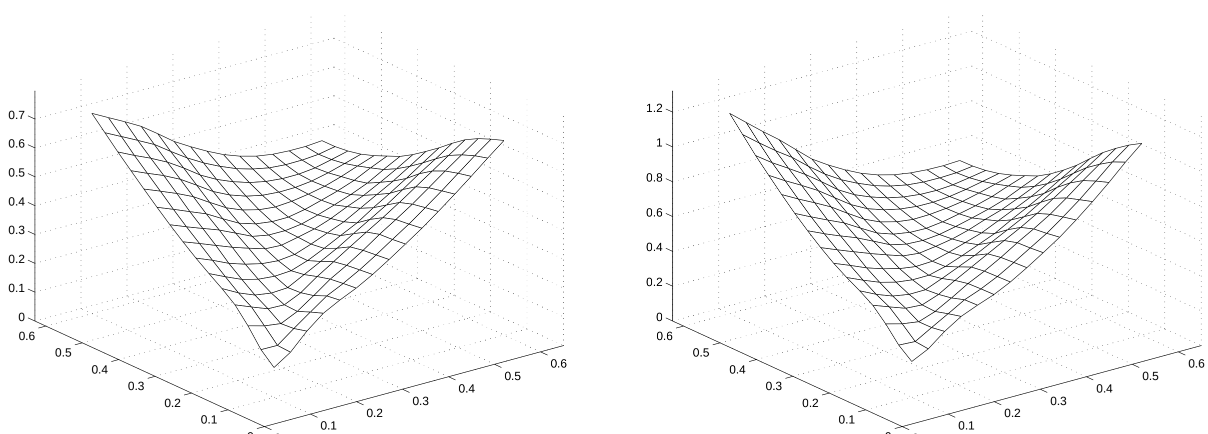

Figure 7: The Hellinger (left) and Kullback-Leibler (right) distances between the two-moment matching and the basket distributions, for fully negative correlation ($\rho = -0.99$), as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 16.

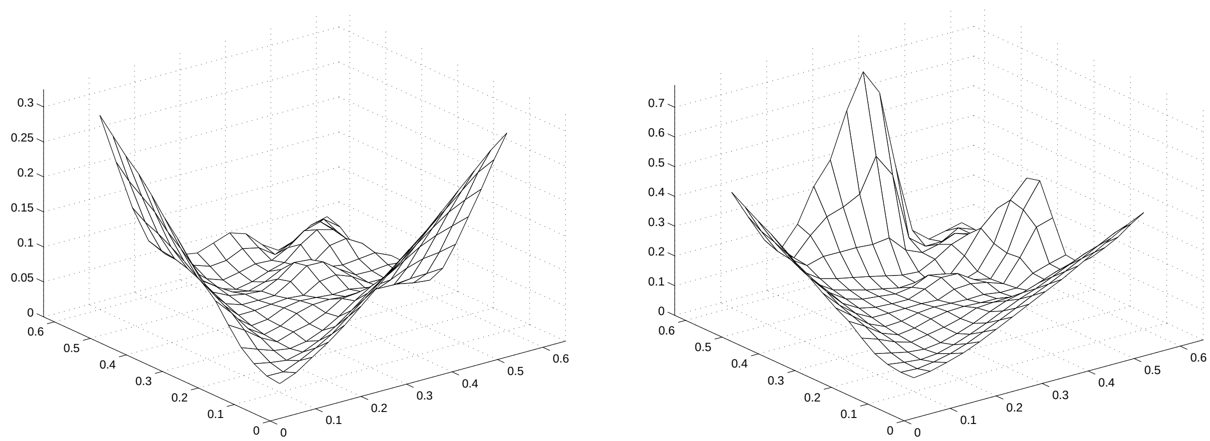

Figure 8: The Hellinger (left) and Kullback-Leibler (right) distances between the three-moment matching and the basket distributions, for fully negative correlation ($\rho = -0.99$), as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 17.

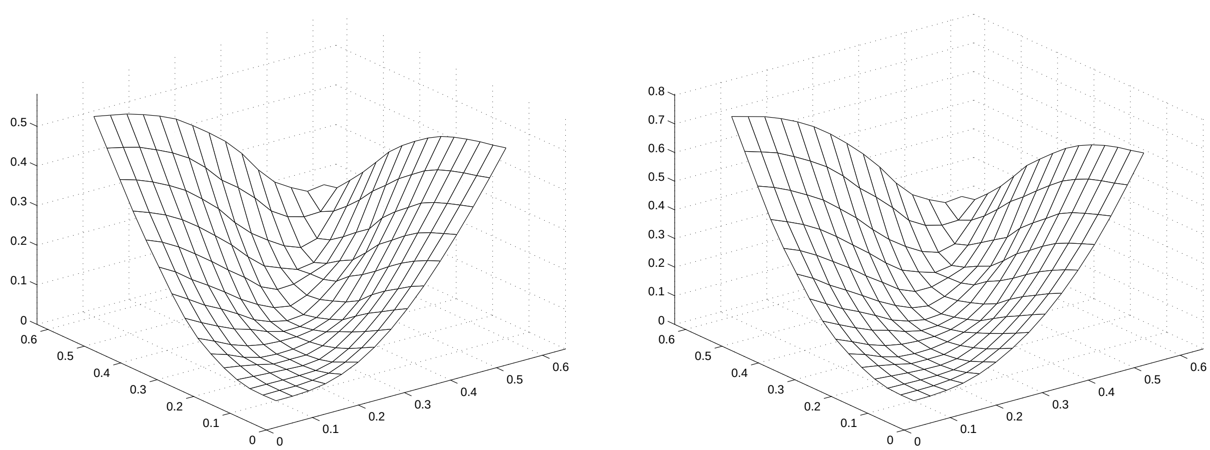

Figure 9: The Hellinger (left) and Kullback-Leibler (right) distances between the two- and the three-moment matching distributions, for zero correlation as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 18.

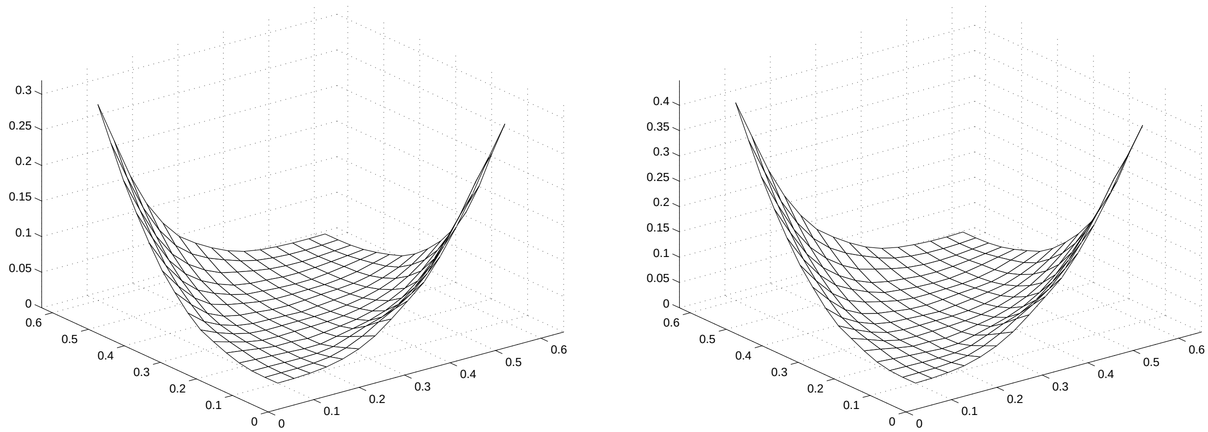

Figure 10: The Hellinger (left) and Kullback-Leibler (right) distances between the two-moment matching and the basket distributions, for zero correlation as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 19.

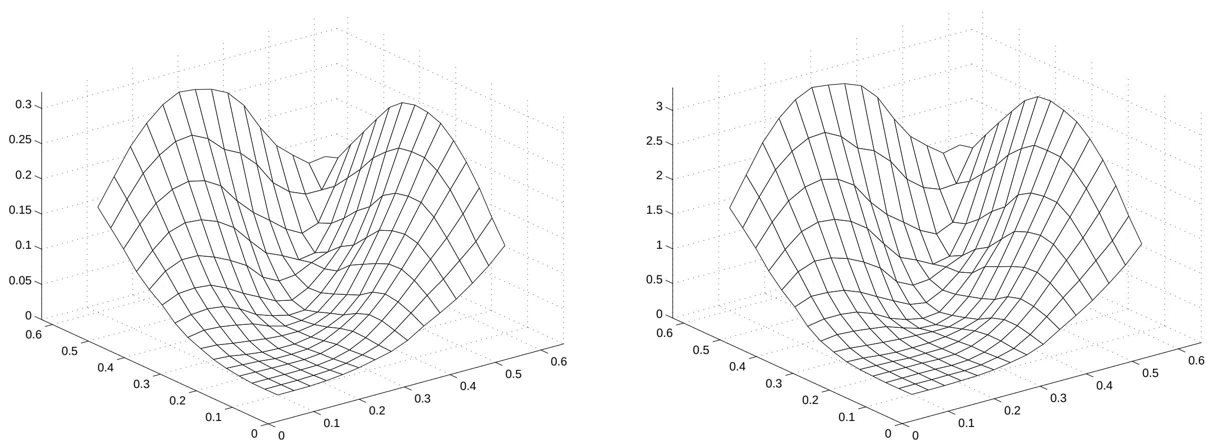

Figure 11: The Hellinger (left) and Kullback-Leibler (right) distances between the three-moment matching and the basket distributions, for zero correlation as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 20.

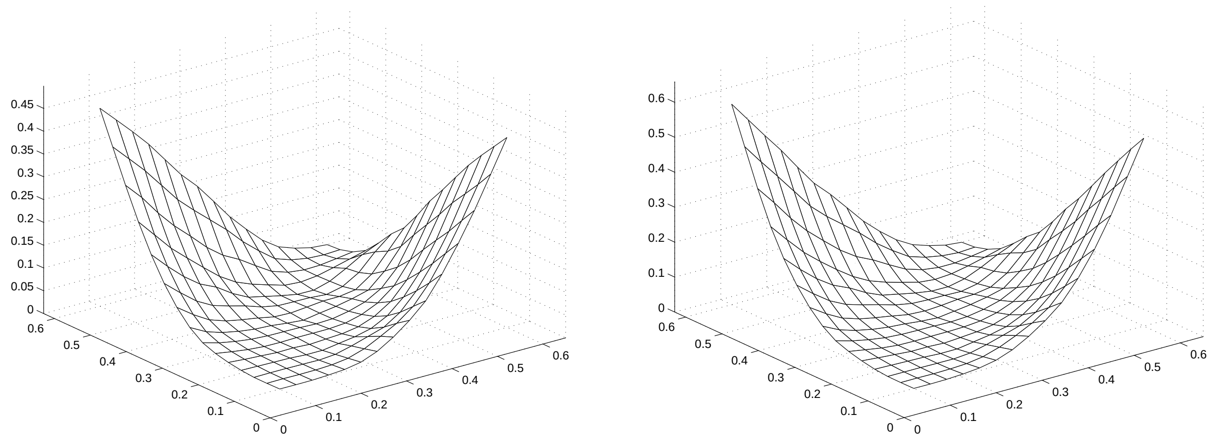

Figure 12: The Hellinger (left) and Kullback-Leibler (right) distances between the two- and the three-moment matching distributions, for fully positive correlation ($\rho = 0.99$), as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 21.

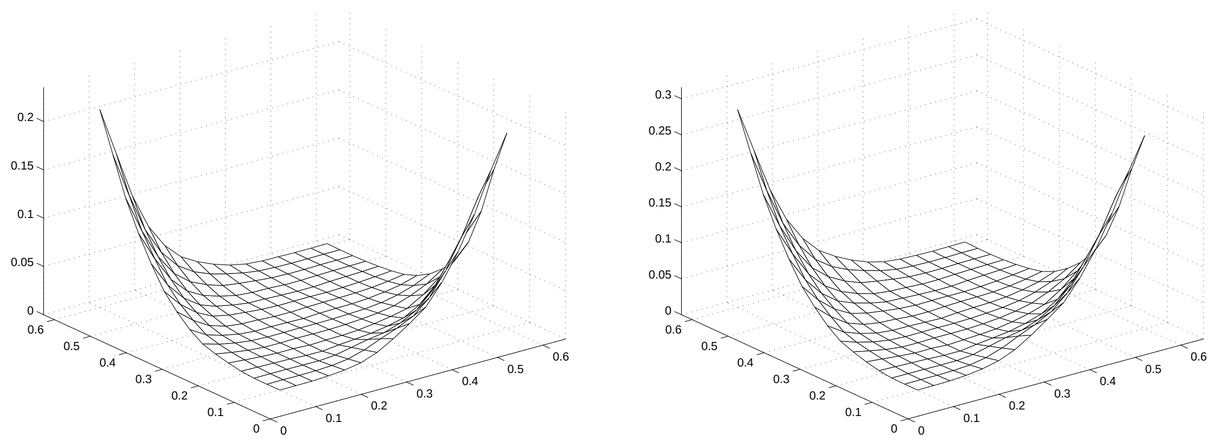

Figure 13: The Hellinger (left) and Kullback-Leibler (right) distances between the two-moment matching and the basket distributions, for fully positive correlation ($\rho = 0.99$), as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 22.

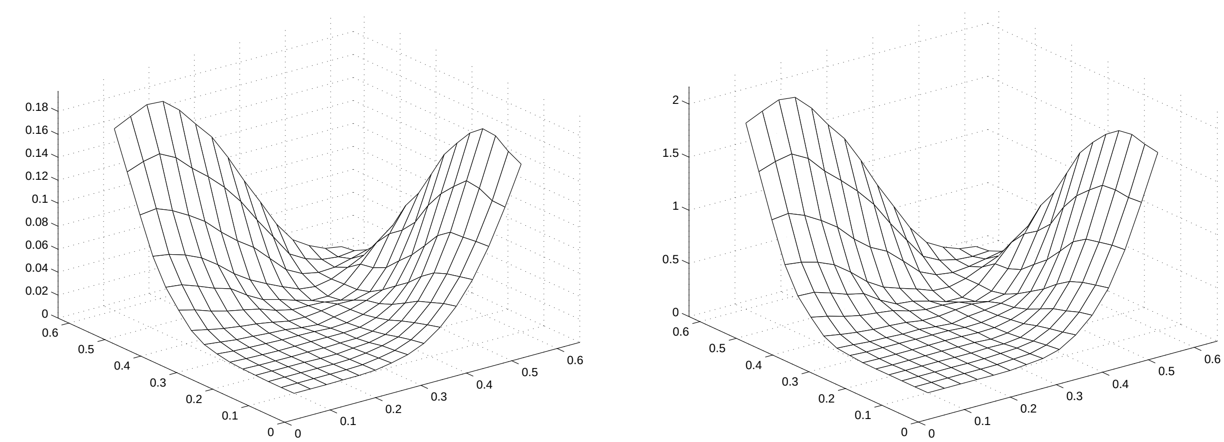

Figure 14: The Hellinger (left) and Kullback-Leibler (right) distances between the three-moment matching and the basket distributions, for fully positive correlation ($\rho = 0.99$), as functions of the two individual volatilities in the range (0,60%). The corresponding contour plots can be seen in Fig. 23.

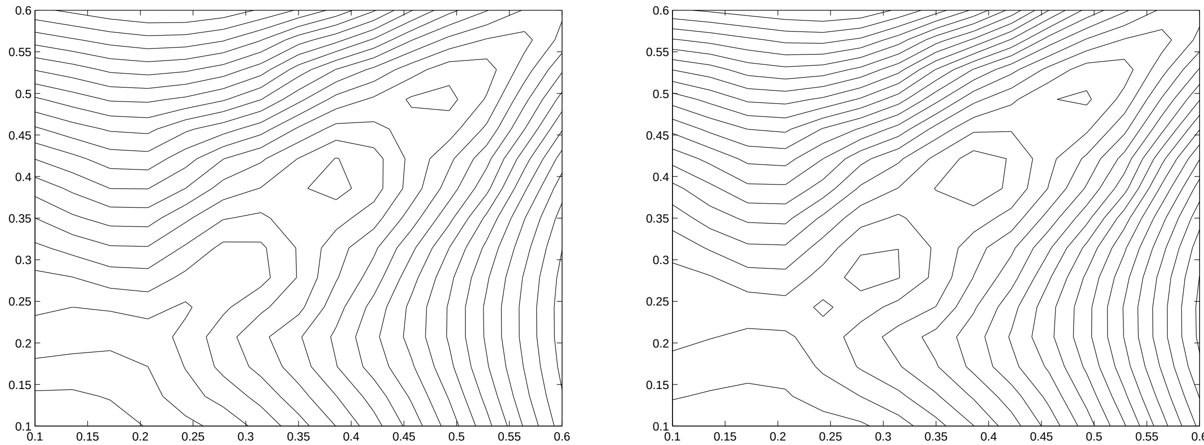

Figure 15: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the two- and the three-moment matching distributions, for fully negative correlation ($\rho = -0.99$), as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 6.

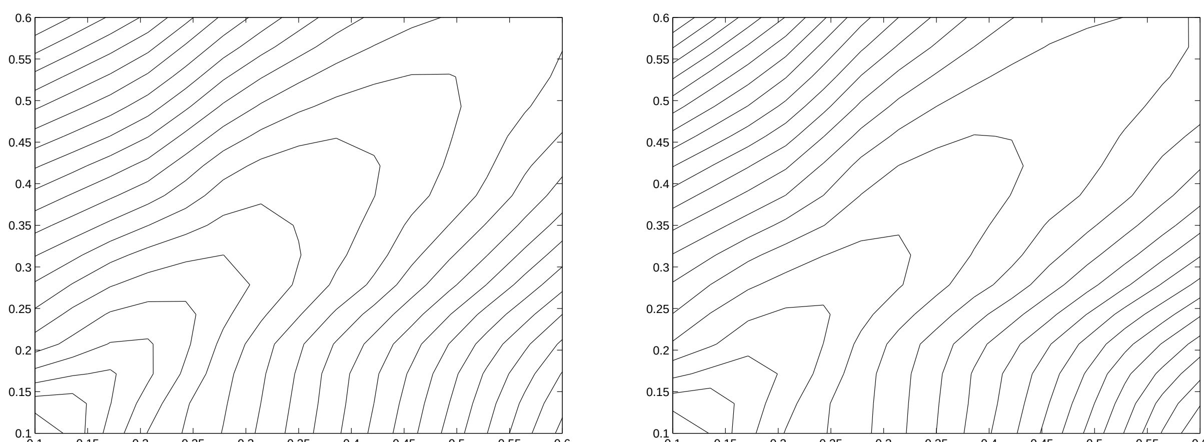

Figure 16: Contour plot of the Hellinger (left) and Kullback-Leibler (right) distances between the two-moment matching and the basket distributions, for fully negative correlation ($\rho = -0.99$), as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 7.

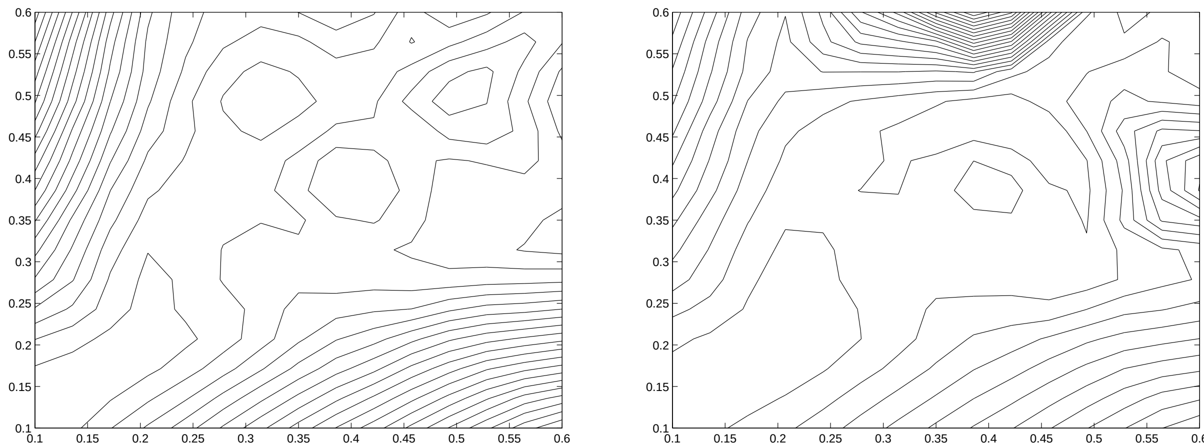

Figure 17: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the three-moment matching and the basket distributions, for fully negative correlation ($\rho = -0.99$), as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 8.

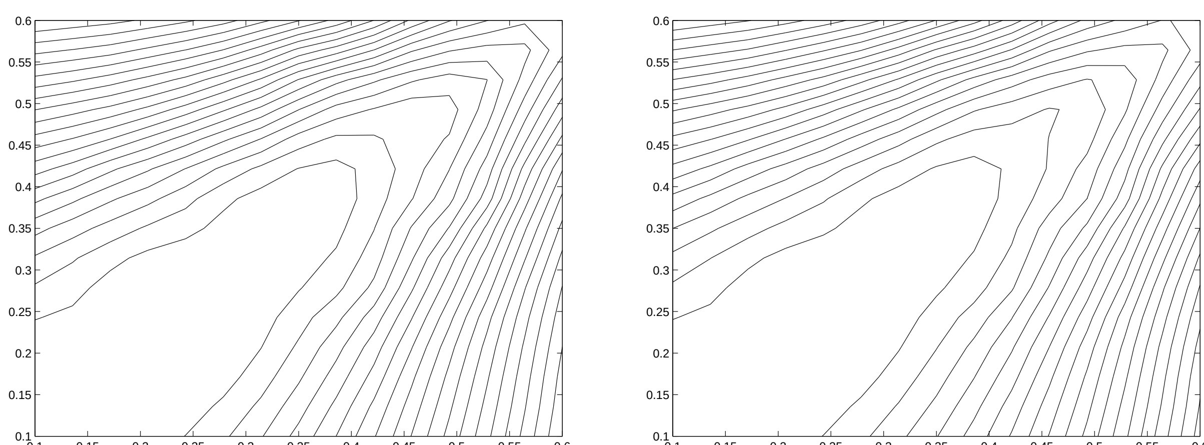

Figure 18: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the two- and the three-moment matching distributions, for zero correlation as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 9.

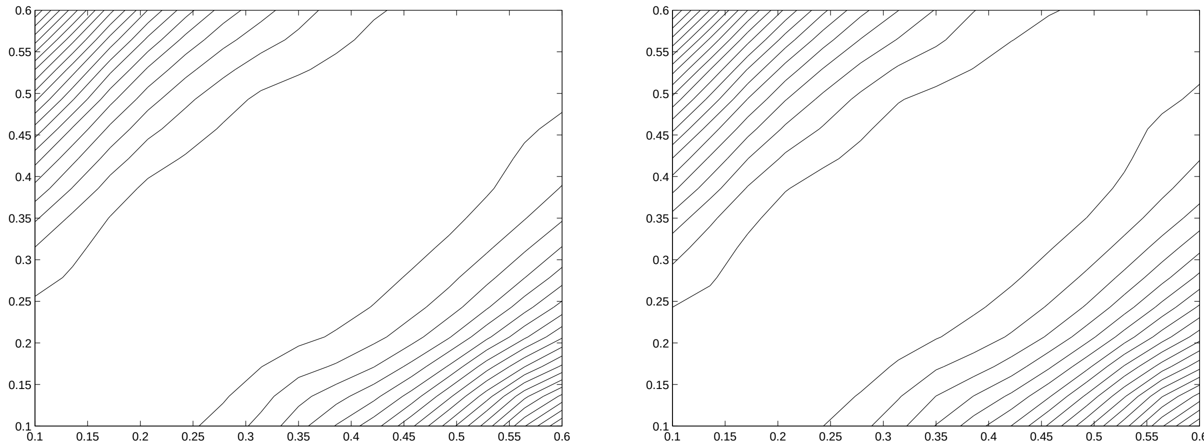

Figure 19: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the two-moment matching and the basket distributions, for zero correlation as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 10.

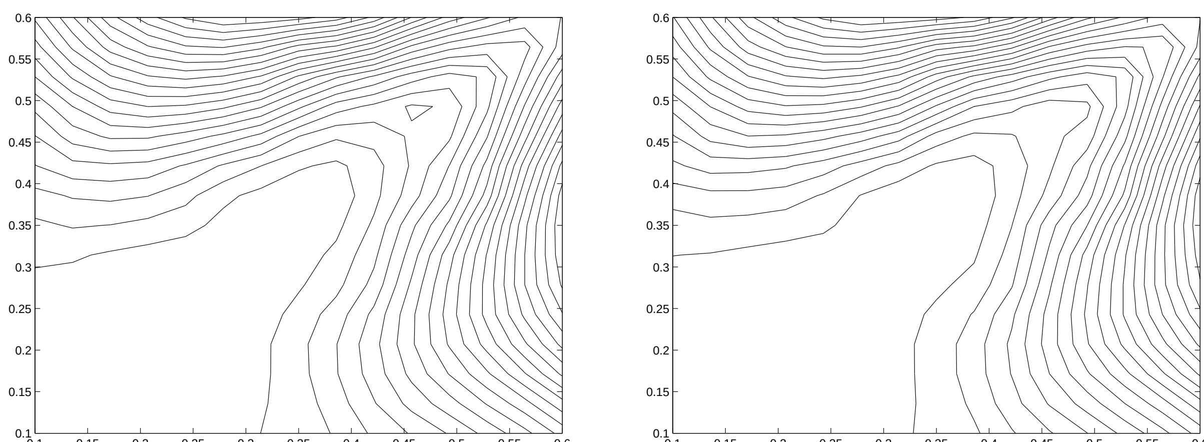

Figure 20: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the three-moment matching and the basket distributions, for zero correlation as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 11.

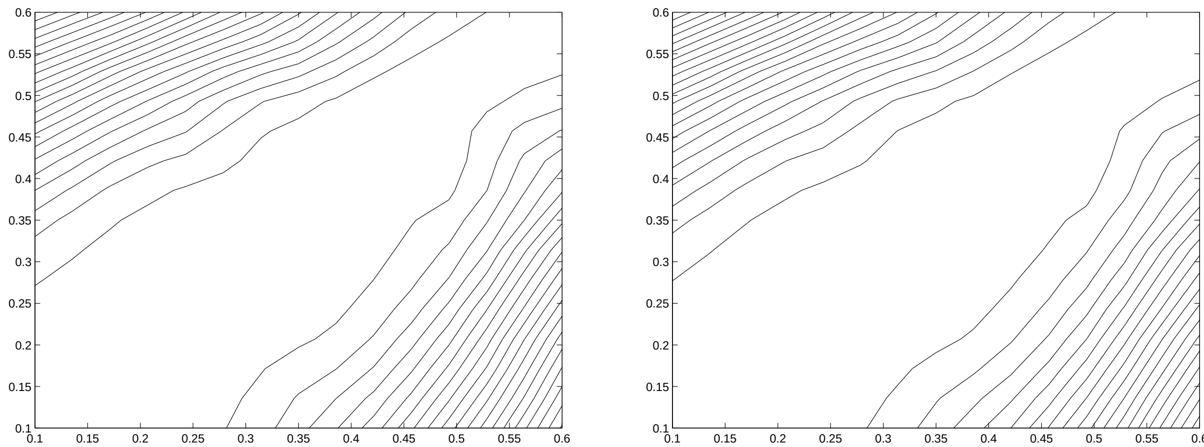

Figure 21: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the two- and the three-moment matching distributions, for fully positive correlation ($\rho = 0.99$), as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 12.

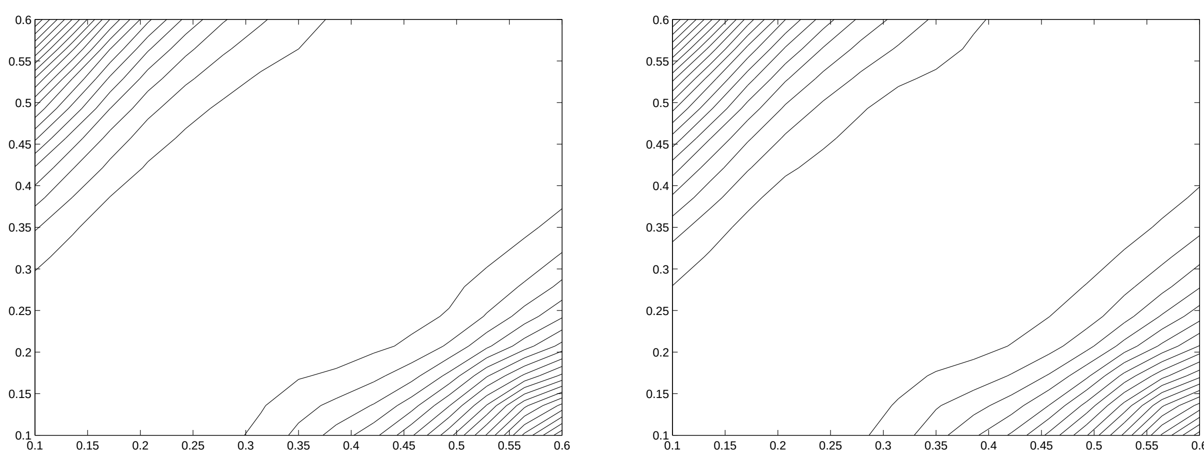

Figure 22: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the two-moment matching and the basket distributions, for fully positive correlation ($\rho = 0.99$), as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 13.

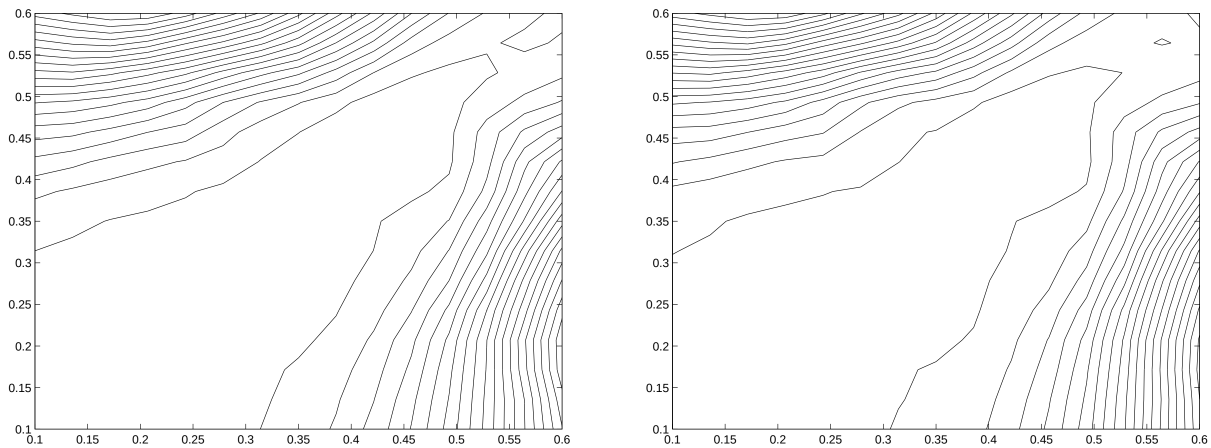

Figure 23: Contour plots of the Hellinger (left) and Kullback-Leibler (right) distances between the three-moment matching and the basket distributions, for fully positive correlation ($\rho = 0.99$), as functions of the two individual volatilities in the range (0,60%). These contour plots correspond to the surfaces shown in Figs. 14.