CDS Calibration with tractable structural models under uncertain credit quality*

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Abstract

In this paper we develop structural first passage models (AT1P and SBTV) with time-varying volatility and characterized by high tractability. The models can be calibrated exactly to credit spreads using efficient closed-form formulas for default probabilities.

In these models default events are caused by the value of the firm assets hitting a safety threshold, which depends on the financial situation of the company and on market conditions. In AT1P this default barrier is deterministic. Instead SBTV assumes two possible scenarios for the initial level of the default barrier, for taking into account uncertainty on balance sheet information and in particular risk of fraud.

We apply the models to exact calibration of Parmalat Credit Default Swap (CDS) data during the months preceding default. In some cases these models show more calibration capability than a reduced-form model. The results we obtain with AT1P and SBTV have reasonable economic interpretation, and are particularly realistic when SBTV is considered. These results are analyzed in relation with the progressive unfolding of news on Parmalat crisis, and compared to the results we obtain for a company with higher credit quality.

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1 Introduction

In this paper we develop two families of structural models (AT1P and SBTV) with time-varying volatility and characterized by high fit to market quotes and high tractability. The models can be calibrated exactly to credit spreads using efficient closed-form formulas for default probabilities. In these models default events are caused by the value of the firm assets hitting a safety threshold, which depends on the financial situation of the company and on market conditions.

The Analytically Tractable First Passage Model (AT1P), presented in Section 2, has a deterministic default barrier and is a generalization of the classic Black and Cox (1976) Model. Brigo and Tarenghi (2004) introduced the model and proved that, differently from the Black and Cox Model, in AT1P we maintain analytic formulas for default probabilities even allowing asset volatility to be realistically time-varying . This allows AT1P to calibrate efficiently and precisely Credit Default Swap (CDS) market quotes for a range of maturities.

In Section 2.1 we apply AT1P to Parmalat CDS data during the months preceding default, with some differences in the calibration procedure compared to tests in Brigo and Tarenghi (2004). AT1P can calibrate exactly all CDS quotes considered and moreover here the output is regular and financially significant. The comparison with the output of a calibration with an intensity based reduced-form model, belonging to a class usually considered more suitable than standard structural models to credit spread calibration, highlights two positive features of AT1P. First, AT1P has an intuitive economic interpretation, which is lacking in reduced-form models. Secondly, AT1P shows high tractability and empirical fit, similarly to reduced-form models, and in some cases it has even more calibration capability than a simple reduced-form model.

The analysis of the model behaviour in representing Parmalat approaching default leads us to introduce in Section 3 an extension of the model, aimed at increasing further the realism of the model. This extension, called Scenario Barrier Time-Varying Volatility AT1P model (SBTV in the following), assumes two possible scenarios for the initial level of the default barrier, to take into account uncertainty on balance sheet information and in particular risk of fraud. Also in this case we maintain analytic default probabilities. The SBTV Model is applied in Section 3.1 to the same Parmalat data previously considered, obtaining again exact calibration. The results have a more realistic interpretation than in AT1P. We analyze SBTV results in relation with the progressive unfolding of news on Parmalat real crisis, and compare them with the output of the calibration of the same model to the CDS quotes of a company with higher credit quality (Vodafone). The CDS-calibrated AT1P and SBTV can be helpful in pricing credit derivatives, and also hybrid products and counterparty risk. We hint at this aspect in the conclusions.

2 Analytically Tractable First Passage Model

The models considered in this work are based on a stochastic process A_t for the assets of a company and on a default boundary H_t, such that when A_t hits H_t for the first time (from above) this is interpreted as the company being forced to declare bankruptcy. This represents the default event relevant for market credit derivative contracts such as CDS. In the AT1P Model the assets value process A_t is modelled as a Geometric Brownian Motion under the risk neutral measure

$$dA_t = A_t\,(r_t - q_t)\,dt + A_t\,\sigma_t\,dW_t \tag{1}$$

where r_t is the risk free rate, q_t is the payout ratio and σ_t is the instantaneous volatility. We allow all parameters to be deterministic functions of time, with the purpose of increasing the realism and the flexibility of the model in calibrating real market data.

The default boundary H_t may depend on different aspects of the financial situation of the firm. We allow it to vary in time, following company and market conditions. The parametric form we assume, following Brigo and Tarenghi (2004), is

$$\begin{aligned} H_t &= H \exp\left(-\int_0^t \left(q_s - r_s + B\sigma_s^2\right) ds\right) \\ &= \frac{H}{A_0} \mathbb{E}\left[A_t\right] \exp\left(-B \int_0^t \sigma_s^2 ds\right), \end{aligned} \tag{2}$$

where $H = H_0$ and $\mathbb{E}$ is expectation under the risk-neutral measure $\mathbb{Q}$. Therefore the behaviour of H_t has a simple economic interpretation. The backbone of the default barrier at t is a proportion, controlled by the parameter H, of the expected value of the company assets at t. H may depend on the level of liabilities, on safety covenants and in general on the characteristics of the capital structure of the company. This is in line with observations in Giesecke (2004), pointing out that some discrepancies between the Black and Cox Model and empirical regularities may be addressed with the realistic assumption that, like the firm value, the total debt grows at a positive rate, or that firms maintain some target leverage ratio as in Collin-Dufresne and Goldstein (2001).

Depending on the value of the parameter B, it is possible that this backbone is modified by accounting for the volatility of the company's assets. In all the following tests we simply take $B = 1$, corresponding to the interpretation that when volatility increases - which can be independent of credit quality - the barrier is slightly lowered to cut some more slack to the company before forcing the company to declare bankruptcy.

In spite of the fact that we allow for time-dependence of the main parameters of the model, with the above default barrier we can always provide closed-form formulas for survival probabilities. This advantage is due to the barrier option technology introduced in Lo et al (2003) and Rapisarda (2003), and allows us to apply efficiently AT1P to the ideal "territory" of reduced-form models, i.e. to calibrate CDS quotes. For the details on the analytic derivation of the formulas we refer to Brigo and Tarenghi (2004), who prove the following (where $(1 + 2\beta)/2 = B$).

Proposition 1 ***(Analytic Survival Probabilities in AT1P)*** *Assume that A_t evolves under $\mathbb{Q}$ according to (1) and that the default barrier is*

$$H_t = H \exp\left(-\int_0^t \left(q_s - r_s + B\sigma_s^2\right) ds\right).$$

Let the default time τ be defined as the first time where A_t hits H_t from above, starting from $A_0 > H$: $\tau = \inf\{t \geq 0 : A_t \leq H_t\}$. Then the survival probabilities are given analytically by

$$\mathbb{Q}\{\tau > T\} = \left[\Phi\left(\frac{\log\frac{A_0}{H} + \frac{2B-1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}\right) - \left(\frac{H}{A_0}\right)^{2B-1} \Phi\left(\frac{\log\frac{H}{A_0} + \frac{2B-1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}\right)\right], \tag{3}$$

where Φ is the cumulative distribution function of a standard normal random variable.

Notice that the survival probability depends only on the ratio $\frac{A_0}{H}$ and not on A_0 or H separately, so we assume a unit A_0 and express H with respect to "1".

Now we recall the basics on the Credit Default Swaps. Consider a unit notional and two companies A (the *protection buyer*) and B (the *protection seller*) who agree on the following. If a third company C (the *reference credit*) defaults at a time $\tau_C \in (T_a, T_b]$, B pays to A at τ_C a protection amount LGD (Loss Given the Default of C), supposed to be deterministic in the present paper. Typically LGD is equal to one minus the recovery rate REC. In exchange for this protection, A pays to B a fixed premium rate $R_{a,b}$ at a set of times $\{T_{a+1}, \ldots, T_b\}$, $\alpha_i = T_i - T_{i-1}$, $T_0 = 0$. These payments are interrupted in case of default. $R_{a,b}$ is fixed at time 0 to make the contract fair at inception. As usual the (payer) CDS price is the risk neutral expectation of the discounted payoff, leading, under independence of interest rates and default time, to the model independent formula

$$\mathbf{CDS}_{a,b}(0, R_{a,b}, \mathrm{LGD}) = R_{a,b} \int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d\mathbb{Q}(\tau > t)$$
$$- R_{a,b} \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \mathbb{Q}(\tau \geq T_i) - \mathrm{LGD} \int_{T_a}^{T_b} P(0,t) d\mathbb{Q}(\tau > t), \tag{4}$$

where $P(s,T)$ is the price at s of a T-maturity zero-coupon bond, and, for a generic time t, $T_{\beta(t)}$ is the first date among the payment dates T_i that follows t. When a closed-form formula for survival probabilities $\mathbb{Q}(\tau > t)$ is available, as in AT1P, pricing a CDS is computationally simple and efficient. In particular, when the protection payment is postponed to the first payment date T_i following default, the price is expressed only in terms of simple analytic survival probabilities that need not be differentiated as in the exact case above.

A remark on the data we use in the following empirical tests is in order. The parameters in the specification of A_t and H_t are computed in this work by calibration to the market quotes of the most actively traded single name credit derivatives, the CDS, and not on balance sheet values. When we need an initial guess for the parameters of A_t we use historical equity statistics to have a proxy for the order of magnitude of the parameters (volatilities and correlations) in the dynamics of a company's assets. Other choices might be possible, for example based on using a precise relationship between asset value and equity value. Like in Giesecke (2002), in this paper we do not make use of such relationships. In fact it would require further assumptions and data on firm fundamentals, while here we want the model calibration to require only CDS quotes, usually reliable in terms of liquidity, and an initial guess based on equity statistics. However, in light of the positive results shown in the following cases of calibration to market quotes, and also for a possible application of the models here presented to hybrid credit/equity products, further analysis of the relationships with the value of company fundamentals appears a relevant aspect for subsequent research.

2.1 Parmalat Calibration with AT1P

We consider the case of Parmalat SpA, an Italian food company, in the months preceding default. After a period of uncertainty on Parmalat's real financial situation, due to deceptive accounting, the real depth of the financial crisis came to light at the end of 2003 and rapidly led to bankruptcy. On September 12, 2003, Parmalat drops plan for a EUR 300 million debt sale. On November 14 the chief financial officer resigns after questions have been raised on Parmalat financial transactions. On December 9 Parmalat misses a EUR 150 million bond payment, while the management claims this is due to a customer not paying its bills. On December 19 a claimed USD 3.9 billion liquidity is revealed not to exist. On December 24 Parmalat goes into administration. For more information on Parmalat fraud and default story see "Parmalat: timeline to turmoil", BBC news, 2005.

Due to the fact that the discovery of the truth on the crisis was reflected by large and rapid movements in market quotes, it is a difficult task for any model to calibrate accurately and with reasonable outputs the behaviour of Parmalat CDS data in the end of 2003. This is particularly true for standard structural models such as AT1P, which assume the default time to be a predictable stopping time. With the AT1P model we have in principle enough flexibility to calibrate precisely and efficiently the available quotes. Now we plan to analyze in practice on Parmalat CDS market quotes how the structural model behaves as the credit quality implied by CDS data rapidly deteriorates in time. We aim at assessing whether the resulting parameters are regular and financially realistic. We do so by checking whether the default barriers we obtain have a reasonable economic interpretation and in particular by assessing if the resulting term structures of asset volatility are stable and consistent with the input market data. In all following empirical tests the volatility σ_t is assumed to be piecewise constant in suitable intervals $[T_\alpha, T_\beta)$ partitioning the time horizon, $\sigma_t = \sigma([T_\alpha, T_\beta))$ for $T_\alpha \leq t < T_\beta$. We consider data from three different days in 2003:

1. September 10, just before the beginning of the final Parmalat default story recalled above.
2. November 28, after the story of the Parmalat crisis began to unfold but before the pitch of the crisis.
3. December 10, one day after Parmalat had missed a payment, and the fraud was not clear yet but the company was openly suspected to be on the verge of bankruptcy.

For each considered day, the quoted CDS have $T_a = 0$ and T_b ranging in the following set of standard maturities: $T_1 = 1y$, $T_2 = 3y$, $T_3 = 5y$, $T_4 = 7y$, $T_5 = 10y$. Since the perceived recovery decreases as the credit quality deteriorates, for the recovery rate we use 40% on the first two dates and 15% for December 10. The payout ratio q_t is set to zero for all days, while r_t is set according to the zero-coupon curve. Various calibration procedures are considered in Brigo and Tarenghi (2004). The procedure we use here has some differences and returns values which are economically reasonable.

In our calibration procedure CDS data are matched via the volatility of the company asset. The procedure is two-stage. A preliminary stage is required only for setting the initial level of the barrier H. This does not affect directly the calibration quality since market quotes are met at the second stage when calibrating volatilities; however this can affect the resulting parameters. One may use balance sheet values to set H at this preliminary stage, but in order to obtain more realistic outputs we prefer to use market data.

Knowing $\sigma([0,1))$, $R_{0,1}$ and H we can price a 1-year CDS with AT1P. When pricing a CDS with the corresponding current market CDS rate, the price of the CDS should equal zero. Thus, having an exogenous guess for $\sigma([0,1))$, we find H as the initial level of the barrier required to minimize the square of the 1-year CDS price when $R_{0,1}$ is used in pricing. The minimization of this metrics (sum of square errors) is the procedure we consider also in all following calibration tests.

In order to have an initial exogenous guess on $\sigma([0,1))$ we use historical analysis of Parmalat equity data. For September 10 we have $\sigma([0,1)) = 5\%$. This value appears quite low since based on the month preceding September 10, a quiet period. We use it whenever we have to guess volatility for "normal" trading periods far from the crisis pitch. Then we can adjust this value for taking into account the increase in volatility typical of crisis periods: equity volatility reaches even 50% when estimated only on the month of data preceding December 10. Taking into account these results on the last month in the guess for the future one year volatility we have $\sigma([0,1)) = 6.3\%$ on November 28 and $\sigma([0,1)) = 15\%$ on December 10.

The values for H we obtain are all close to 0.9. For September 10: $H = 0.8987705$; for November 28: $H = 0.9050667$; for December 10: $H = 0.8794308$. Once the barrier parameter H has been set, we are left with calibrating the volatility buckets to the quoted CDS. For all days considered we calibrate the entire term structure of CDS with a calibration error on a CDS quote never higher than 10^{-32}, therefore we have in practice exact calibration. Now it is of interest to check whether the output is financially realistic.

The CDS data, in basis points, are presented in Table 1, while in Table 2 we report the calibrated volatilities.

CDS Maturity T_b	1y	3y	5y	7y	10y
Sep 10: Premium rate $R_{0,b}$	192.5	215	225	235	235
Nov 28: Premium rate $R_{0,b}$	725	630	570	570	570
Dec 10: Premium rate $R_{0,b}$	5050	2100	1500	1250	1100

Table 1: Parmalat CDS data (basis points).

CDS Maturity T_b	1y	3y	5y	7y	10y
Sep 10: $\sigma([T_{b-1}, T_b))$	5.0%	3.0%	3.1%	3.5%	3.5%
Nov 28: $\sigma([T_{b-1}, T_b))$	6.3%	4.4%	4.7%	6.3%	7.8%
Dec 10: $\sigma([T_{b-1}, T_b))$	15.2%	5.7%	6.4%	7.1%	9.1%

Table 2: AT1P Calibration to Parmalat CDS: Volatilities.

First, notice that all values for volatility are reasonable and in line with the historical estimations that we have taken as a proxy for the volatility of the company's assets. Secondly, notice that we see clearly the increase in volatility as the crisis approaches. Each volatility bucket shows an increasing behaviour from September 10 to December 10, following the decreasing perceived credit quality of the company.

Now we compare with the results from an intensity based reduced-form model which are presented in Brigo and Tarenghi (2004). Reduced-form models are usually considered more flexible than standard structural models in calibration. In addition, since reduced-form models represent the default time as the time of the first jump of a stochastic process, they appear more suitable to represent sudden default events. The instantaneous intensity of the considered reduced-form model is assumed to be piecewise linear, the simplest choice for having a continuous, albeit flexible, intensity and differentiable hazard function or cumulated intensity. Consistently with the above assumptions for the volatility of the structural model, the slope of the linear intensity is different in the different intervals $[T_\alpha, T_\beta)$ partitioning the time horizon. The parameters in the different intensity buckets are calibrated to the corresponding CDS market quotations, analogously to the calibration procedure of AT1P.

To compute CDS prices with AT1P we need computing (analytical) default probabilities. We recall that all probabilities we obtain in this work are computed under the risk neutral probability measure $\mathbb{Q}$. Probabilities follow the decrease in credit quality indicated by the movements of CDS spreads, and are very close to those implied by the reduced-form model, as is natural from formula (4) and its analogous for postponed defaults, in particular before the pitch of the crisis. See results for September 10 on the left in Figure 1, where we show survival probabilities, connecting linearly the values at the relevant CDS maturities. This confirms that the two models detect an analogous information on CDS implied default probabilities.

However the survival probabilities computed with the reduced-form model become quite irregular when calibrating the very high CDS spreads of December 2003, as shown on the right in Figure 1.

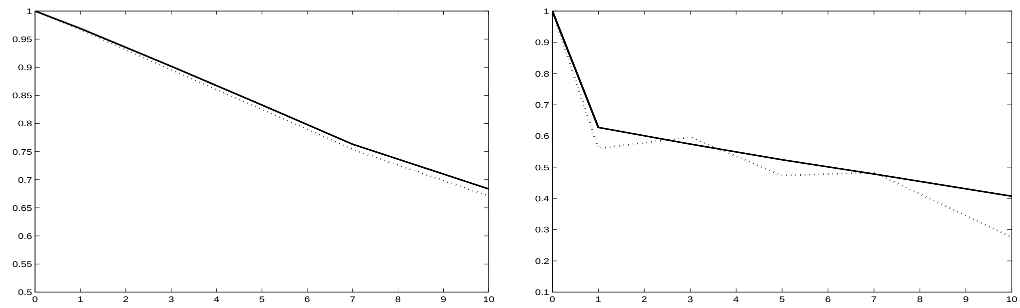

Figure 1: Survival Probabilities for AT1P and for the Piecewise Linear Intensity Model (dotted). On the left we plot our results for September 10, on the right for December 10.

In particular we see that with the reduced-form model survival probabilities become slightly increasing in two of the intervals considered, which implies that the instantaneous intensity becomes negative, an unacceptable result. On the other hand, results obtained with AT1P model are always regular and reasonable. It appears that our first passage model has a more stable structure than a reduced-form model when dealing with an extreme situation. In contrast with the usual folklore, here a simple structural model has more calibration capability than a simple reduced-form model. Further tests of ours show that negative intensity can be avoided using the less regular but more robust piecewise constant intensity formulation. Also in this case, however, results are extremely irregular, with a jump in intensity from 55.483% to 0.807% when moving from the first to the second intensity bucket, thus confirming the above conclusions.

We see in Figure 2 the logarithm of the barrier (continuous line), $t \mapsto \ln(H_t)$, and the expected value of the logarithm of the firm asset value (dashed), $t \mapsto \mathbb{E}[\ln(A_t)]$. We use logarithms because the logarithm of the firm asset value has a normal distribution, a fact that will be exploited below. Notice first that the time-behaviour of the barriers confirms our initial interpretation. However, across the different trading days considered, the barriers do not show relevant changes for adapting to the flow of information on the financial situation of the company.

It is in fact the volatility which changes so as to adapt to the decreasing credit reliability. Observe the dotted lines in Figure 2. These represent confidence regions built around the expected value of the logarithm of the firm asset value, based on usual statistics for normal random variables. We plot $t \mapsto (\mathbb{E}[\ln(A_t)] \mp Std[\ln(A_t)])$ (black dots) and $t \mapsto (\mathbb{E}[\ln(A_t)] \mp 2.33 \cdot Std[\ln(A_t)])$ (red dots), where Std indicates standard deviation. In the first confidence region are nearly 68% of the possible scenarios, in the second one about 98%. Therefore the time when the lower boundary of the confidence regions hits the logarithm of the barrier gives us a rough indication of the likelihood of early default. We observe that as the credit quality of the company decreases this hitting times get closer and closer to the current time.

The scarce relevance of the default barriers and the strong reliance on volatilities for representing changes in credit quality can appear a partial representation of reality, and in addition it can lead to some aspects in the volatility behaviour which are not fully satisfactory. Notice first that the volatility $\sigma([0,1))$ is for all days remarkably higher than the volatility in the following years. This can appear to be correctly related to the incorporation in the first volatility bucket of the high historical volatility of the last months. However notice that this happens also for September 10, when historical data embed no recent increase in volatility. In addition, in some cases the discrepancy between $\sigma([0,1))$ and the immediately following $\sigma([1,3))$ found in calibration appears suspiciously high. This could rather depend on a feature typical of first passage models based on diffusion processes with perfect knowledge of the default boundary, such as AT1P. As Duffie and Singleton (2003) say, in such models "the likelihood of default within a short time horizon is extremely small because the asset process has continuous paths that take time to cross the default boundary". Thus, when the default boundary is deterministic, it is hard for models to calibrate a considerable probability of default in a very short horizon (justifying non-null short-term credit spreads), without supposing particularly high short-term volatility.

Hence the problem is also related to the fundamental assumption that the default threshold is a deterministic, known function of time, based on reliable accounting data. The Parmalat case shows clearly that in the most dangerous defaults this assumption does not hold, since books are not reliable and there is uncertainty about the financial situation of a company. Therefore in the following section we aim at increasing the realism of the model by modelling explicitly the possible uncertainty on the real situation of a company.

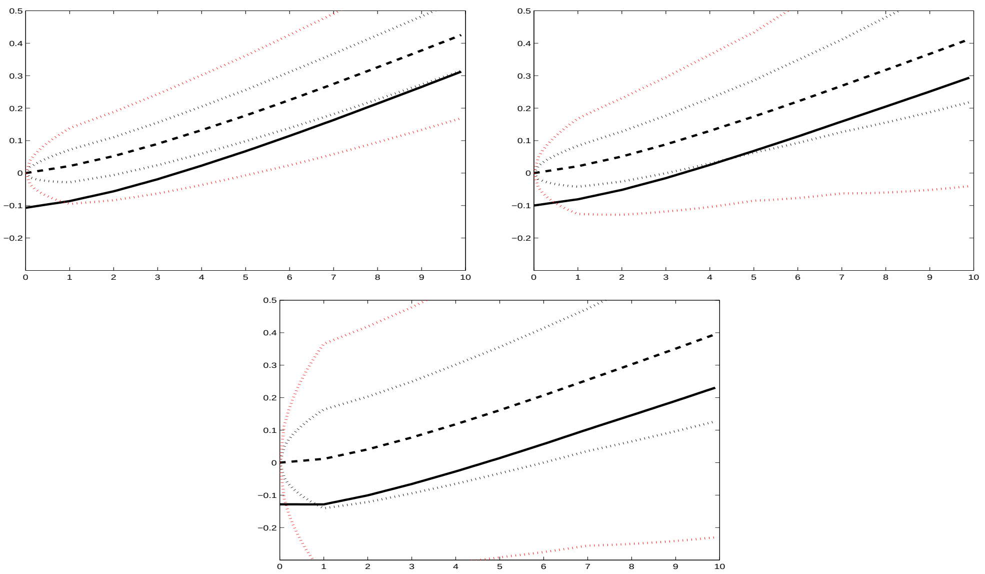

Figure 2: We plot $\ln(H_t)$, $\mathbb{E}[\ln(A_t)]$ (dashed), $\mathbb{E}[\ln(A_t)] \mp Std[\ln(A_t)]$ (black dotted), and $\mathbb{E}[\ln(A_t)] \mp 2.33 \cdot Std[\ln(A_t)]$ (red dotted) for September 10 (top left), November 28 (top right) and December 10 (bottom).

3 Scenario Barrier and Time-varying Volatility Model

Where may one consider explicitly market uncertainty on the situation of the company, due to the fact that balance sheet information is not always reliable, possibly because the company is hiding information? In light of the above analysis, and following for example Giesecke (2004) who refers to scandals such as Enron, Tyco and WorldCom, a crucial aspect in market uncertainty is that public investors have only a partial and coarse information about the true value of the firm assets or the related liability-dependent firm condition that would trigger default. This leads to focus attention on the parameter H, which in our models is seen as the ratio between the initial level of the default barrier and the initial value of company assets. In order to take market uncertainty into account in a realistic, albeit simple, manner, in the following H is replaced by a random variable assuming different values in different scenarios, each scenario with a different probability.

Similar possibilities are considered in Brigo and Tarenghi (2005), including a barrier depending on scenarios with time-constant volatility or barrier and time-constant volatility depending both on scenarios. Instead in this work we deem scenarios on the barrier to be an efficient representation of the uncertainty on the balance sheet of a company, while deterministic *time-varying* volatility can be required for precise and efficient calibration of CDS quotes. The resulting model is called Scenario Barrier Time-Varying Volatility AT1P Model (SBTV). Differently from model specifications in Brigo and Tarenghi (2005), and thanks to realistically time-varying volatility, our model allows negligible calibration errors.

Let the assets value A_t risk neutral dynamics be given by (1). The default time τ is again the first time where A_t hits the barrier from above, but now we have a scenario barrier

$$H_t^I = H^I \exp\left(-\int_0^t \left(q_s - r_s + B\sigma_s^2\right) ds\right) = \frac{H^I}{A_0}\mathbb{E}\left[A_t\right] \exp\left(-B\int_0^t \sigma_s^2 ds\right),$$

where H^I assume scenarios H^1, $H^2, \ldots, H^N$ with $\mathbb{Q}$ probabilities $p_1, \ldots, p_N$. All probabilities are in $[0,1]$ and add up to one, and H^I is independent of W. Thus now the ratio H^I/A_0 depends on the scenario. If we are to price a default-sensitive discounted payoff Π, by iterated expectation we have

$$\mathbb{E}\left[\Pi\right] = \mathbb{E}\left[\mathbb{E}\left[\Pi | H^I\right]\right] = \sum_{i=1}^N p_i \mathbb{E}\left[\Pi | H^I = H^i\right],$$

so that the price of a security is a weighted average of the prices of the security in the different scenarios, with weights equal to the probabilities of the different scenarios. For CDS, the price with the SBTV model is

$$SBTV\mathbf{CDS}_{a,b} = \sum_{i=1}^N p_i \cdot AT1P\mathbf{CDS}_{a,b}(H^i) \tag{5}$$

where $AT1P\mathbf{CDS}_{a,b}(H^i)$ is the CDS price computed according to the AT1P survival probability formula (3) when H is set to H^i.

3.1 Parmalat Calibration with SBTV

In the following tests we consider $N = 2$, namely only two possible scenarios. This is consistent with situations where the market assigns a probability to the balance sheet information of a company to be accurate, but this probability is lower than 1. The remaining probability is attributed to one scenario where real figures are actually neatly worse than they appear in the balance sheet. Preliminary work of ours seems to confirm that an increase in the number of scenarios does not alter substantially the results or their interpretation, while they are very different when moving from one to two scenarios.

Our calibration procedure is again two-stage. Now in the first preliminary stage we have to set three parameters: two barrier levels H^1 and H^2 and one probability p_1, since $p_2 = 1 - p_1$. We need therefore three market quotes for calibration, and an exogenous value for the volatility, taken constant at this first preliminary stage. We consider the CDS with maturity 1y, 3y and 5y, so that for finding the required parameters we need a guess for $\sigma([0,5))$. Since it is unlikely that, if a company survives, short-term market turmoils have a high influence on the expectations about very long-term volatility, we use here the historical estimate for a "normal" period, $\sigma([0,5)) = 5\%$.

Via calibration to the CDS market we obtain the results reported in Table 3. Now the interpretation of results on the initial barrier obtained at different dates is easier. The lower barrier level, representing a healthier situation and an optimistic scenario, possibly corresponding to reliable accounting, has slight variations in both directions, as we had in AT1P tests. But the initial barrier values and the probabilities for the pessimistic scenario (the one with higher barrier), possibly corresponding to a fraud in accounting, represent clearly the unfolding of the Parmalat crisis.

September 10 2003	Scenario $i = 1$	Scenario $i = 2$
Initial Default Barrier H^i	0.8089966064	0.9305503038
Scenario Probability p_i	0.7852110931	0.2147889069

November 28 2003	Scenario 1	Scenario 2
Initial Default Barrier	0.8219377007	0.9433919414
Scenario Probability	0.5448892061	0.4551107939

December 10 2003	Scenario 1	Scenario 2
Initial Default Barrier	0.8094537194	0.9878007501
Scenario Probability	0.5347894636	0.4652105364

Table 3: SBTV Calibration to Parmalat CDS: Scenarios

On September 10, when no crisis has started although there are already doubts on the solidity of the Parmalat finances, the market is quoting a 0.215 probability of a very pessimistic situation. This result recalls the findings reported by Cherubini and Manera (2005), who perform a historical analysis of risk neutral probabilities of fraud via a modification of the Merton model. They consider two years of data preceding December 9, 2003, and find that market quotes imply a probability of fraud of 0.2. Although there are some differences in the assumptions and in the analysis between our work and that of Cherubini and Manera (2005) preventing from exact comparison, it appears that we detect with one single day of data a representation of the market expectations qualitatively similar to the one they obtain via historical analysis. In particular it is not surprising that a result similar to theirs is found when we calibrate to market data of September 10, which are more in line with the market situation of the preceding years and less influenced by the irregularities typical of the final few hectic months.

On November 28 the pessimistic scenario, possibly representing misreporting, is slightly more negative on the real company situation in case of misreporting, since the initial barrier is slightly higher. But the real effect of the recent events has been a more than double probability of this worst-scenario situation to be the true one.

On December 10, when the company has already missed a payment but the fraud has not yet been clearly revealed, the probability of fraud appears only slightly increased compared to November, but the market feels that, in case of misreporting, the situation is so bad as to lead to immediate default, and therefore the pessimistic barrier starts at a level which is almost 99% of the current value of the company assets.

Preliminary tests of ours also considered different values for the volatility input or the recovery rate, for example setting them to one half of the values used here. Unnatural choices appear to make calibration at times less easy and thus results less regular, but we still have similar interpretations of the scenarios behaviour as new information reaches the market.

Once scenario barriers and probabilities are set, we have the second stage of calibration, where the time-varying volatility is calibrated to all available market quotes. We have exact calibration as with AT1P, and for the volatility we have the results reported in Table 4.

CDS Maturity T_b	1y	3y	5y	7y	10y
Sep 10: $\sigma([T_{b-1}, T_b))$	5.0%	5.2%	5.6%	6.2%	6.2%
Nov 28: $\sigma([T_{b-1}, T_b))$	5.0%	5.4%	6.4%	8.4%	10.1%
Dec 10: $\sigma([T_{b-1}, T_b))$	5.0%	5.6%	6.2%	8.6%	8.0%

Table 4: SBTV Calibration to Parmalat CDS: Volatilities under Scenarios of Table 3

The movements of the barriers and of the scenario probabilities for representing proximity to default allow now implied volatility to behave more regularly. Variations are lower and values are in line with long-term volatility. No higher volatility in the short term is required to match the market expectation on the possibility of an early default, since this is already implied by the proximity of assets value and default barrier in case the pessimistic scenario is true.

In order to assess the capability of the SBTV model to adapt to different corporate situations, we test the model on the market CDS data for a company with a high credit quality, Vodafone on March 10, 2004, leaving the other inputs unchanged. The CDS data are reported in Table 5. We are interested in checking if the barriers and probabilities obtained for the two possible scenarios actually represent the higher credit quality implied by the Vodafone CDS quotes. Results are reported in Table 6.

CDS Maturity T_b	1y	3y	5y	7y	10y
March 10: Premium rate $R_{0,b}$	21.5	33	43	49	61

Table 5: Vodafone CDS data

March 10 2004	Scenario 1	Scenario 2
Initial Default Barrier	0.3841124933	0.8724843344
Scenario Probability	0.8462346685	0.1537653315

Table 6: SBTV Calibration to Vodafone

With Vodafone CDS data, the optimistic scenario implied by market data represents a fully healthy company, with a level for the default barrier much lower than the barriers we found for Parmalat even before the crisis started. However a pessimistic scenario is considered too, and this allows to explain easily the market short-term credit spreads. This scenario has a lower probability to be true than in Parmalat tests, and in case it is true the situation is worse than it might appear from balance sheet information but the company is still quite far from default. These results confirm that the model allows a meaningful representation of the market feelings about the credit quality of a company also when considering companies with perceived credit reliability higher than Parmalat.

A remark on the use of the model when dealing jointly with different companies is in order. The “cyclical default correlation”, given by dependence on common smoothly varying macroeconomic factors, can be introduced naturally by correlating the dynamics of assets values, as is usual for structural models (see Giesecke (2002) on these issues). Furthermore, the randomness of the barrier introduced in SBTV allows to model not only idiosyncratic uncertainty factors related for example to the firm’s specific risk of fraud, but also those systematic uncertainty factors related to the concept of “default contagion”, an interdependence usually due to business or financial links between companies and very different in nature from cyclical default correlation.

The SBTV model gives us the flexibility to introduce in the model the elements of systematic risk that could emerge when analysing jointly different companies, in fact in SBTV we can correlate the asset value processes and model barrier uncertainty for the plurality of firms considered through a general multivariate discrete random variable.

4 Conclusions

In this paper we develop structural first passage models (AT1P and SBTV) with time-varying volatility which can be calibrated exactly to credit spreads using tractable closed-form formulas for default probabilities, a characteristic which is considered typical of the reduced-form models. In addition in our models, unlike reduced-form models, default events are caused by the value of the firm assets hitting a safety threshold, thus maintaining a link with the economy of the firm. In AT1P this barrier is deterministic, while in SBTV its initial value is a random variable, to take into account the uncertainty on company balance sheets that emerges observing some recent market default events.

With both models we show how to calibrate exactly the CDS market data, and we analyze the output in relation with the flow of information on the credit crisis of Parmalat in the months before default. Results can be interpreted economically, and are particularly realistic if SBTV is considered, also when comparing with the results we obtain for a company with higher credit quality.

Due to their Black and Scholes tractability and their calibration power, the structural models we present here can be applied to precise and efficient valuation of credit derivatives. An application to be addressed in further research is in particular the efficient valuation of hybrid credit/equity payoffs, such as equity default swaps, and of counterparty risk in equity payoffs. Counterparty risk is important also because of the implications of the Basel II framework and of the IAS 39 accounting standards. Structural models are particularly suited for this task since they can provide a realistic link between default probabilities and the economic fundamentals of a company. For example Brigo and Tarenghi (2004, 2005) and Brigo and Masetti (2005) consider the application of tractable structural models to the valuation of an equity return swap.

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