# Option pricing range under statistically indistinguishable models: a new look at historical and implied volatilities

Options: 45 Years after the Publication of the Black Scholes Model,

Gershon Center, Hebrew University, Jerusalem, Dec 4-5, 2018

*Damiano Brigo*
Dept. of Mathematics, Imperial College, London
Joint *J. Armstrong, C. Bellani & T. Cass (2018), F. Mercurio (1998)*

—

**http://arxiv.org/abs/1808.09378** **http://arxiv.org/abs/0812.4010**
and Finance & Stochastics (2000), 4, pp. 147-159

# Agenda I

1. Statistical estimation and valuation
2. Two indistinguishable processes
   - Matching margins
   - Matching the whole law on a $\Delta$ grid
   - Arbitrarily different option prices
   - Surprised?
3. Reconciling historical and implied volatility
4. Possible explanation of arbitrary option prices?
   - Rough paths and option pricing
   - 1998, 2008, 2018: 20 years of pathwise pricing
5. Conclusions and references

# Statistics/econometrics vs pricing

In this work we focus on the following question.

- Take two models $S$ and $Y$ of stock price dynamics under the objective / statistical / historical / physical measure $\mathbb{P}$.

# Statistics/econometrics vs pricing

In this work we focus on the following question.

- Take two models $S$ and $Y$ of stock price dynamics under the objective / statistical / historical / physical measure $\mathbb{P}$.
- Fix a discrete time trading grid with (even very small) step $\Delta$.

# Statistics/econometrics vs pricing

In this work we focus on the following question.

- Take two models $S$ and $Y$ of stock price dynamics under the objective / statistical / historical / physical measure $\mathbb{P}$.
- Fix a discrete time trading grid with (even very small) step $\Delta$.
- Price options on the stock via the continuous time theory of Black, Scholes and Merton (BSM) and Harrison and Kreps, Pliska etc.

# Statistics/econometrics vs pricing

In this work we focus on the following question.

- Take two models $S$ and $Y$ of stock price dynamics under the objective / statistical / historical / physical measure $\mathbb{P}$.
- Fix a discrete time trading grid with (even very small) step $\Delta$.
- Price options on the stock via the continuous time theory of Black, Scholes and Merton (BSM) and Harrison and Kreps, Pliska etc.
- Can we find situations where $S$ and $Y$ are statistically very close (under $\mathbb{P}$), having very close laws in the $\Delta$ grid, but they imply very different option prices (under the pricing / risk-neutral/martingale measure $\mathbb{Q}$)?

# Statistics/econometrics vs pricing

In this work we focus on the following question.

- Take two models $S$ and $Y$ of stock price dynamics under the objective / statistical / historical / physical measure $\mathbb{P}$.
- Fix a discrete time trading grid with (even very small) step $\Delta$.
- Price options on the stock via the continuous time theory of Black, Scholes and Merton (BSM) and Harrison and Kreps, Pliska etc.
- Can we find situations where $S$ and $Y$ are statistically very close (under $\mathbb{P}$), having very close laws in the $\Delta$ grid, but they imply very different option prices (under the pricing / risk-neutral/martingale measure $\mathbb{Q}$)?
- Can we do this in a constructive way, rather than just proving existence theorems?

# The basic idea

- Start from the Black-Scholes-Merton model $dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t, \quad S_0$ (abbreviated BSM$(\mu, \bar{\sigma})$) under the objective measure $\mathbb{P}$.

# The basic idea

- Start from the Black-Scholes-Merton model $dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t, \;\; S_0$ (abbreviated BSM$(\mu, \bar{\sigma})$) under the objective measure $\mathbb{P}$.
- Look for a process $Y$, $dY = u(Y, \ldots)dt + \sigma_t(Y_t)dW$ with local volatility $\sigma_t(Y)$ and with the same *margins* as $S$.

# The basic idea

- Start from the Black-Scholes-Merton model $dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t, \quad S_0$ (abbreviated BSM$(\mu, \bar{\sigma})$) under the objective measure $\mathbb{P}$.
- Look for a process $Y$, $dY = u(Y, \ldots)dt + \sigma_t(Y_t)dW$ with local volatility $\sigma_t(Y)$ and with the same *margins* as $S$.
- To find this, invert the Fokker Planck (FP) equation for $Y$ to find the drift $u$ for $Y$ such that the FP equation has solution $p_{S_t}$, the lognormal density of the original $S$.

# The basic idea

- Start from the Black-Scholes-Merton model $dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t, \quad S_0$ (abbreviated BSM$(\mu, \bar{\sigma})$) under the objective measure $\mathbb{P}$.
- Look for a process $Y$, $dY = u(Y, \ldots)dt + \sigma_t(Y_t)dW$ with local volatility $\sigma_t(Y)$ and with the same *margins* as $S$.
- To find this, invert the Fokker Planck (FP) equation for $Y$ to find the drift $u$ for $Y$ such that the FP equation has solution $p_{S_t}$, the lognormal density of the original $S$.

This is done in B. and Mercurio (1998, 2000) [1, 4] using previous results on diffusions with laws on exponential families (B. (1997) [5] and (2000) [6]).We obtain the following

# The basic idea: matching margins

$$dY_t = u^\sigma_t(Y_t, s_0, 0)dt + \sigma_t(Y_t)dW_t, \quad Y_\epsilon = S_\epsilon, \quad \epsilon \le t \le T, \tag{1}$$

$$\begin{aligned} u^\sigma_t(x,y,\alpha) &:= \frac{1}{2}\frac{\partial(\sigma_t^2)}{\partial x}(x) + \frac{1}{2}\frac{(\sigma_t(x))^2}{x}\left[\frac{\mu}{\bar\sigma^2} - \frac{3}{2} - \frac{1}{\bar\sigma^2(t-\alpha)}\ln\frac{x}{y}\right] \\ &+ \frac{x}{2(t-\alpha)}\left[\ln\frac{x}{y} - \frac{\frac{\mu}{\bar\sigma^2}-\frac{1}{2}}{2-\frac{1}{2\bar\sigma^2(t-\alpha)}}\right]. \end{aligned} \tag{2}$$

where the definition of $Y$ is then extended to the whole interval $[0, T]$ by setting $dY_t = \mu Y_t\, dt + \bar\sigma Y_t dW_t, \ 0 < t < \epsilon, \ Y_0 = s_0$.

# The basic idea: matching margins

$$dY_t = u_t^\sigma(Y_t, s_0, 0)dt + \sigma_t(Y_t)dW_t, \quad Y_\epsilon = S_\epsilon, \quad \epsilon \le t \le T, \tag{1}$$

$$\begin{aligned} u_t^\sigma(x,y,\alpha) &:= \frac{1}{2}\frac{\partial(\sigma_t^2)}{\partial x}(x) + \frac{1}{2}\frac{(\sigma_t(x))^2}{x}\left[\frac{\mu}{\bar\sigma^2} - \frac{3}{2} - \frac{1}{\bar\sigma^2(t-\alpha)}\ln\frac{x}{y}\right] \\ &+ \frac{x}{2(t-\alpha)}\left[\ln\frac{x}{y} - \frac{\frac{\mu}{\bar\sigma^2}-\frac{1}{2}}{2-\frac{1}{2\bar\sigma^2(t-\alpha)}}\right]. \end{aligned} \tag{2}$$

where the definition of $Y$ is then extended to the whole interval $[0, T]$ by setting $dY_t = \mu Y_t\, dt + \bar\sigma Y_t dW_t,\ 0 < t < \epsilon,\ Y_0 = s_0$.

The process $Y$, *if the related SDE is regular enough* (we'll show this to hold in a fundamental case below), has the same marginal distribution as BSM$(\mu, \bar\sigma)$: $\boxed{p_{S_t} = p_{Y_t}}$ for all $t$.

# Matching the whole law

A further fundamental property of the BSM$(\mu, \bar{\sigma})$ model is that its log-returns satisfy

$$\ln \frac{S_{t+\delta}}{S_t} \sim \mathcal{N}\left( (\mu - \frac{1}{2}\bar{\sigma}^2)\delta,\ \bar{\sigma}^2\delta \right), \quad \delta > 0, \quad t \in [0, T-\delta].$$

Alternative models such as our $Y$ above do not share this property because identity of the marginal laws alone does not suffice to ensure it. We need equality of second order laws or of transition densities.

# Matching the whole law

A further fundamental property of the BSM$(\mu, \bar{\sigma})$ model is that its log-returns satisfy

$$\ln \frac{S_{t+\delta}}{S_t} \sim \mathcal{N}\left((\mu - \frac{1}{2}\bar{\sigma}^2)\delta,\ \bar{\sigma}^2\delta\right), \quad \delta > 0, \quad t \in [0, T-\delta].$$

Alternative models such as our $Y$ above do not share this property because identity of the marginal laws alone does not suffice to ensure it. We need equality of second order laws or of transition densities.

To tackle this issue, we restrict the set of dates for which the log-return property must hold true. Modify the definition of $Y$ so that, given $\mathcal{T}^\Delta := \{0, \Delta, 2\Delta, \ldots, N\Delta\}$, $\Delta = T/N$, $\Delta > \epsilon$, we have

$$\ln \frac{Y_{i\Delta}}{Y_{j\Delta}} \sim \mathcal{N}((\mu - \frac{1}{2}\bar{\sigma}^2)(i-j)\Delta,\ \bar{\sigma}^2(i-j)\Delta), \ \ i > j. \tag{3}$$

# Matching the whole law

Limiting such key property to a finite set of times is not so dramatic. Indeed, only discrete time samples are observed in practice, so that once the time instants are fixed, our process $Y$ can not be distinguished from Black and Scholes process.

# Matching the whole law

Limiting such key property to a finite set of times is not so dramatic. Indeed, only discrete time samples are observed in practice, so that once the time instants are fixed, our process $Y$ can not be distinguished from Black and Scholes process.

The new definition of $Y$ is still based on our earlier $Y$. However, we use the earlier $Y$ process "locally" in each time interval $[(i-1)\Delta,\ i\Delta)$. In such interval we define iteratively the drift $u^\sigma$ as in the earlier $Y$ but

# Matching the whole law

Limiting such key property to a finite set of times is not so dramatic. Indeed, only discrete time samples are observed in practice, so that once the time instants are fixed, our process $Y$ can not be distinguished from Black and Scholes process.

The new definition of $Y$ is still based on our earlier $Y$. However, we use the earlier $Y$ process "locally" in each time interval $[(i-1)\Delta,\ i\Delta)$. In such interval we define iteratively the drift $u^\sigma$ as in the earlier $Y$ but

- we translate back the time–dependence of a time amount $(i-1)\Delta$ (thus locally restoring the dynamics of the original result) and

# Matching the whole law

Limiting such key property to a finite set of times is not so dramatic. Indeed, only discrete time samples are observed in practice, so that once the time instants are fixed, our process $Y$ can not be distinguished from Black and Scholes process.

The new definition of $Y$ is still based on our earlier $Y$. However, we use the earlier $Y$ process "locally" in each time interval $[(i-1)\Delta,\ i\Delta)$. In such interval we define iteratively the drift $u^\sigma$ as in the earlier $Y$ but

- we translate back the time–dependence of a time amount $(i-1)\Delta$ (thus locally restoring the dynamics of the original result) and
- we replace $Y_0$ with the final value of $Y$ relative to the previous interval. This will also replace $p_0$ with $p_Y$ at the end of last interval.

# Matching the whole law

$$dY_t = u_t^\sigma(Y_t, Y_{\alpha(t)}, \alpha(t))dt + \sigma_t(Y_t)dW_t, \quad t \in [i\Delta + \epsilon, (i+1)\Delta), \tag{4}$$

$$dY_t = \mu Y_t dt + \bar{\sigma} Y_t dW_t, \quad t \in [i\Delta, i\Delta + \epsilon),$$

where $u_t^\sigma(x, y, \alpha)$ was defined in the earlier $Y$. Note that $\alpha(t) = i\Delta$ for $t \in [i\Delta,\ (i+1)\Delta)$.

# Matching the whole law

$$dY_t = u_t^\sigma(Y_t, Y_{\alpha(t)}, \alpha(t))dt + \sigma_t(Y_t)dW_t, \quad t \in [i\Delta + \epsilon, (i+1)\Delta), \tag{4}$$

$$dY_t = \mu Y_t dt + \bar{\sigma} Y_t dW_t, \quad t \in [i\Delta, i\Delta + \epsilon),$$

where $u_t^\sigma(x, y, \alpha)$ was defined in the earlier $Y$. Note that $\alpha(t) = i\Delta$ for $t \in [i\Delta,\ (i+1)\Delta)$.

It is clear by construction that the transition densities of $S$ and $Y$ satisfy $p_{Y_{(i+1)\Delta}|Y_{i\Delta}}(x; y) = p_{S_{(i+1)\Delta}|S_{i\Delta}}(x; y)$.

Note that the new process $Y$ is not a Markov process in $[0, T]$. However, it is Markov in all time instants of $\mathcal{T}^\Delta$ ($\Delta$–*Markovianity*).

# Matching the whole law

$$dY_t = u_t^\sigma(Y_t, Y_{\alpha(t)}, \alpha(t))dt + \sigma_t(Y_t)dW_t, \quad t \in [i\Delta + \epsilon, (i+1)\Delta), \tag{4}$$

$$dY_t = \mu Y_t dt + \bar{\sigma} Y_t dW_t, \quad t \in [i\Delta, i\Delta + \epsilon),$$

where $u_t^\sigma(x, y, \alpha)$ was defined in the earlier $Y$. Note that $\alpha(t) = i\Delta$ for $t \in [i\Delta,\ (i+1)\Delta)$.

It is clear by construction that the transition densities of $S$ and $Y$ satisfy $p_{Y_{(i+1)\Delta}|Y_{i\Delta}}(x; y) = p_{S_{(i+1)\Delta}|S_{i\Delta}}(x; y)$.

Note that the new process $Y$ is not a Markov process in $[0, T]$. However, it is Markov in all time instants of $\mathcal{T}^\Delta$ ($\Delta$–*Markovianity*).

Note that now the two models $S$ (BSM$(\mu, \bar{\sigma})$) and $Y$ are *statistically indistinguishable* in $\mathcal{T}^\Delta$ since there they share the same finite dimensional distributions. But *what option prices do they imply?*

# A fundamental case

We take now $\sigma(Y) = \nu Y$, so that also the volatility of $Y$ is of BSM type, but with vol $\nu$ instead of $\bar{\sigma}$. Still, with the drift $u$, $S$ and $Y$ will be indistinguishable in $\mathcal{T}^{\Delta}$.

# A fundamental case

We take now $\sigma(Y) = \nu Y$, so that also the volatility of $Y$ is of BSM type, but with vol $\nu$ instead of $\bar{\sigma}$. Still, with the drift $u$, $S$ and $Y$ will be indistinguishable in $\mathcal{T}^{\Delta}$.

In this case the equation for $u$ specializes to

$$u_t^{\nu}(y, y_\alpha, \alpha) = y\left[\frac{1}{4}(\nu^2 - \bar{\sigma}^2) + \frac{\mu}{2}(\frac{\nu^2}{\bar{\sigma}^2} + 1)\right] + \frac{y}{2(t-\alpha)}(1 - \frac{\nu^2}{\bar{\sigma}^2}) \ln \frac{y}{y_\alpha},$$

and one can show that the SDE for $Y$ has a unique strong solution.

# A fundamental case

We take now $\sigma(Y) = \nu Y$, so that also the volatility of $Y$ is of BSM type, but with vol $\nu$ instead of $\bar{\sigma}$. Still, with the drift $u$, $S$ and $Y$ will be indistinguishable in $\mathcal{T}^{\Delta}$.

In this case the equation for $u$ specializes to

$$u_t^{\nu}(y, y_{\alpha}, \alpha) = y \left[ \frac{1}{4}(\nu^2 - \bar{\sigma}^2) + \frac{\mu}{2}\left(\frac{\nu^2}{\bar{\sigma}^2} + 1\right) \right] + \frac{y}{2(t-\alpha)}\left(1 - \frac{\nu^2}{\bar{\sigma}^2}\right) \ln \frac{y}{y_{\alpha}},$$

and one can show that the SDE for $Y$ has a unique strong solution.

Moreover, the change of measure that replaces the drift $u$ with $rY$ is well defined and regular, so that it is possible to change probability measure from $\mathbb{P}$ to $\mathbb{Q}$ for the model $Y$.

# A fundamental case

We take now $\sigma(Y) = \nu Y$, so that also the volatility of $Y$ is of BSM type, but with vol $\nu$ instead of $\bar{\sigma}$. Still, with the drift $u$, $S$ and $Y$ will be indistinguishable in $\mathcal{T}^{\Delta}$.

In this case the equation for $u$ specializes to

$$u_t^{\nu}(y, y_\alpha, \alpha) = y\left[\frac{1}{4}(\nu^2 - \bar{\sigma}^2) + \frac{\mu}{2}(\frac{\nu^2}{\bar{\sigma}^2} + 1)\right] + \frac{y}{2(t-\alpha)}(1 - \frac{\nu^2}{\bar{\sigma}^2}) \ln \frac{y}{y_\alpha},$$

and one can show that the SDE for $Y$ has a unique strong solution.

Moreover, the change of measure that replaces the drift $u$ with $rY$ is well defined and regular, so that it is possible to change probability measure from $\mathbb{P}$ to $\mathbb{Q}$ for the model $Y$.

But what happens when we change measure?

# Indistinguishable under $\mathbb{P}$, different under $\mathbb{Q}$

$$dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t^{\mathbb{P}} \quad \Delta - indistinguish.from \quad dY_t = u_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{P}}$$

but under $\mathbb{Q}$

$$dS_t = rS_t dt + \bar{\sigma} S_t dW_t^{\mathbb{Q}} \quad very\ different\ from \quad dY_t^{\nu} = rY_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{Q}}$$

# Indistinguishable under $\mathbb{P}$, different under $\mathbb{Q}$

$$dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t^{\mathbb{P}} \quad \Delta - indistinguish.from \quad dY_t = u_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{P}}$$

but under $\mathbb{Q}$

$$dS_t = rS_t dt + \bar{\sigma} S_t dW_t^{\mathbb{Q}} \quad very\ different\ from \quad dY_t^{\nu} = rY_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{Q}}$$

If we price a call option:

$$E^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = BScholes(\bar{\sigma}),\ \ E^{\mathbb{Q}}[e^{-rT}(Y_T - K)^+] = BScholes(\nu)$$

# Indistinguishable under $\mathbb{P}$, different under $\mathbb{Q}$

$$dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t^{\mathbb{P}} \quad \Delta - indistinguish.from \quad dY_t = u_t^{\nu} dt + \nu \; Y_t \; dW_t^{\mathbb{P}}$$

but under $\mathbb{Q}$

$$dS_t = rS_t dt + \bar{\sigma} S_t dW_t^{\mathbb{Q}} \quad very \; different \; from \quad dY_t^{\nu} = rY_t^{\nu} dt + \nu \; Y_t \; dW_t^{\mathbb{Q}}$$

If we price a call option:

$$E^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = BScholes(\bar{\sigma}), \;\; E^{\mathbb{Q}}[e^{-rT}(Y_T - K)^+] = BScholes(\nu)$$

Since the indistinguishability holds for every $\nu$, we can take $\nu \downarrow 0$ and $\nu \uparrow +\infty$. This way we find that

# Indistinguishable under $\mathbb{P}$, different under $\mathbb{Q}$

$$dS_t = \mu S_t dt + \bar\sigma S_t dW_t^{\mathbb{P}} \quad \Delta - indistinguish.from \quad dY_t = u_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{P}}$$

but under $\mathbb{Q}$

$$dS_t = rS_t dt + \bar\sigma S_t dW_t^{\mathbb{Q}} \quad very\ different\ from \quad dY_t^{\nu} = rY_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{Q}}$$

If we price a call option:

$$E^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = BScholes(\bar\sigma), \ E^{\mathbb{Q}}[e^{-rT}(Y_T - K)^+] = BScholes(\nu)$$

Since the indistinguishability holds for every $\nu$, we can take $\nu \downarrow 0$ and $\nu \uparrow +\infty$. This way we find that

*statistically indistinguishable stock price models* imply options prices so different to span the whole no arbitrage interval $[(S_0 - K)^+, S_0]$.

# Indistinguishable under $\mathbb{P}$, different under $\mathbb{Q}$

$$dS_t = \mu S_t dt + \bar{\sigma} S_t dW_t^{\mathbb{P}} \quad \Delta - indistinguish.from \quad dY_t = u_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{P}}$$

but under $\mathbb{Q}$

$$dS_t = rS_t dt + \bar{\sigma} S_t dW_t^{\mathbb{Q}} \quad very\ different\ from \quad dY_t^{\nu} = rY_t^{\nu} dt + \nu\ Y_t\ dW_t^{\mathbb{Q}}$$

If we price a call option:

$$E^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = BScholes(\bar{\sigma}), \;\; E^{\mathbb{Q}}[e^{-rT}(Y_T - K)^+] = BScholes(\nu)$$

Since the indistinguishability holds for every $\nu$, we can take $\nu \downarrow 0$ and $\nu \uparrow +\infty$. This way we find that

*statistically indistinguishable stock price models* imply options prices so different to span the whole no arbitrage interval $[(S_0 - K)^+, S_0]$.

Perhaps surprisingly, they span a range that is not related to $\Delta$.

# Surprised?

In 1997, this was presented at a mainstream math finance conference. A leading MF academic stood up and said “I don’t believe it”, interrupting the presentation half-way.

# Surprised?

In 1997, this was presented at a mainstream math finance conference. A leading MF academic stood up and said “I don’t believe it”, interrupting the presentation half-way.

We already knew that in discrete time the market is incomplete, but we illustrated this in a constructive and graphic way, and found that incompleteness is quite extreme.

# Surprised?

In 1997, this was presented at a mainstream math finance conference. A leading MF academic stood up and said “I don’t believe it”, interrupting the presentation half-way.

We already knew that in discrete time the market is incomplete, but we illustrated this in a constructive and graphic way, and found that incompleteness is quite extreme.

Our result shows that conjugating discrete and continuos time modeling (e.g. econometrics and option pricing) might be quite problematic.

# Consistent historical and implied volatility

Possible interpretation: what if what we have seen is a way to account separately for *historical* and *implied* volatility?

- Option prices trade independently of the underlying stock price

# Consistent historical and implied volatility

Possible interpretation: what if what we have seen is a way to account separately for *historical* and *implied* volatility?

- Option prices trade independently of the underlying stock price
- We have been able to construct a stock price process $Y^{\nu}$ whose marginal distribution and transition density depend on the volatility coefficient $\bar{\sigma}$, whereas the corresponding option price only depends on the volatility coefficient $\nu$.

# Consistent historical and implied volatility

Possible interpretation: what if what we have seen is a way to account separately for *historical* and *implied* volatility?

- Option prices trade independently of the underlying stock price
- We have been able to construct a stock price process $Y^\nu$ whose marginal distribution and transition density depend on the volatility coefficient $\bar{\sigma}$, whereas the corresponding option price only depends on the volatility coefficient $\nu$.
- As a consequence, we can provide a consistent theoretical framework which justifies the differences between historical and implied volatility that are commonly observed in real markets.

# We got rough volatility too (well kind of) I

As $Y^\nu$ has the same margins as $S$, $Y_t^\nu > 0$. Then take $Z_t = \ln Y_t^\nu$:

$$Z_t = Z_{j\Delta} + (\mu - \frac{1}{2}\bar{\sigma}^2)(t - j\Delta) \quad (5)$$
$$+ \begin{cases} \bar{\sigma}(W_t - W_{j\Delta}) & \text{for } t \in [j\Delta, j\Delta + \epsilon), \\ \left(\frac{t-j\Delta}{\epsilon}\right)^{\beta/2} \left[\bar{\sigma}(W_{j\Delta+\epsilon} - W_{j\Delta}) + \nu \int_{j\Delta+\epsilon}^t \left(\frac{u-j\Delta}{\epsilon}\right)^{-\beta/2} dW_u\right] \end{cases}$$

the second for $t \in [j\Delta + \epsilon, (j+1)\epsilon)$ and where $\beta = 1 - \frac{\nu^2}{\bar{\sigma}^2}$.
In [1] we show that we can take $\epsilon \to 0$ in the regularization:

$$Z_t = Z_{j\Delta} + (\mu - \frac{\bar{\sigma}^2}{2})(t - j\Delta) + \nu \int_{j\Delta}^t \left[\frac{t - j\Delta}{u - j\Delta}\right]^{\frac{\beta}{2}} dW_u, \quad t \in [j\Delta, (j+1)\Delta).$$

This process is well defined since the integral in the right-hand side exists finite a.s. even though its integrand diverges when $u \to j\Delta^+$.

# Pathwise approach using rough paths

Since probability and statistics have proven to be deceiving when working in discrete time (as is unavoidable) under $\mathbb{P}$, we try now to strip the valuation from probability and statistics.

# Pathwise approach using rough paths

Since probability and statistics have proven to be deceiving when working in discrete time (as is unavoidable) under $\mathbb{P}$, we try now to strip the valuation from probability and statistics.

Applying rough paths theory of Foellmer, Lyons, Davie, Friz, Gubinelli et al, in Armstrong et al. (2018) [3] we manage to re-interpret the Black Scholes formula & option pricing in a purely pathwise sense.

# Pathwise approach using rough paths

Since probability and statistics have proven to be deceiving when working in discrete time (as is unavoidable) under $\mathbb{P}$, we try now to strip the valuation from probability and statistics.

Applying rough paths theory of Foellmer, Lyons, Davie, Friz, Gubinelli et al, in Armstrong et al. (2018) [3] we manage to re-interpret the Black Scholes formula & option pricing in a purely pathwise sense.

In our work [3] we abandon even semimartingales: using Davie's rough differential equations and rough brackets we leave probability theory altogether, giving an extreme version of the result of Bender et al. [2]

# Rough paths and option pricing

We denote in general $X_{s,t} = X_t - X_s$. Recall the $BSM(\mu, \sigma)$ model

$$dB_t = B_t r dt, \;\; B_0 = 1, \quad dS_t = S_t[\mu dt + \sigma dW_t^{\mathbb{P}}], \;\; 0 \le t \le T.$$

As we give up probability, we won't be able to use stochastic integrals any more. To compensate for this, we will need to add information on the price trajectory in the form of a lift. We need to provide the input

$$\mathbb{S}_{s,t} = \int_s^t S_{s,u} dS_u.$$

This is really an input: if the signal $S$ has finite $p$-variation for $2 < p < 3$, as in case of paths in the Black Scholes model, it is too rough to define the above intergral as a Stiltjes or Young integral. We need therefore to add it ourselves. But how does $\mathbb{S}_{s,t}$ help in defining other integrals?

# Rough paths and option pricing

Why does this help? Consider $\int F(S_r)dS_r$ and try to write it as a Young integral. Take Taylor expansion $F(S_r) \approx F(S_u) + DF(S_u)S_{u,r}$.

# Rough paths and option pricing

Why does this help? Consider $\int F(S_r)dS_r$ and try to write it as a Young integral. Take Taylor expansion $F(S_r) \approx F(S_u) + DF(S_u)S_{u,r}$.

The Young integral can be seen as approximating $F(S_r)$, in each $[u,t] \in \pi$ with the zero-th order term $F(S_u)$. Hence

$$\int_0^T F(S_r)dS_r = \lim_{|\pi|\to 0} \sum_{[u,t]\in\pi} \int_u^t F(S_u)dS_r = \lim_{|\pi|\to 0} \sum_{[u,t]\in\pi} F(S_u)S_{u,t}.$$

# Rough paths and option pricing

Why does this help? Consider $\int F(S_r)dS_r$ and try to write it as a Young integral. Take Taylor expansion $F(S_r) \approx F(S_u) + DF(S_u)S_{u,r}$.

The Young integral can be seen as approximating $F(S_r)$, in each $[u,t] \in \pi$ with the zero-th order term $F(S_u)$. Hence

$$\int_0^T F(S_r)dS_r = \lim_{|\pi|\to 0} \sum_{[u,t]\in\pi} \int_u^t F(S_u)dS_r = \lim_{|\pi|\to 0} \sum_{[u,t]\in\pi} F(S_u)S_{u,t}.$$

(limit is on *all partitions* whose mesh size tends to zero). If we can't use Young because $S$ is too rough, try a 1st order expansion

$$\int_0^T F(S_r)d\mathbf{S}_r = \lim_{|\pi|\to 0} \sum_{[u,t]\in\pi} \int_u^t (F(S_u) + DF(S_u)S_{u,r})\, dS_r =$$

$$= \lim_{|\pi|\to 0} \sum_{[u,t]\in\pi} (F(S_u)S_{u,t} + DF(S_u)\boxed{\mathbb{S}_{u,t}}).$$

This intuition can be made rigorous. Now going back to BSM:

# Rough paths and option pricing

Take $S_t$ as a path of finite $p$ variation, $2 < p < 3$ (Brownian motion has finite $p$ variation for $p > 2$, so $S$ is potentially rougher than BSM).

# Rough paths and option pricing

Take $S_t$ as a path of finite $p$ variation, $2 < p < 3$ (Brownian motion has finite $p$ variation for $p > 2$, so $S$ is potentially rougher than BSM).

Consider the lifted $\mathbf{S}_t := (S_t, \mathbb{S}_t)$, where $\mathbb{S}$ is our input for $\int S\, dS$.

# Rough paths and option pricing

Take $S_t$ as a path of finite $p$ variation, $2 < p < 3$ (Brownian motion has finite $p$ variation for $p > 2$, so $S$ is potentially rougher than BSM).

Consider the lifted $\mathbf{S}_t := (S_t, \mathbb{S}_t)$, where $\mathbb{S}$ is our input for $\int S\, dS$.

Technical note: we work with *reduced* rough paths, obtained from the pair $(S, \mathbb{S})$ by considering only the symmetric part of $\mathbb{S}$. This is equivalently described by the rough bracket defined in

$$[\mathbf{S}]_{u,t} = S_{u,t} S_{u,t} - 2\, \mathbb{S}_{u,t}$$

A *reduced* rough path with bounded variation bracket is a path where $[\mathbf{S}]_t$ is a continuous path of finite (1–) variation.

# Rough paths and option pricing

If $[\mathbf{S}]_{u,t}$ is regular enough to define a measure of $[u, t]$ with density $a(S_t)$ with $a(x)$ also regular, then PDE for the option price is defined entirely in terms of the *purely pathwise* $[\mathbf{S}]$, without probability.

# Rough paths and option pricing

If $[\mathbf{S}]_{u,t}$ is regular enough to define a measure of $[u, t]$ with density $a(S_t)$ with $a(x)$ also regular, then PDE for the option price is defined entirely in terms of the *purely pathwise* $[\mathbf{S}]$, without probability.

It follows that the option price will not depend on the probabilistic setting but only on path properties. Notice that we don't need semimartingales quadratic variation, our definition is more general.

# Rough paths and option pricing

If $[\mathbf{S}]_{u,t}$ is regular enough to define a measure of $[u, t]$ with density $a(S_t)$ with $a(x)$ also regular, then PDE for the option price is defined entirely in terms of the *purely pathwise* $[\mathbf{S}]$, without probability.

It follows that the option price will not depend on the probabilistic setting but only on path properties. Notice that we don't need semimartingales quadratic variation, our definition is more general.

The purely pathwise property $[\mathbf{S}]_{u,t}$ takes the place of implied volatility in determining the option price as a path property rather than a statistical property. The latter would be associated with historical volatility as a standard deviation (statistics).

# Rough paths and option pricing

This is consistent with B. and Mercurio 1998 result above [1] and with Bender, Sottinen and Valkeila (2008) [2] who observe, in a reference we found after writing [3]:

# Rough paths and option pricing

This is consistent with B. and Mercurio 1998 result above [1] and with Bender, Sottinen and Valkeila (2008) [2] who observe, in a reference we found after writing [3]:

*"[...] the covariance structure of the stock returns is not relevant for option pricing, but the quadratic variation is.*

# Rough paths and option pricing

This is consistent with B. and Mercurio 1998 result above [1] and with Bender, Sottinen and Valkeila (2008) [2] who observe, in a reference we found after writing [3]:

*"[...] the covariance structure of the stock returns is not relevant for option pricing, but the quadratic variation is. So, one should not be surprised if the historical and implied volatilities do not agree: the former is an estimate of the variance and the latter is an estimate of the [semimartingale] quadratic variation".*

# Rough paths and option pricing

This is consistent with B. and Mercurio 1998 result above [1] and with Bender, Sottinen and Valkeila (2008) [2] who observe, in a reference we found after writing [3]:

*"[...] the covariance structure of the stock returns is not relevant for option pricing, but the quadratic variation is. So, one should not be surprised if the historical and implied volatilities do not agree: the former is an estimate of the variance and the latter is an estimate of the [semimartingale] quadratic variation".*

**For us [3] historical vol is a stats of the variance too, while implied vol is associated with a pathwise lift [no semimartingales].**

# Rough paths and option pricing

This is consistent with B. and Mercurio 1998 result above [1] and with Bender, Sottinen and Valkeila (2008) [2] who observe, in a reference we found after writing [3]:

*"[...] the covariance structure of the stock returns is not relevant for option pricing, but the quadratic variation is. So, one should not be surprised if the historical and implied volatilities do not agree: the former is an estimate of the variance and the latter is an estimate of the [semimartingale] quadratic variation".*

**For us [3] historical vol is a stats of the variance too, while implied vol is associated with a pathwise lift [no semimartingales].**

20 years of "pathwise" pricing

1998: [1] $\longrightarrow$ 2008: [2] $\longrightarrow$ 2018: [3]

# References I


[1] Brigo, D. and Mercurio, F. (1998). Discrete time vs continuous time stock price dynamics and implications for option pricing. arXiv.org and SSRN.com

[2] Bender, C., Sottinen, T., and Valkeila, E. (2008). Pricing by hedging and no-arbitrage beyond semimartingales. Finance and Stochastics, Vol. 12(4), pp 441–468

[3] Armstrong, J., Bellani, C., Brigo, D., and Cass, T. (2018). Gamma-controlled pathwise hedging in generalised Black-Scholes models. https://arxiv.org/abs/1808.09378

[4] Brigo, D. and Mercurio, F. (2000). Option pricing impact of alternative continuous time dynamics for discretely observed stock prices. Finance & Stochastics (2000), 4, pp. 147-159

# References II

[5] Brigo, D. (1997). On nonlinear SDEs whose densities evolve in a finite–dimensional family. In: Stochastic Differential and Difference Equations, Progress in Systems and Control Theory 23: 11–19, Birkhäuser, Boston.

[6] Brigo, D (2000). On SDEs with marginal laws evolving in finite-dimensional exponential families, Statistics and Probability Letters, 49: 127 – 134