Consistent iterated simulation of MV defaults: Markovian indicators characterization and Marshall Olkin law

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Innovations in Insurance, Risk & Asset Mgmt, TUM, 5-7/04/2017

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Joint work with
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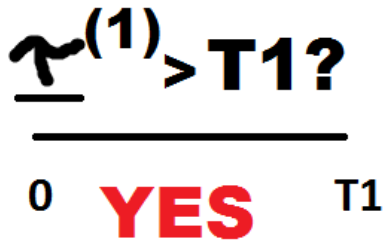
Based on the papers
B. & Chourdakis (2012) [8], B., Mai & Scherer (2013, 2016) [9, 10]

Earlier version presented at:

- Bachelier Seminar, Institut Henri Poincare, Paris, April 8, 2016
- RiskMinds, Amsterdam, Dec 9, 2015
- Nomura, Internal Risk Seminar, London, Aug 12, 2015
- Quant Congress Europe, London, April 15, 2015
- Imperial-ETH worskshop, London, March 6, 2015.

Content I

Iterating default simulations: Survival of all



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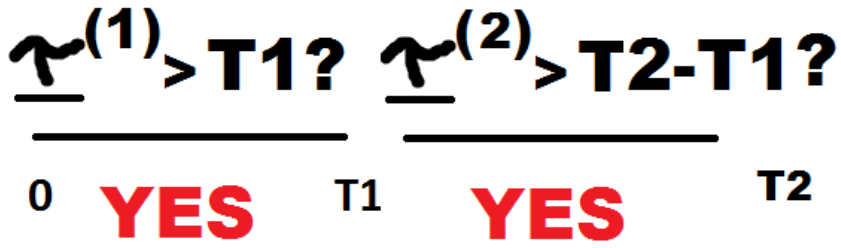
Iterating default simulations: Survival of all



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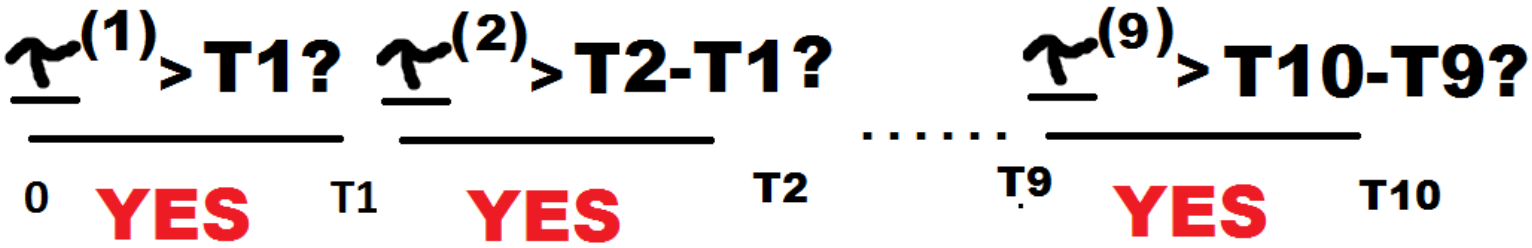

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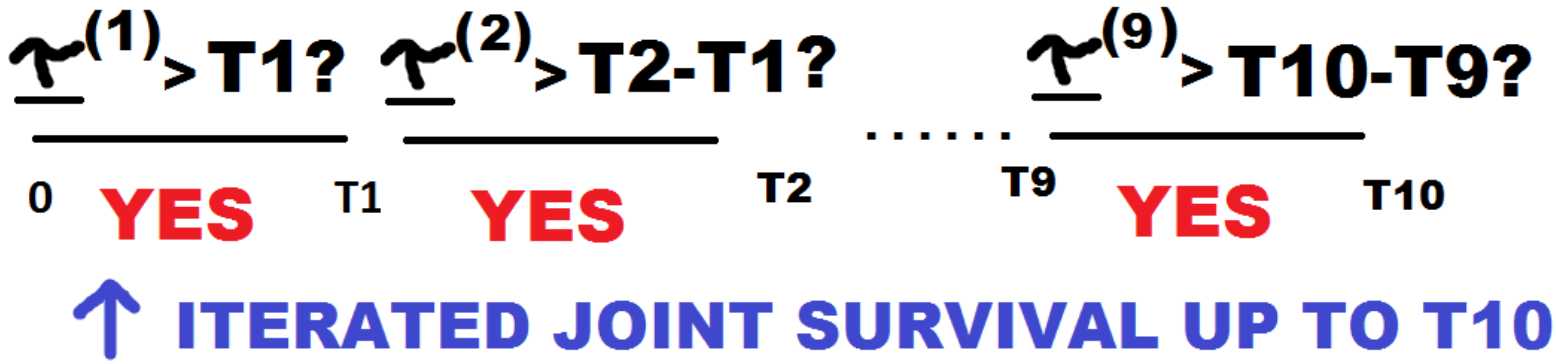

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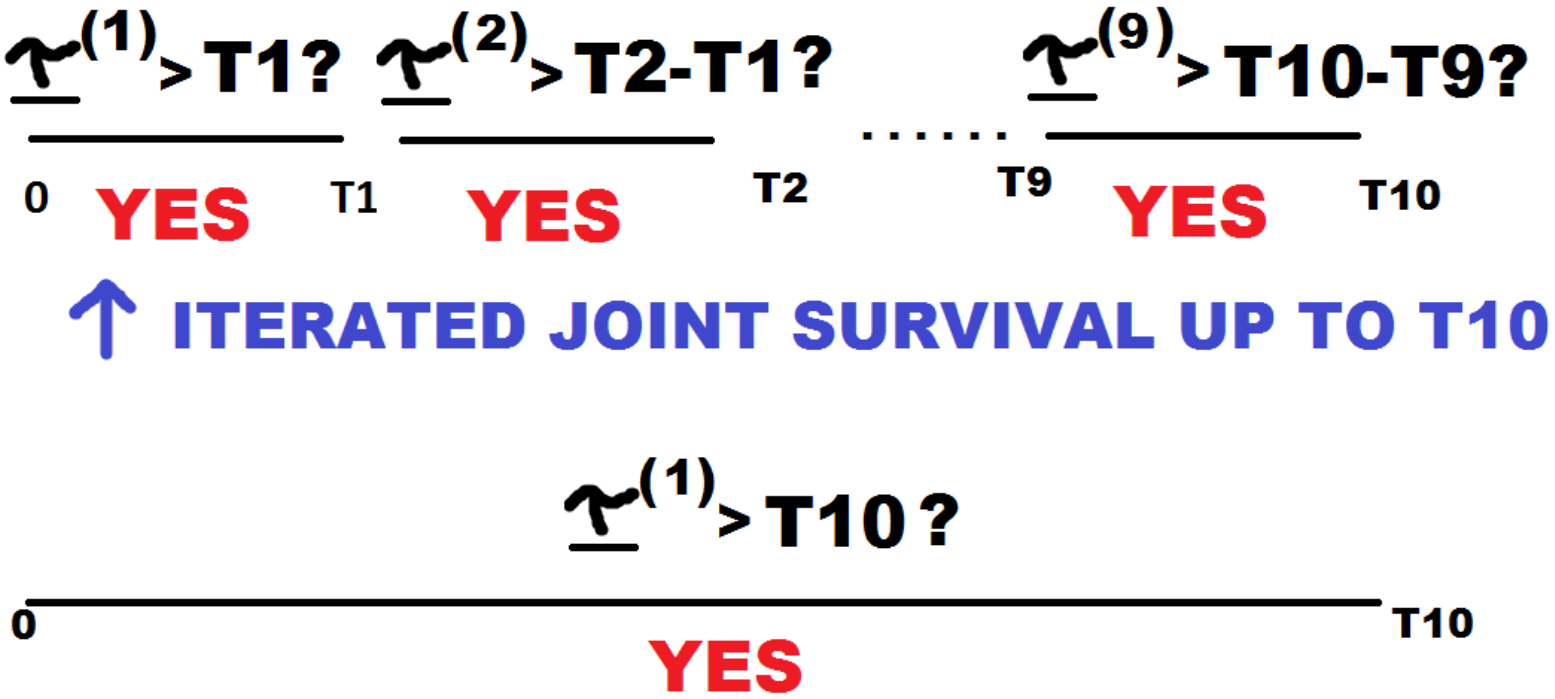

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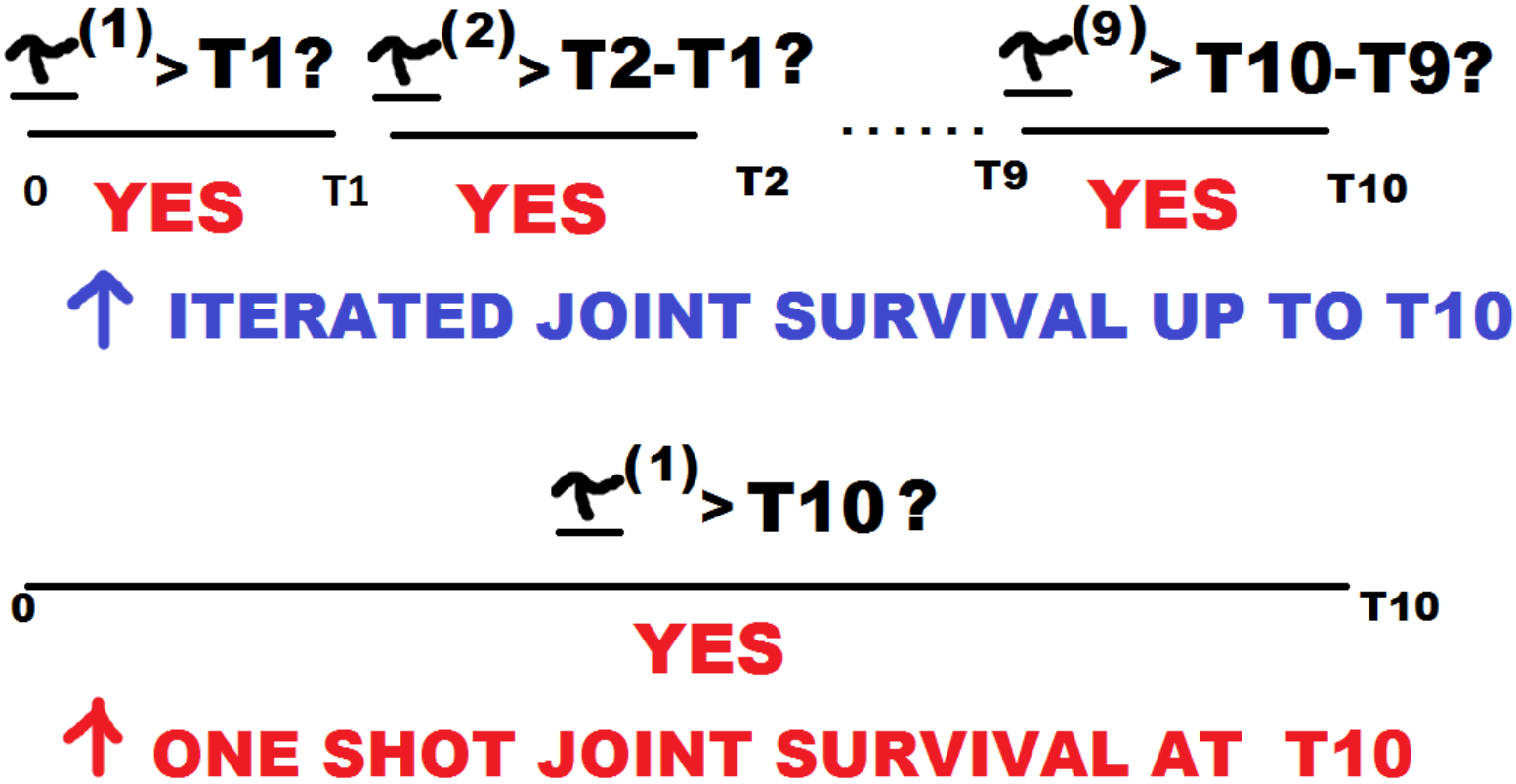

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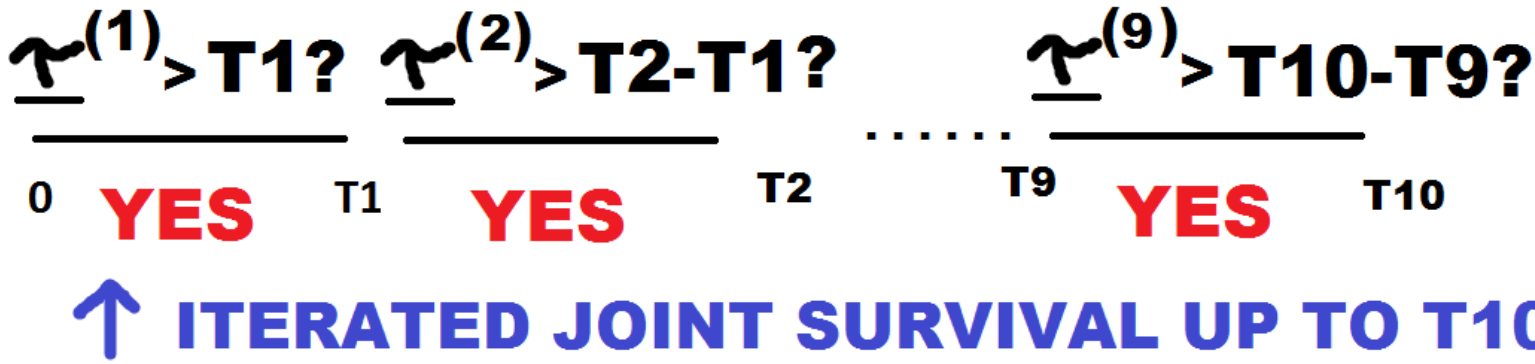
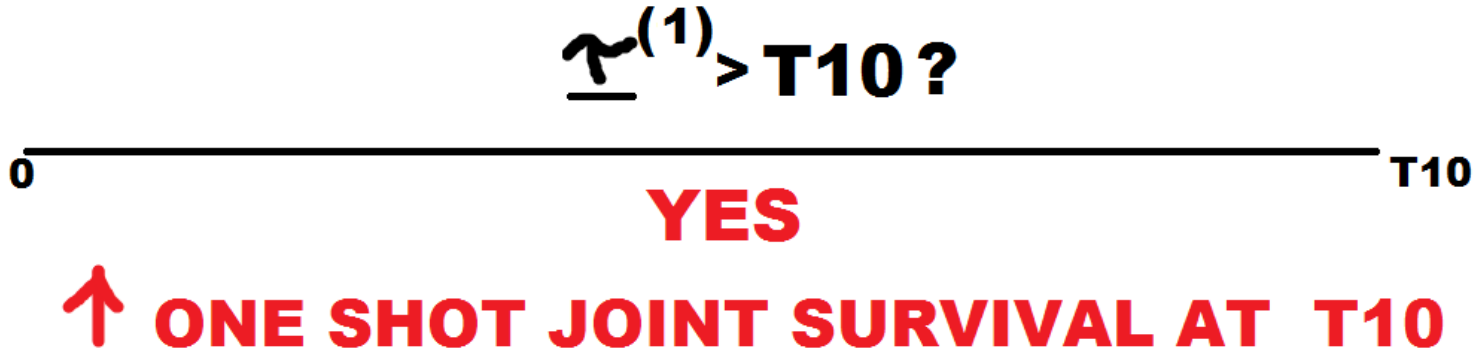




WHEN ARE THE TWO METHODS EQUIVALENT ?

$\underline{\tau} = [\tau_1, \tau_2, \ldots, \tau_d]$; $\underline{\tau}^{(i)}$: iid copies of $\underline{\tau}$

Problems: (1) all-survival and (2) intermediate defaults

In fact a full specification will lead us to two related problems.

Problem (1): The (easier) problem of all survival in the earlier picture: when iterated (wrong) equivalent to one shot (correct)?

Problem (2): The general problem with in-between defaults. Can we transform the one shot (correct) into an iterated process (wrong) when we care about more general survival/default patterns?

Notation. $\underline{\tau} = [\tau_1, \tau_2, \ldots, \tau_d]$ vector of default times for names $1, \ldots, d$.

(1): look at joint survival at deterministic $\underline{S} = [S_1, S_2, \ldots, S_d]$:

$$\mathbb{P}(\underline{\tau} > \underline{S}) := \mathbb{P}(\tau_1 > S_1,\ \tau_2 > S_2, \ldots,\ \tau_d > S_d) =: G(S_1, \ldots, S_d)$$

joint survival function. We abbreviate $\mathbb{P}(\underline{\tau} > S\underline{1})$ with $\mathbb{P}(\underline{\tau} > S)$.

Practical interest?

- **Basel III requirement for risk horizons:** BIS suggests *"The Committee has agreed that the differentiation of market liquidity across the trading book will be based on the concept of liquidity horizons. It proposes that banks' trading book exposures be assigned to a small number of liquidity horizon categories. [10 days, 1 month, 3 months, 6 months, 1 year] [...]".* A bank will need to simulate the risk factors of the portfolio across a grid including the standardized holding periods above.
- **Consistency with "Brownian-driven" asset classes simulation**. Risk measure or valuation adjustment simulation. Evolve risk factors according to common controlled time steps. Natural for asset models that are Brownian driven but harder when trying to include defaults. This is because default times, typically in intensity models, should be simulated just once, being static random variables as opposed to random processes.

The univariate case: key property is lack of memory

Definition. In the univariate case $d = 1$ we say that the distribution of τ has lack of memory (LOM) when for all $S, U > 0$

$$\mathbb{P}(\tau > S + U | \tau > S) = \mathbb{P}(\tau > U) \quad (\iff \quad G(S+U) = G(S)G(U)).$$

If $0 < G \leq 1$ we can take logs and get Cauchy's functional equation. With continuity in at least one point $\Rightarrow \; G(t) = \exp(-\lambda t)$.

As we know well, lack of memory is a characterization of the exponential distribution in case $d = 1$.

This solves problem (1) in the univariate case. Indeed,

$$\text{One-shot prob} = \mathbb{P}(\tau > S + U) = \mathbb{P}(\tau > S + U | \tau > S)\mathbb{P}(\tau > S) =$$

$$= (LOM) = \mathbb{P}(\tau^{(1)} > U)\mathbb{P}(\tau^{(2)} > S) = \text{iterated prob}$$

Lack of memory in the multivariate case

Definition. The distribution of $\underline{\tau}$ has multivariate homogeneous lack of memory (MHLOM) when, given $T > 0$, for any two integers i, j, $i < j$

$$\mathbb{P}(\underline{\tau} > jT | \underline{\tau} > iT) = \mathbb{P}(\underline{\tau} > (j-i)T) \quad (\iff \quad G((jT)\underline{1}) = G(iT\underline{1})G((j-i)T\underline{1})$$

The definition is formally the same as for the univariate case when $S = iT, U = jT$, ie under "homogeneity" (of the time step T).

One might try to adopt a more general definition of lack of memory, namely for all $\underline{S} = [S_1, \ldots, S_d]$, $\underline{U} = [U_1, \ldots, U_d]$ deterministic times

$$\mathbb{P}(\underline{\tau} > \underline{S} + \underline{U} | \underline{\tau} > \underline{S}) = \mathbb{P}(\underline{\tau} > \underline{U}).$$

This however is too strong and results in the trivial case of independence of exponential univariates, see [24].

Lack of memory in the multivariate case

The most general definition of multivariate lack of memory, without collapsing into indepedence, assumes uniformity in conditioning time $\underline{S}$ but not in increment time $\underline{U}$:

Definition MLOM: Every subvect $\underline{\tau}_I$ of $\underline{\tau}$, $I \subset \{1, 2, \ldots, d\}$ satisfies

$$\mathbb{P}(\underline{\tau}_I > S\underline{1} + \underline{U} | \underline{\tau}_I > S\underline{1}) = \mathbb{P}(\underline{\tau}_I > \underline{U}) \quad (\iff \quad G_I(S\underline{1} + \underline{U}) = G_I(S\underline{1})G_I(\underline{U})).$$

Theorem (Marhall Olkin [24]).

$\underline{\tau}$ satisfies MLOM $\iff$ $\underline{\tau} \sim$ Marshall Olkin multivariate distribution.

Multivariate lack of memory: Marshall Olkin

Recall the MO distribution (in the case $d = 2$ for simplicity).

$$\mathbb{P}(\tau_1 > S_1, \tau_2 > S_2) = G(\underline{S}) = \exp\left[-\lambda_1 S_1 - \lambda_2 S_2 - \lambda_{1,2} \max(S_1, S_2)\right]$$

where all λ's are non-negative parameters. Exponential margins:

$$\mathbb{P}(\tau_1 > S_1) = \exp\left[-(\lambda_1 + \lambda_{1,2})S_1\right], \ \mathbb{P}(\tau_2 > S_2) = \exp\left[-(\lambda_2 + \lambda_{1,2})S_2\right].$$

Important: Notice the very specific link between the joint distribution and the margins.

Moreover, This distribution has an important property: the probability to have simultaneous defaults is not zero: $\mathbb{P}(\tau_1 = \tau_2) > 0$. This is due to the max function which is not smooth. For typically smooth multivariate densities the simultaneous default probability would be zero.

Multivariate lack of memory: Marshall Olkin

The MO distribution can also be obtained as follows. Given 3 independent exponential margins $\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_{1,2}$, with parameters $\lambda_1, \lambda_2, \lambda_{1,2}$ respectively, then MO is consistent with

$$\tau_1 = \min(\bar{\tau}_1, \bar{\tau}_{1,2}), \quad \tau_2 = \min(\bar{\tau}_2, \bar{\tau}_{1,2})$$

Note: MO COPULA with arbitrary exponential margins has no MLOM.

The link in MO between margins and dependence is broken (ruining MLOM) if we replace the consistent margins having intensities $\lambda_1 + \lambda_{1,2}$ and $\lambda_2 + \lambda_{1,2}$ with different exponentials/intensities.

Arbitrarily decoupling the marginals & the dependence structure may result in paradoxical results when analyzing wrong way risk, see for example B. & Chourdakis [7] or Morini [27].

The homogeneous multivariate case: EV Copulas

For the same reasons why LOM solves Problem (1) in the univariate case, MHLOM solves Problem (1) in multivariate:

$$\text{One-shot prob} = \mathbb{P}(\underline{\tau} > T + T) = \mathbb{P}(\underline{\tau} > T + T | \underline{\tau} > T)\mathbb{P}(\underline{\tau} > T) =$$

$$= (MHLOM) = \mathbb{P}(\underline{\tau}^{(1)} > T)\mathbb{P}(\underline{\tau}^{(2)} > T) = \text{iterated prob}$$

Since MHLOM is weaker than MLOM=Marshall-Olkin, we can hope for solutions of Problem (1) given by distributions of $\underline{\tau}$ different from the Marshall Olkin multivariate distribution.

For this, we analyze MHLOM condition $G((jT)\underline{1}) = G(iT\underline{1})G((j-i)T\underline{1})$.

Assume G is associated with a *survival copula* function C.

$$G(iT\underline{1}) = \mathbb{P}(\underline{\tau} > iT) = \mathbb{P}(G_m(\underline{\tau}) < G_m(iT)) = C(G_m(iT))$$

where $G_m(t)$ is the vector of the marginal survival functions of the components of $\underline{\tau}$, all computed in t.

The homogeneous multivariate case: EV Copulas

Hence we can rewrite MHLOM as

$$G(jT\underline{1}) = G(iT\underline{1})\, G((j-i)T\underline{1}) \text{ iff } C(G_m(jT)) = C(G_m(iT))\, C(G_m((j-i)T))$$

We require that the marginal distributions satisfy lack of memory, so this means, due to the univariate characterization, that $G_m(kT) = G_m(T)^k$ (they are exponential functions), and hence the MHLOM condition reads

$$C(G_m(T)^j) = C(G_m(T)^i)\, C(G_m(T)^{j-i})$$

where the product of the G_m is component-wise.
Write the above equation for $i = 1, j = 2$ to get

$$C(G_m(T)^2) = C(G_m(T))^2 \,.$$

The homogeneous multivariate case: EV Copulas

Then $i = 1, j = 3$, substituting the one just found, and iterating, gives

$$\text{1-shot prob } C(G_m(T)^k) = C(G_m(T))^k \text{ iterated prob}, \quad k \in \mathbb{N}$$

Given the arbitrariness of the marginal intensities in G_m, we conclude

$$C(\underline{x}^t) = C(\underline{x})^t \iff \boxed{C(\underline{x}) = (C(\underline{x}^t))^{1/t}} \text{ for all } t > 0, \quad \underline{x} \in [0,1]^d$$

This is a characterization of **extreme value copulas**.

Theorem (B. Chourdakis [8]). In the multivariate setting, and under a common time step, Problem (1) is solved if G has exponential margins and an extreme value survival copula (self-chaining copula).

The homogeneous multivariate case: EV Copulas

Corollaries: Given exponential margins, the only solution in the archimedean sub-family is the Gumbel Copula.
Marshall Olkin copula with arbitrary exponential margins (not necessarily consistent a multivariate MO law) ok too.
In $d = 2$ Pickands functions and exponential margins are general sol.

Summing up, iterating all-survival simulation is only possible under special dependence. Gaussian copula cannot be iterated in principle.

I have witnessed use of iteration both in CVA valuation and in default simulation for Risk measurement.
(Blade Runner 1982)

Beware iterating Gaussian C: $P = P(\tau_1 > T, \tau_2 > T)$

$$T = 5y; \quad \Delta t = 0.0050y = 1.825d \quad (N = 1000)$$

λ_1	λ_2	ρ	P_{1shot}	$P_{iterated}$	Perc. Diff
0.0100	0.0100	0.2500	0.9084	0.9049	0.3810
0.0100	0.0100	0.7500	0.9238	0.9103	1.4840
0.0100	0.0300	0.7500	0.8482	0.8278	2.4626
0.0100	0.0500	0.2500	0.7495	0.7410	1.1397
0.0100	0.0500	0.7500	0.7722	0.7514	2.7771
0.0300	0.0300	0.7500	0.7984	0.7572	5.4441
0.0300	0.0500	0.2500	0.6885	0.6708	2.6475
0.0500	0.0300	0.7500	0.7392	0.6902	7.0893
0.0500	0.0500	0.2500	0.6303	0.6071	3.8133
0.0500	0.0500	0.7500	0.6943	0.6312	9.9925

P_{1shot} & $P_{iterated}$ would coincide with EV copulas & exp margins.

Beware iterating Gaussian C: $P = P(\tau_1 > T, \tau_2 > T)$

$$T = 30y; \quad \Delta t = 0.1y = 36.5d \quad (N = 300)$$

λ_1	λ_2	ρ	P_{1shot}	$P_{iterated}$	Perc. Diff
0.0100	0.0100	0.2500	0.5765	0.5503	4.7604
0.0100	0.0100	0.7500	0.6483	0.5835	11.1033
0.0100	0.0300	0.2500	0.3322	0.3032	9.5613
0.0100	0.0300	0.7500	0.3919	0.3365	16.4782
0.0100	0.0500	0.2500	0.1880	0.1669	12.6371
0.0100	0.0500	0.7500	0.2205	0.1901	16.0382
0.0300	0.0300	0.7500	0.2949	0.2069	42.4875
0.0500	0.0300	0.2500	0.1205	0.0929	29.7150
0.0500	0.0300	0.7500	0.1891	0.1224	54.4983
0.0500	0.0500	0.7500	0.1382	0.0751	84.0337

P_{1shot} & $P_{iterated}$ would coincide with EV copulas & exp margins.

Beware Gaussian C: $P = P(\tau_1 > T, \tau_2 > T, \tau_3 > T)$

Trivariate case. Margins satisfy LOM but 2- **and now 3-** dimensional distributions won't, so we expect **larger errors**.

$$T = 5y; \quad \Delta t = 0.005y = 1.825d \quad (N = 1000)$$

λ_1	λ_2	λ_3	ρ	P_{1shot}	$P_{iterated}$	Perc. Diff
0.03	0.03	0.03	0.10	0.6507	0.6378	2
0.03	0.03	0.03	0.50	0.7107	0.6451	10
0.03	0.05	0.03	0.50	0.6600	0.5855	13
0.03	0.05	0.05	0.50	0.6167	0.5319	16
0.05	0.03	0.05	0.10	0.5400	0.5222	3
0.05	0.05	0.05	0.10	0.4930	0.4726	4
0.05	0.05	0.05	0.50	0.5792	0.4834	20

P_{1shot} & $P_{iterated}$ would coincide with EV copulas & exp margins.

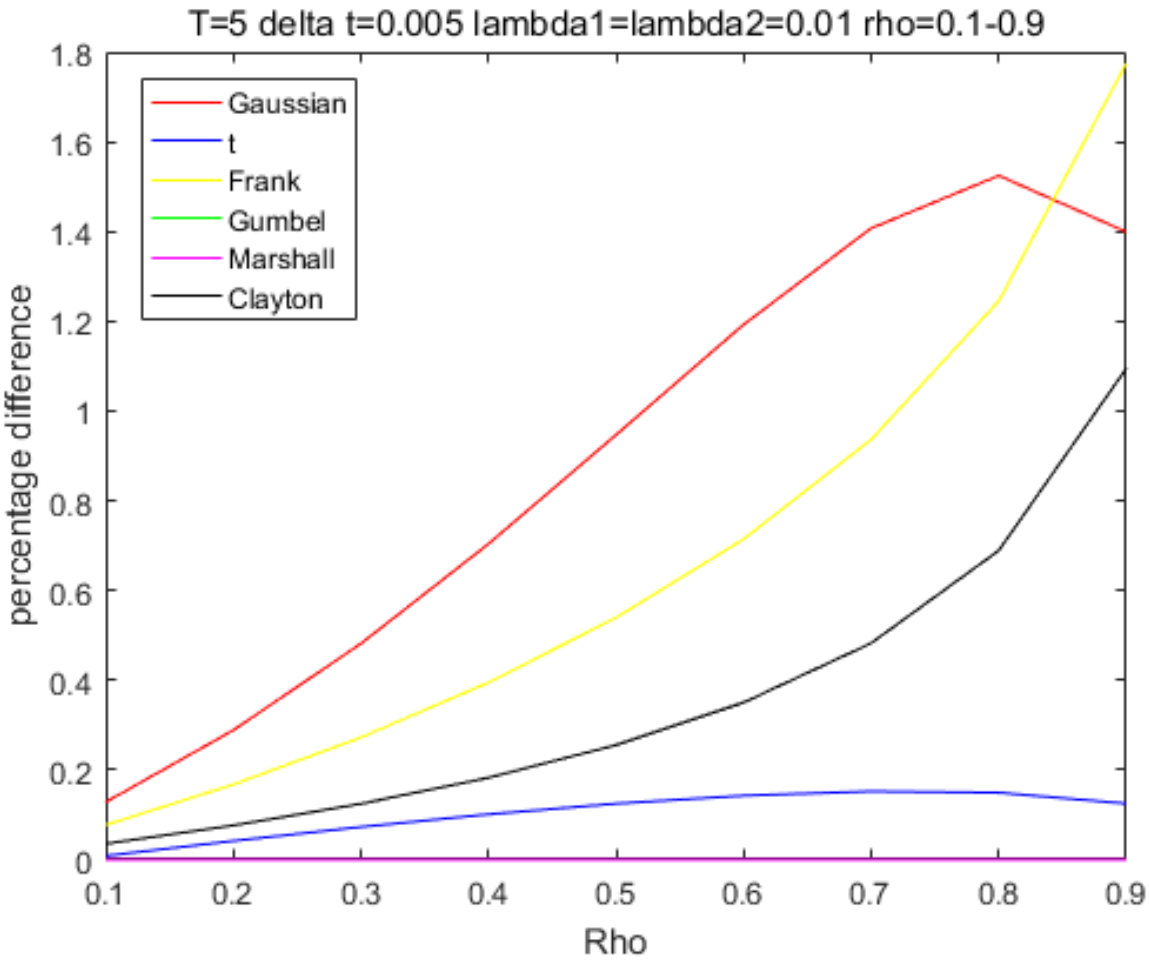
Beware Gaussian C: $P = P(\tau_1 > T, \tau_2 > T, \tau_3 > T)$

$$T = 30y; \quad \Delta t = 0.1y = 36.5d \quad (N = 300)$$

λ_1	λ_2	λ_3	ρ	P_{1shot}	$P_{iterated}$	Perc. Diff
0.03	0.03	0.03	0.10	0.0860	0.0679	27
0.03	0.03	0.05	0.50	0.1156	0.0474	144
0.03	0.05	0.03	0.10	0.0503	0.0374	34
0.03	0.05	0.03	0.50	0.1156	0.0474	144
0.03	0.05	0.05	0.50	0.0818	0.0278	194
0.05	0.03	0.05	0.50	0.0818	0.0278	194
0.05	0.05	0.05	0.10	0.0178	0.0114	56
0.05	0.05	0.05	0.50	0.0604	0.0164	267

P_{1shot} & $P_{iterated}$ would coincide with EV copulas & exp margins.

100*(Pr1shot - PrIter) /PrIter vs Spearman's rho



$T = 5y$, $\Delta t = 0.005$y, $\lambda_1 = \lambda_2 = 0.01$, 3 degrees for t-dist [13].
See [28] for an analogous analysis with Kendall's tau

What if we increase iterations more and more?

Fixing a terminal time T, we may be tempted to increase the number of iterations k to get to T with steps T/k. Based on the above examples, we suspect we would destroy dependence by doing this. Let's check.

Definition. For an extreme value copula C there exists a copula C_F such that

$$C_F(u_1^{1/k}, ..., u_d^{1/k})^k \to C(u_1, ..., u_d) \quad (k \to \infty)$$

for all $(u_1, ...u_d) \in [0,1]^d$. The copula C_F is said to be in the *domain of attraction* of C. We are interested in cases where C is the **independence copula**. The LHS of the arrow is the iterated survival. Does it converge to independence?

What if we increase iterations more and more?

Theorem

In the bivariate case the Clayton copula, the Frank copula and the Gaussian copula for $\rho < 1$ are all in the domain of attraction of the independence copula $u_1 u_2$ (see for example [14] or [13]).

This means that

$$C_F(u_1^{1/k}, u_2^{1/k})^k \to u_1 u_2 \quad (k \to \infty)$$

for C_F either Gaussian ($\rho < 1$), Clayton or Frank. Recall iterated prob =

$$= \mathbb{P}\left(\underline{\tau}^{(1)} > \frac{T}{k}\right) \mathbb{P}\left(\underline{\tau}^{(2)} > \frac{T}{k}\right) \cdots \mathbb{P}\left(\underline{\tau}^{(k)} > \frac{T}{k}\right) = C_F(G_m(T)^{1/k})^k$$

So, in these cases, in the limit when we iterate indefinitely we end up completely destroying the correlation.

A second and more-ambitious problem

Problem 1: when can we split a "survival of all" sampling in several equal time steps? *Solution: Exponential margins and EV copula.*

PROBLEM 2

Don't look just at the "survival of all" event but consider any possible mix of states (including removal of defaulted/liquidated components) and check when a terminal simulation of this can be split into different time steps for $\underline{\tau}$ and its sub-vectors.

A second and more-ambitious problem

We have already seen the theorem where M-O satisfies the most general multivariate lack of memory, including removal of components.

So we may guess that M-O will play a key role here and will be a solution. However, we can say more. Define

$$Z_t = [1_{\tau_1 > t}, 1_{\tau_2 > t}, \dots, 1_{\tau_d > t}]$$

(notice that earlier we were considering $1_{\tau_1 > t \cap \tau_2 > t \dots \cap \tau_d > t}$).

We would like to work with a Markovian Z (MCZ=Markov Chain Z). Under which conditions on the distribution of $\underline{\tau}$ do we get Markovian Z?

This would be a great step forward for us since Markov Chains can be managed efficiently via matrices and are well understood.

Nested Margining?

Nested Margining:

When names *I* default, we remove τ_I from the total vector $\underline{\tau}$, but the remaining vector *has the same type of distribution as the original* $\underline{\tau}$*, only with dimension decreased by the size of I and with updated parameters*.

What type of distributions can give us a Markov chain for Z and nested margining for the $\underline{\tau}$ distribution?

Lack of memory & new characterization of MO

The Matrix (1999)

New result showing that the full MO law (the MO copula is not enough) is *characterized* by nested margining within MCZ. The only model solving Problem 2 is MO. New characterization of MO via MCZ.

Theorem (B. Mai & Scherer [9, 10]) : New Characterization of MO.

The survival indicator processes $\mathbf{Z}_I$ are time-homogeneous Markovian for all subsets $\emptyset \neq I \subset \{1, \ldots, d\}$

$$\Leftrightarrow$$

$(\tau_1, \ldots, \tau_d)$ has a Marshall–Olkin distribution

Thus we can use a Markov chain for default simulation.

Alternatively, sample a MO law repeatedly, & whenever name(s) default remove the component(s) & simulate sub-vector law, that is still MO.

Proof ideas:

$\Rightarrow$ intuitive, using combinatorics Markov property can be shown to imply MLOM, lack of memory for τ's, which characterizes MO (above).

$\Leftarrow$ is less intuitive. Based on alternative MO stochastic construction from Arnold [1], which shows that a dynamic simulation of survival indicators is actually Markovian.

Lack of memory & new MO characterization: Proof I

Proof:

"$\Rightarrow$" By the time-homogeneous Markov property, there is a transition function $p_{\mathbf{x},\mathbf{y}}(t)$ for $\mathbf{x}, \mathbf{y} \in \{0,1\}^d$ and $t \geq 0$ such that

$$\mathbb{P}(\mathbf{Z}(t_n) = \mathbf{x}_n, \ldots, \mathbf{Z}(t_1) = \mathbf{x}_1) = p_{(1,\ldots,1),\mathbf{x}_1}(t_1) \prod_{l=2}^{n} p_{\mathbf{x}_{l-1},\mathbf{x}_l}(t_l - t_{l-1})$$

for $t_n > \ldots > t_1 > 0$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \{0,1\}^d$. Let $t, s_1, \ldots, s_d \geq 0$ be arbitrary and denote by π a permutation s.t. $s_{\pi(1)} \leq s_{\pi(2)} \leq \ldots \leq s_{\pi(d)}$ is the ordered list of $s_1, \ldots, s_d$. Define the following subsets of $\{0,1\}^d$:

$$A_1 := \{(1,\ldots,1)\}, \; A_k := \left\{\mathbf{x} \in \{0,1\}^d \; : \; x_{\pi(l)} = 1 \text{ for all } l \geq k\right\}, \; k = 2..d.$$

In words, A_k denotes the subset of $\{0,1\}^d$ in which all components $\pi(k), \ldots, \pi(d)$ are still alive.

Lack of memory & new MO characterization: Proof II

There is a finite number M of distinct paths $(\mathbf{x}_2^{(i)}, \dots, \mathbf{x}_d^{(i)}) \in A_2 \times \dots \times A_d$, $i = 1, \dots, M$, that avoid inconsistent patterns in time (such as default resurrections etc), i.e. such that

$$0 < \mathbb{P}\{\mathbf{Z}(t + s_{\pi(1)}) = (1, \dots, 1), \mathbf{Z}(t + s_{\pi(2)}) = \mathbf{x}_2^{(i)}, \dots, \mathbf{Z}(t + s_{\pi(d)}) = \mathbf{x}_d^{(i)}\}$$

This set of paths depends on $s_1, \dots, s_d$, but it does not depend on t by the time-homogeneity property of $\mathbf{Z}$. We have

$$\begin{aligned}
&\mathbb{P}(\tau_1 > t, \dots, \tau_d > t)\, \mathbb{P}(\tau_1 > s_1, \dots, \tau_d > s_d) \\
&\quad = \mathbb{P}(\mathbf{Z}(t) \in A_1)\, \mathbb{P}\big(\mathbf{Z}(s_{\pi(1)}) \in A_1, \mathbf{Z}(s_{\pi(2)}) \in A_2, \dots, \mathbf{Z}(s_{\pi(d)}) \in A_d\big)
\end{aligned}$$

$$= \mathbb{P}(\mathbf{Z}(t) \in A_1) \sum_{i=1}^{M} \mathbb{P}(\mathbf{Z}(s_{\pi(1)}) = (1..1), \mathbf{Z}(s_{\pi(2)}) = \mathbf{x}_2^{(i)}, \dots, \mathbf{Z}(s_{\pi(d)}) = \mathbf{x}_d^{(i)})$$

Lack of memory & new MO characterization: Proof III

$$= p_{(1,\dots,1),(1,\dots,1)}(t) \sum_{i=1}^{M} p_{(1,\dots,1),(1,\dots,1)}(s_{\pi(1)})\, p_{(1,\dots,1),\mathbf{x}_2^{(i)}}(s_{\pi(2)} - s_{\pi(1)}) \cdot$$

$$\cdot \prod_{k=3}^{N} p_{\mathbf{x}_{k-1}^{(i)},\mathbf{x}_k^{(i)}}(s_{\pi(k)} - s_{\pi(k-1)})$$

$$= \sum_{i=1}^{M} p_{(1,\dots,1),(1,\dots,1)}(t + s_{\pi(1)})\, p_{(1,\dots,1),\mathbf{x}_2^{(i)}}(t + s_{\pi(2)} - (t + s_{\pi(1)})) \cdot$$

$$\cdot \prod_{k=3}^{N} p_{\mathbf{x}_{k-1}^{(i)},\mathbf{x}_k^{(i)}}(t + s_{\pi(k)} - (t + s_{\pi(k-1)}))$$

$$= \mathbb{P}(\mathbf{Z}(t + s_{\pi(1)}) \in A_1, \mathbf{Z}(t + s_{\pi(2)}) \in A_2, \dots, \mathbf{Z}(t + s_{\pi(d)}) \in A_d)$$

$$= \mathbb{P}(\tau_1 > t + s_1, \dots, \tau_d > t + s_d)$$

Lack of memory & new MO characterization: Proof IV

Repeating the above derivation for every subset $I \subset \{1, \ldots, d\}$ we obtain the equation

$$\mathbb{P}(\tau_{i_1} > t+s_{i_1}, \ldots, \tau_{i_k} > t+s_{i_k}) = \mathbb{P}(\tau_{i_1} > t, \ldots, \tau_{i_k} > t)\, \mathbb{P}(\tau_{i_1} > s_{i_1} .. \tau_{i_k} > s_{i_k})$$

for arbitrary $1 \leq i_1, \ldots, i_k \leq d$ and $t, s_{i_1}, \ldots, s_{i_k} \geq 0$. This is precisely the functional equality describing the multi-variate lack-of-memory property MLOM, which is well-known to characterize the Marshall–Olkin exponential distribution as we have seen earlier in this talk (see [24, 25]). □

Lack of memory & new MO characterization: Proof V

"$\Leftarrow$" Assume $(\tau_1, \ldots, \tau_d)$ has a Marshall–Olkin distribution with parameters $\{\lambda_I\}$, $\emptyset \neq I \subset \{1, \ldots, d\}$ satisfying $\sum_{I:k\in I} \lambda_I > 0$ for all $k = 1, \ldots, d$. We prove Markovianity of $\mathbf{Z}_I$ for an arbitrary non-empty subset I of components.

Lack of memory & new MO characterization: Proof VI

Without loss of generality, we may assume that $(\tau_1, \ldots, \tau_d)$ is defined on the following probability space, as first considered in [1]: we consider an iid sequence $\{E_n\}_{n\in\mathbb{N}}$ of exponential random variables with rate $\lambda := \sum_{\emptyset\neq K\subset\{1,\ldots,d\}} \lambda_K$ and an independent iid sequence $\{Y_n\}_{n\in\mathbb{N}}$ of set-valued random variables with distribution given by

$$\mathbb{P}(Y_1 = K) = p_K := \frac{\lambda_K}{\lambda}, \quad \emptyset \neq K \subset \{1, \ldots, d\}.$$

The random vector $(\tau_1, \ldots, \tau_d)$ is then defined as $\tau_k := E_1 + \ldots + E_{\min\{n\,:\,k\in Y_n\}}$, $k = 1, \ldots, d$. Introduce the notation

$$N_t := \sum_{k=1}^{\infty} 1_{\{E_1+\ldots+E_k\leq t\}}, \quad t \geq 0,$$

which is a Poisson process with intensity λ.

Lack of memory & new MO characterization: Proof VII

Fix a non-empty set $I \subset \{1, \ldots, d\}$, say $I = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \ldots < i_k \leq d$. Denoting the power set of $\{1, \ldots, d\}$ by $\mathcal{P}_d$, we define the function $f_I : \{0,1\}^k \times \mathcal{P}_d \to \{0,1\}^k$ as follows:

$$j\text{-th component of } f_I(\vec{x}, J) := 1_{\{x_j=1 \text{ and } i_j \notin J\}}, \quad j = 1, \ldots, k,$$

for $\vec{x} = (x_1, \ldots, x_k) \in \{0,1\}^k$ and $J \in \mathcal{P}_d$. It is now readily observed – in fact just a rewriting of Arnold's model – that

$$\mathbf{Z}_I(t) = f_I\Big(\mathbf{Z}_I(s), \bigcup_{k=N_s+1}^{N_t} Y_k\Big), \quad t \geq s \geq 0. \tag{1}$$

Lack of memory & new MO characterization: Proof VIII

$$\mathbf{Z}_I(t) = f_I\Big(\mathbf{Z}_I(s), \bigcup_{k=N_s+1}^{N_t} Y_k\Big), \quad t \geq s \geq 0.$$

This stochastic representation implies the claim, since the second argument of f_I is independent of $\mathcal{F}_I(s) := \sigma(\mathbf{Z}_I(u) \,:\, u \leq s)$ by the Poisson property of $\{N_t\}$. To see this, it suffices to observe that $Z_I(s)$ is a function of N_s and $Y_1, \ldots, Y_{N_s}$ (which can be seen by setting $t = s$ and $s = 0$ in (1)), whereas the second argument is a function of $Y_{N_s+1}, \ldots, Y_{N_t}$. Consequently, the independent random variables N_s and $N_t - N_s$ only serve as a random pick of two independent (because disjoint) partial sequences of the iid sequence $Y_1, Y_2, \ldots$.

Conclusions

- If we simulate“survival of all names” multi-step, we kill correlation unless default correlation is an extreme value copula.
- In particular, iterating the *Gaussian copula* kills the depedence (“correlation”) and should be avoided.
- If we are concerned about other events than just “all-survival”, then to avoid correlation killing...
- ... the vector of default times has to be Marshall–Olkin distributed.
- This in turn allows us to work with simple Markov chains via a new characterization of M-O.
- M-O has come out in many different contexts (reliability theory, frailty analysis, credit derivatives models).
- *Safer to simulate defaults one shot if possible*, in this case there is no limitation.
- If iterating necessary, M-O or at least EV copula should be used.
- Failing this, *check error involved in multi-step process*.

Thank you for your attention!

Questions?

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