Optimal Execution Comparison Across Risks and Dynamics, with Solutions for Displaced Diffusions

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Agenda I

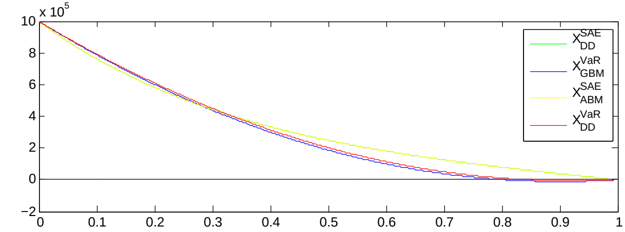
1. The trade execution problem under different dynamics
   - Context and earlier results for ABM / GBM
   - DD dynamics for the unaffected stock S
   - The trade execution problem
   - Risk Functions
2. Solution of the trade execution problem for a displaced diffusion
   - Adapting Gatheral and Schied's theorem to DD
   - SAE risk criterion
3. Comparing different Dynamics and Risk Criteria
   - Numerical examples
4. Conclusions

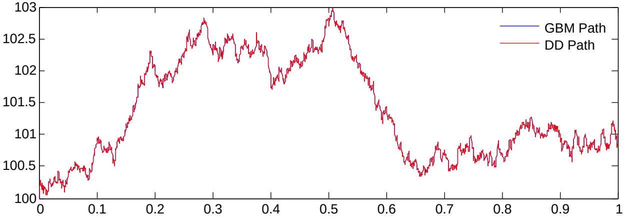
Optimal Trade Execution: Context

Partition a large trade into smaller trades so as to minimize the effect of market impact.

Eg: Sell X shares by the time T by minimizing cost and risk in the execution. Costs are defined in terms of instantaneous market impact and permanent market impact. Risk may be defined in different ways.

An Example: Selling 1 million shares, with initial price 100 each, over 1 day, miniminzing cost and risk





Optimal Trade Execution: Context I

We are aware that this problem should be addressed with intensive data analysis to detect the empirical features about market impact and asset dynamics first.

However in this talk we rather turn to the search for an exact mathematical solution to the problem.

This will only hold under very simple assumptions on dynamics and criteria but may help us in understanding the contributions of different factors in a simplified setting.

Optimal Trade Execution: Context II

In [1] and [2], Almgren and Chriss (AC) combine expected execution cost and execution risk. Linearly increasing execution costs (in the trading rate) whereas the risk criterion is related to the cost variance.

Advantage: closed-form analytical solution. This is deterministic and is usually in the class of the Volume Weighted Average Price (VWAP) solutions. Disadvantage: extremely stylized dynamics

Gatheral and Schied (GS) [4] solve the problem under the more realistic assumption of geometric Brownian motion (GBM) but under different risk, since cost variance as risk is not tractable for GBM.

GBM with Variance as risk: Numerical approach in Forsyth [5] and [6]. **The cost-risk efficient frontier is almost the same under ABM and GBM: the deterministic solution with ABM is not very suboptimal.**

Optimal Trade Execution: Context III

Summing up:

- ABM with Cost Variance Risk Criterion : Tractable (AC)
- GBM with Cost Variance Risk Criterion : Not tractable (F)
- GBM with Value at Risk (future loss percentile) : Tractable (GS)
- GBM with Expected shortfall (mean after percent) : Tractable (GS)

Here we extend the GS [4] analytic result of the last two cases to a more general dynamics, namely the displaced diffusion (DD) model.

- DD with VaR or ES : Tractable
- DD with Squared Asset Expectation (SAE) Risk: semitractable

How does the optimal execution problem solution change when we change dynamics and criteria? Extend analysis to DD & SAE

Optimal Trade Execution: Variables I

- The initial time is 0, the final time is T, and usually $t \in [0, T]$;
- S_t is the unaffected pre-impact share price at time t;
- $x(t)$: shares left to be sold at time t; Assumed absolutely continuous and adapted;
- $x(0) = X$ (sell X shares in total), $x(T) = 0$ (all shares sold by T).
- Selling shares impacts the shares price. Affected share price:

$$\widetilde{S}_t = S_t + \eta\, \dot{x}(t) - \gamma(x_0 - x(t))$$

- $\eta\, \dot{x}(t)$ is the instantaneous impact of trading $dx(t) = \dot{x}(t)dt$ shares in $[t, t+dt)$ and only affects the $[t, t+dt)$ order.
- $-\gamma(x(0) - x_t)$ represents the permanent impact price that has been accumulating over $[0, t]$ by all transactions up to t.
- Cost and Risk of x: $C(x) := \int_0^T \widetilde{S}_t\, dx(t)$; $R(x)$? (several poss.)
- Find x that minimizes $\mathbb{E}[C(x) + L\, R(x)]$ with L leverage param

Optimal Trade Execution: Modelling Framework I

Following [1], [2] & especially [4] we assume that the trader's number of shares follows an absolutely continuous and adapted trajectory

$$t \mapsto x(t), \quad x(0) = X, \quad x(T) = 0.$$

Given this trading path, the price at which the transaction occurs is

$$\widetilde{S}_t = S_t + \eta\, \dot{x}(t) - \gamma(x_0 - x(t))$$

where η and γ are constants and S is the process for the unaffected stock price level.

As explained earlier, η is a instantaneous impact parameter. If η increases, the trading activity will affect $\tilde{S}$ more and costs will increase.

γ is a cumulative impact parameter. If γ increases, the impacted price $\tilde{S}$ decreases.

Optimal Trade Execution: Modelling Framework II

Our minimiz prob for $\mathbb{E}[C(x) + LR(x)]$ is almost completely specified:

- We explicit the cost function $C(x)$
- The instantaneous cost of the strategy is the cost of buying $dx(t) = \dot{x}(t)dt$ shares at time $[t, t + dt)$ at the impacted stock price $\widetilde{S}_t$, spending $\widetilde{S}_t \dot{x}(t)dt$. **Costs** arising from the strategy are then

$$
\begin{aligned}
C(x) := \int_0^T \widetilde{S}_t \, dx(t) &= \int_0^T [S_t + \eta \, \dot{x}(t) + \gamma(x(t) - x_0)]\dot{x}(t)dt \\
&= -XS_0 - \int_0^T x(t)dS_t + \eta \int_0^T (\dot{x}(t))^2 dt + \gamma X^2/2
\end{aligned}
$$

  where we have used an integration by parts. This calculation does not involve the specific dynamics of S yet and is general.
- We need to postulate a Stochastic dynamics for S (and hence $\tilde{S}$).
- We also need to decide on the Risk Function $R(x)$.

Optimal Trade Execution: The DD Dynamics I

Let's start with S's dynamics. We assume zero interest rates. Consider now DD. Unaffected stock price S: given a GBM Y_t with volatility parameter σ, ie $dY_t = \sigma Y_t dW_t$, ($Y_0 = S_0 - K$), we define

$$S_t = K + Y_t \text{ or equivalently } dS_t = \sigma(S_t - K)dW_t, \; S_0 \text{ (DD)}$$

S is a shifted GBM. The shift is a constant K and the GBM is Y_t.

DD mimicks features of GBM and ABM in a single model.

$$\sigma(S_t - K) = \sigma S_t(1 - K/S_t)$$

$$\sigma S_t(1 - K/S_t) \approx \sigma S_t \text{ for } S_t >> K \quad (GBM).$$

$$\sigma(S_t - K) \approx \sigma(-K) \text{ for } S_t \approx 0 \quad (ABM).$$

This is usually summarized by practioners by saying that "DD behaves like ABM for small stock values and as GBM for large ones".

Optimal Trade Execution: The DD Dynamics II

DD also allows to set a minimum allowed value for the share price.

This is set at K, the shift.

Optimal Execution problem for DD: Risk I

To complete the problem specification, let us now look at possible risk functions $R(x)$.
To have tractability with GBM, Gatheral and Schied adopt the position VaR or ES as **Risk** criteria, rather than Cost Variance as in AC.

Let $\nu_{\alpha,t,h}$ be the Value at Risk measure computed at time t, for the position, for a given confidence level α over a time horizon h.

$$\mathbb{P}\{S_t - S_{t+h} \leq \nu_{\alpha,t,h} | \mathcal{F}_t\} = \alpha.$$

Optimal Execution problem for DD: Risk II

If at t we have $x(t)$ shares with price S_t, the time t VaR measure for a risk horizon h under DD dynamics at confidence level α would be

$$\begin{aligned}
\nu_t[x(t)(S_t - S_{t+h})] &= x(t)\nu_t[(S_t - S_{t+h})] \\
= x(t)\nu_t[(Y_t + K - Y_{t+h} - K)] &= x(t)\nu_t[(Y_t - Y_{t+h})] \\
&= x(t)\nu_t[Y_t(1 - \exp(-\sigma^2 h/2 + \sigma(W_{t+h} - W_t)))] \\
&= x(t)Y_t q_\alpha[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}\epsilon)] = \\
= x(t)Y_t[1 - \exp(-\sigma^2 h/2 + \sigma\sqrt{h}q_{1-\alpha}(\epsilon))] &=: \tilde{\lambda}_\alpha x(t)(S_t - K).
\end{aligned}$$

where ϵ is a standard normal, where we have used the homogeneity of VaR, and where $q_\alpha(X)$ is the α quantile of the distribution of X. This is the VaR measure for the instantaneous position at time t. If we average VaR over the life of the strategy we obtain the risk criterion

$$R^{\mathrm{VaR}_\alpha}(x) := \tilde{\lambda}\int_0^T x(t)(S_t - K)dt.$$

Optimal Execution problem for DD: Risk III

The expected shortfall risk criteria is the same with different λ.

Optimal Execution problem for DD: Risk IV

We are now ready to define our criterion to be minimized. We put together costs and risks in a single criterion

$$\mathbb{E}\left[C(x) + L\,R(x)\right] = \mathbb{E}\left[C(x) + L\bar{\lambda}\int_0^T x(t)(S_t - K)dt\right].$$

In this criterion:

- L is a cost/risk leverage parameter that measures risk aversion in executing the order.
- $L = 0$ we only look at costs, whereas for large L risk dominates costs.
- $\bar{\lambda}$ has a precise endogenous expression depending on whether we are using the VaR or ES risk function

Optimal Execution problem for DD: Risk V

$$\mathbb{E}\left[C(x) + L\, R(x)\right] = -XS_0 + \gamma X^2/2 + \eta \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt \right.$$

$$\left. + L\frac{\bar{\lambda}}{\eta}\int_0^T x(t)(S_t - K)dt\right]$$

$$= -XS_0 + \gamma X^2/2 + \eta \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\check{\lambda}\int_0^T x(t) Y_t dt\right]$$

where we have set $\check{\lambda} = \bar{\lambda}/\eta$.

The problem is now finding

$$x^* = \text{argmin}_x \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\frac{\bar{\lambda}}{\eta}\int_0^T x(t) Y_t dt\right]. \qquad (1)$$

Optimal Execution: Solution for DD I

$$x^* = \mathrm{argmin}_x \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\frac{\bar{\lambda}}{\eta}\int_0^T x(t) Y_t dt\right]. \qquad (2)$$

Problem (2) has been solved by Gatheral and Schied [4]. If we pretend for a moment that Y is our true unaffected underlying stock, the criterion we have is the same as the criterion for a GBM Y, and this has been solved in [4]. Indeed, Theorem 1 in Gatheral and Schied [4] provides the solution and we may substitute back $Y = S - K$ to obtain the following

Optimal Execution: Solution for DD II

Theorem

*(**Optimal execution strategy for a displaced diffusion).** The unique optimal trade execution strategy attaining the infimum in (2) is*

$$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} \int_0^t (S_u - K)du\right]$$

Furthermore, the value of the minimization problem in (2) is given by

$$\mathbb{E}\left[\int_0^T \dot{x}^*(t)^2 dt + \check{\lambda} L \int_0^T x^*(t)(S_t - K)dt\right]$$

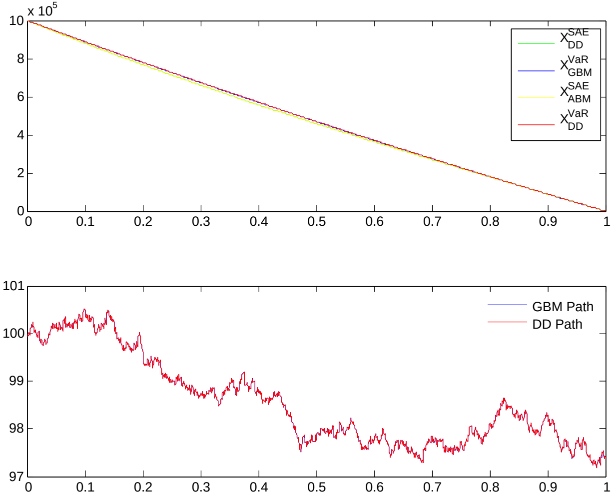
$$= \frac{X^2}{T} + \frac{L\check{\lambda}TX(S_0 - K)}{2} - \frac{(L\check{\lambda})^2}{8\sigma^6}(S_0 - K)^2\left(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{\sigma^4 T^2}{2}\right)$$

Optimal Execution: Solution for DD III

$$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} \int_0^t (S_u - K) du\right]$$

- If $L = 0$ or if instantaneous impact η is very large compared to other parameters, then Risk is not there and we only minimize cost. This leads to
$$x_t^* \approx \frac{T-t}{T} X$$
This is a line in t. Some solutions are very close to this. Note also that this solution is not just adapted, but deterministic.
- Hence the optimal adapted solution with $L = 0$ is also deterministic.

Optimal Execution: Solution for DD IV



Optimal Execution: Solution for DD V

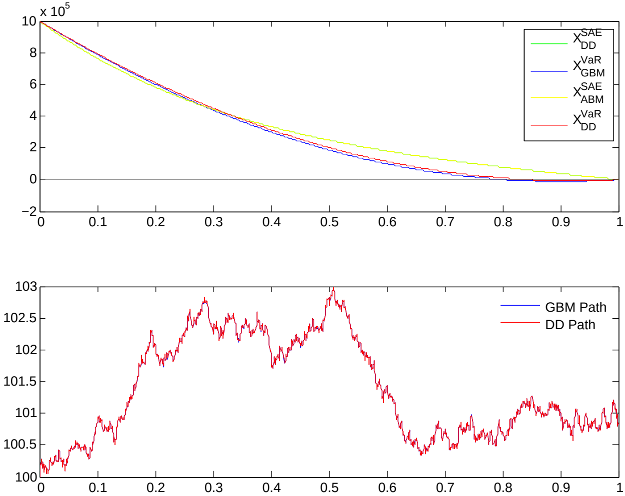
- $$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} \int_0^t (S_u - K) du\right]$$

  If L is not negligible and η is not extremely large, then a relevant component in the solution is $(1/t)\int_0^t S_u du$ which is the average shares price in time up to t. For example, if S were constantly equal to its initial value S_0 we would have

  $$x_t^* = \frac{T-t}{T}\left[X - \frac{\bar{\lambda}}{\eta} L \frac{T}{4} (S_0 - K)\, t\right]$$

  that looks like a convex (quadratic) function of t.

Optimal Execution: Solution for DD VI



A different risk criterion: SAE I

In order to test the robustness of the optimal strategy, we introduce the alternative "squared-asset expectation" (SAE) risk criteria

$$R^{SAE}(x) \equiv \lambda \int_0^T x^2(t)\sigma^2 E[S_t^2]dt \quad (R^{AC} \equiv \lambda \int_0^T x^2(t)\sigma^2 S_0^2 dt \text{ for ABM}) \tag{3}$$

Going back towards considering cost variance as risk.

λ here is exogenous rather than endogenous.

The optimal execution strategy can be derived by solving the optimisation problem ($g(t) = \mathbb{E}[S_t^2]$)

$$x^* = \text{arginf}_x \mathbb{E}\left[\int_0^T \dot{x}(t)^2 dt + L\lambda \int_0^T \sigma^2 x^2(t) g(t) dt\right]. \tag{4}$$

A different risk criterion: SAE II

Using calculus of variation, we show that the optimal solution needs to satisfy the ODE

$$\ddot{x}(t) = k^2 g(t) x(t), \quad k = \sigma\sqrt{L\lambda} \tag{5}$$

and the initial and terminal conditions are given by $x(0) = X$ and $x(T) = 0$ respectively. A solution to the boundary value problem above could be found using standard numerical routines. We derive an approximation based on series expansion in the paper.

Important: The SAE criterion makes the optimal adapted solution a **deterministic one**, since the solution of the above ODE is clearly deterministic. This is not surprising given that the key element in the risk criterion, namely $g(t)$, is deterministic with SAE.

Comparing risk criteria I

To compare optimal strategies under SAE risk and VaR/ES, we need to set the exogenous λ in SAE in a way that makes the comparison sensible ("equalizing the λ's").

A possible way is to check what happens for the VWAP solution

$$x_0(t) = X\frac{T-t}{T}$$

and match the risk functions corresponding to this solution.

$$\mathbb{E}_0\left[\int_0^T \check{\lambda}\, x_0(t)\,(S_t - K)dt\right] = \mathbb{E}_0\left[\int_0^T \lambda_{DD}^{SAE}\, x_0^2(t)\,\sigma^2\mathbb{E}_0[S_t^2]dt\right]$$

Comparing risk criteria II

in the DD case, while using the expansion $e^{\sigma^2 t} \approx 1 + \sigma^2 t$, leads to

$$\lambda_{DD}^{SAE} = \frac{\check{\lambda}}{2X\sigma^2} \frac{1}{(S_0 - K)\left(\frac{\sigma^2 T}{12} + \frac{S_0^2}{3(S_0 - K)^2}\right)}. \tag{6}$$

This is the formula we will use, and we will set also

$$\lambda_{ABM}^{SAE} := \lambda_{DD}^{SAE} \, . \tag{7}$$

In the graphs below, we compare the optimal solution of the following dynamics/risk criterion combinations:
ABM+SAE, DD+SAE, GBM+VaR and DD+VaR.

The λ's will be equalized through Equations (6) and (7).

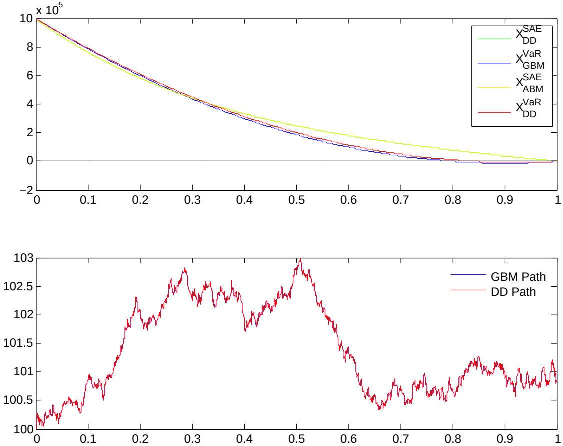
Model Comparison: $S_0 = 100$, $X = 10^6$, $T = 1$day, $\sigma_{1y} = 0.3(\sigma_{1d} = 0.0189)$, $L = 100$

- The time horizon for the execution is set to one day (realistic).
- All models start at the same S_0. The volatility of the asset is $\sigma = 30\%$.
- Absolute vol in ABM is σS_0.
- Instantaneous volatility of the DD is rescaled using formula (8) to ensure that the integrated volatility of the DD and GBM models are roughly of the same order of magnitude,

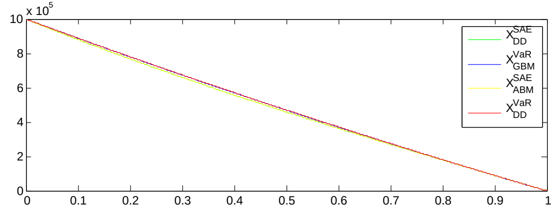
$$\sigma^{DD}(S_0 - K) = \sigma^{GBM} S_0. \tag{8}$$

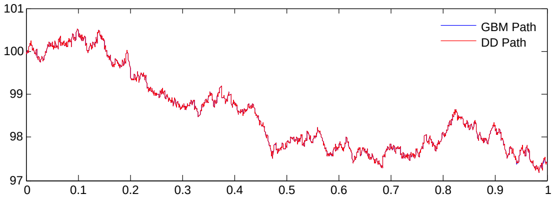
- As we will see, this condition plus imposing the same S_0 and the short T will make DD and GBM paths indistinguishable. $R(x)$ and x^* will be quite different in the two cases nonetheless.
- Most importantly the cost/risk parameter L is relatively low at 100.

Model Comparison: $S_0 = 100$, $X = 10^6$, $T = 1$day, $\sigma_{1y} = 0.3$ ($\sigma_{1d} = 0.0189$), $L = 100$. $K = 5$ in DD. $\eta = 20^{-6}$: impact of an instantaneous sale of 1 million units would be equal to 2 dollars per asset, or 2% of the mid price. The optimal execution is faster for the model that employs the VaR risk function compared to the SAE

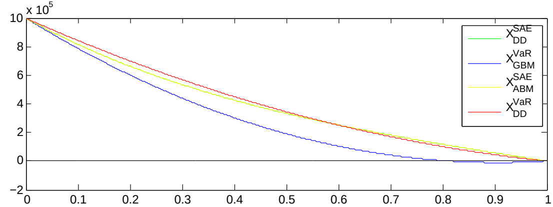


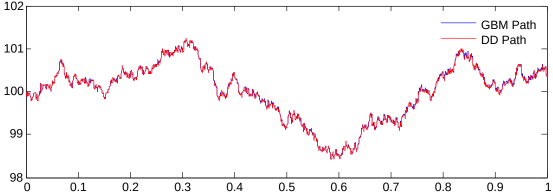
Same parameters except η, which is increased $\times$ 10 to $\eta = 20^{-5}$. Relatively illiquid security. Price impact dominates over the risk component and the execution strategy is very close to linear



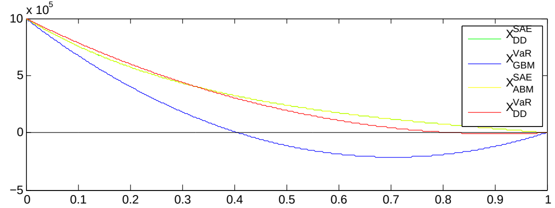


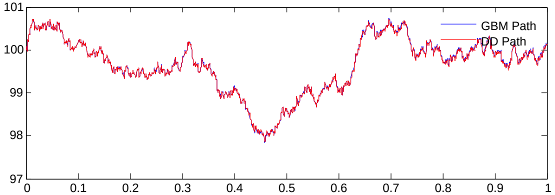
Same parameters as in 1st example but increased DD shift $K = 50$. Optimal policy faster for the GBM+VaR combination compared to DD+VaR. In the 1st case the risk function depends on S_t whereas in the second case it depends on $S_t - K$, which is roughly half (but instantaneous volatility of the DD is rescaled for comparison)





Same parameters as in 1st example but $K = 50$ and η half its previous value, namely 10^{-6}. Note that in this case the GBM+VaR solution is not monotone as no smooth constraint at zero has been imposed on x_t. Note also that a similar behaviour may occur with the DD+VaR combination.





Conclusions

- We solved the optimal trade execution problem under the Value-at-risk or Expected shortfall criteria when the underlying unaffected stock price follows a displaced diffusion model
- From the examples above, it emerges that for high levels of risk aversion and relatively low impact, the optimal execution may be significantly different for the linear solution.
- Optimal policy may exhibit convexity

Conclusions

- High impact η or low risk aversion L may lead to almost VWAP / linear solutions
- Different combinations of asset dynamics and risk function give rise to different solutions in these cases, although this is partly due to the specific choice of parameters mapping between different models and of the subjective parameter L.
- Investigation with less stylized and more data driven dynamics for S is needed
- More general problems with more agents are definitely more interesting, even though one may have to give up tractability completely for that.

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