Risk Management under Liquidity Risk: Liquidity inclusive Risk Measures

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Damiano Brigo
Chair of Mathematical Finance and co-Head of Group
Dept. of Mathematics, Imperial College London
www.damianobrigo.it
and Director of the Capco Institute
http://www.capco.com/capco-insights

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Joint work with Claudio Nordio

Agenda

Liquidity in Risk Measurement and Pricing

Liquidity has been one of the key drivers of the crisis.

- "In 2007 liquidity risk did not rank among the top 30 risks affecting the banking system. In 2008, liquidity risk was numberone" *(PWC & Centre for the study of financial innovation)*
- "The contraction of liquidity in certain structured products... led to severe funding liquidity strains... Banks had made assumptions about asset market liquidity that proved to be overly optimistic. The (committee) will take action aimed at strengthening banks' liquidity risk management in relation to the risks they hold" *(Basel committee on banking supervision: Liquidity risk management and supervisory challenges, 2008)*

Liquidity in Risk Measurement and Pricing

Szego (2009) illustrates, among other factors, a negative loop involving illiquidity as fueling the crisis development. We can consider for example the following schematization:

1. (Further) liquidity reduction on asset trade;
2. (Further) price contraction due to liquidity decline;
3. (Further) decline of value of bank assets portfolio;
4. Difficulty in refinancing, difficulty in borrowing, forced to (further) sale of assets;
5. Assets left? If so, go back to 1. If not:
6. Impossibility of refinancing;
7. Bankruptcy.

Liquidity in Risk Measurement and Pricing

This simplified representation highlights three types of liquidity

- **Market/trading liquidity:** ability to trade quickly at a low cost (O'Hara (1995)). Low transaction costs / bid-ask spreads, low price impact of trading large volumes. Can be applied to different asset classes and to the overall financial markets.
- **Funding liquidity:** liabilities can be easily funded through different financing sources and at a reasonable cost.
- Market and funding liquidity are related since timely funding of liabilities relies on the market liquidity risk of assets, see above loop. The recent crisis prompted discussion on new guidelines (see BIS(2008), FSA(2009)).
- A 3d kind of liquidity, implicit in the above schematization, is the **systemic liquidity risk** associated to a global financial crisis, characterized by a generalized difficulty in borrowing.

Liquidity: Pricing and Risk Measurement

As with other risks, like Credit Risk, liquidity needs to be analyzed from both a pricing perspective (CVA) and a risk management one (Credit VaR) (actually now even VaR of CVA is paramount as risk and pricing are getting more interconnected...)

- **Risk Measurement:** Brigo and Nordio (2010) analyze the impact of liquidity on holding period for risk measures calculations, adopting stochastic holding period risk measures. Fat tails and tail dependence? This is today's talk.
- Coherent risk measures: the general axioms a liquidity measure should satisfy are discussed in Acerbi and Scandolo (2008). Coherent risk measures defined on the vector space of portfolios (rather than on portfolio values). Portfolio value can be nonlinear. Introduction of a nonlinear value function depending on a notion of liquidity policy based on a general description of the microstructure of illiquid markets.

Liquidity: Pricing and Risk Measurement

- **Risk Measurement, cont'd:** Bangia et al. (1999) classify market liquidity risk in two categories:
  (a) the exogenous illiquidity which depends on general market conditions, is common to all market players and is unaffacted by the actions of any one participant and
  (b) the endogenous illiquidity that is specific to one's position in the market, varies across different market players and is mainly related to the impact of the trade size on the bid-ask spread.

  Bangia et al. (1999) and Earnst et al. (2009) only consider the exogenous illiquidity risk and propose a liquidity adjusted VaR measure built using the distribution of the bid-ask spreads.
- Angelidis and Benos (2005), Jarrow and Protter (2005), Stange and Kaserer (2008) model and account for endogenous risk in the calculation of liquidity adjusted risk measures.

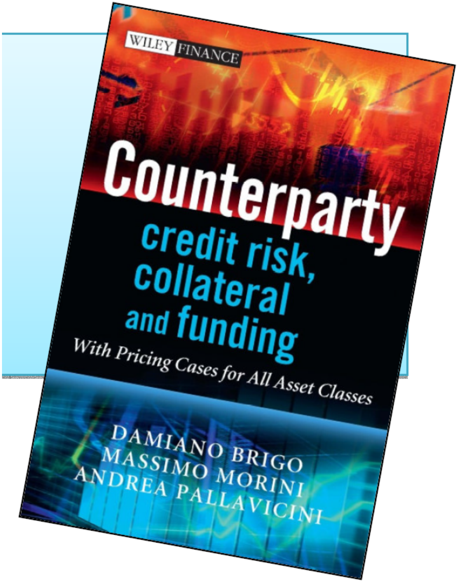
Liquidity: Pricing and Risk Measurement

- **Pricing: Trading Liquidity.** Amihud, Mendelson, and Pedersen (2005). Survey of theoretical and empirical papers that analyze the impact of liquidity on asset prices for traditional securities such as stocks and bonds.
- Cetin, Jarrow, Protter, and Warachka (2005), Garleanu, Pedersen and Poteshman (2006) investigated the impact of liquidity on option prices.
- Cetin, Jarrow and Protter (2004) extends arbitrage theory to include liquidity risk by considering an economy with a stochastic supply curve where the price of a security is a function of the trade size. New definition of self-financing trading strategies and additional restrictions on hedging strategies.
- Brigo, Predescu and Capponi (2010) analyze liquidity pricing for Credit Default Swaps, reviewing in particular the joint modeling of credit and liquidity for pricing.

Liquidity: Pricing and Risk Measurement

- **Pricing: Funding Liquidity.** The industry is working on the introduction of a Funding Valuation Adjustment (FVA) for pricing trades in presence of funding liquidity costs.
- This is proving controversial in a number of ways. See for example the book by Brigo, Morini and Pallavicini, "Counterparty Credit Risk, Collateral and Funding", Wiley, 2013. Approaches:
  - "Call the treasury and ask how much they'll charge us!!"
  - "Add this spread when you discount!"
  - ⋮
  - "Implement this 2nd order Backward Stochastic Differential Equation and get rid of the martingale representation theorem"...
- FVA is in fact a big problem, it makes valuation aggregation-dependent, nonlinear in the extreme and - to some extent - subjective.
- We are not going to discuss FVA here (Thank Goodness!!!)...

... but you are still welcome to buy the book...



Check also the Journal of Financial Transformation

http://www.capco.com/capco-insights/capco-journal

Issue 33 has an article on Liquidity Risk Management

Liquidity Adjusted Risk Measures with Stochastic Holding Period

We consider a possible methodology to extend risk measures to Random Holding Periods, so as to account for liquidity risk.

Basel updates and different holding periods I

The Basel Committee now recognizes that the risk horizon of a portfolio cannot be expressed by a simple unique number in general.

"Consultative document: Fundamental review of the trading book", BIS, May 2012, http://www.bis.org/publ/bcbs219.pdf :

Basel III on Holding Periods.

"The Committee is proposing that varying liquidity horizons be incorporated in the market risk metric under the assumption that banks are able to shed their risk at the end of the liquidity horizon.[...]. This proposed liquidation approach recognises the dynamic nature of banks trading portfolios but, at the same time, it also recognises that not all risks can be unwound over a short time period, which was a major flaw of the 1996 framework".

Basel updates and different holding periods II

The BIS consultative document details some of the ideas to solve this problem in Annex 4.

Annex 4: Different holding periods?

It is proposed to assign a different liquidity horizon to each risk factor.

While this is a step forward, it can be insufficient. How is one to combine the different estimates for different horizons into a consistent and logically sound way?

A possible solution: random holding period

A possible solution (B. and Nordio 2010) is to resort to mixtures, where the holding period assumes several different values with different probabilities. This allows to compute a single risk measure, for example Expected Shortfall, while taking into account different liquidity horizons.

VaR and ES with Stochastic Holding Period I

Back in 2009, according to the Interaction of Market and Credit Risk group of the Basel Committee Banking Supervision

"Liquidity conditions interact with market risk and credit risk through the horizon over which assets can be liquidated"

Risk managers agreed on longer holding periods, for instance 10d instead of 1day; in 2009-2010, BCBS has prudentially stretched such liquidity horizon to 3m. However, even the IMCR group pointed out that

"the liquidity of traded products can vary substantially over time and in unpredictable ways' [...] IMCR studies suggest that banks' exposures to market risk and credit risk vary with liquidity conditions in the mkt".

The former statement suggests a **stochastic description of the time horizon** over which a portfolio can be liquidated, and the latter highlights a **dependence issue**.

VaR and ES with Stochastic Holding Period II

Probably the holding period of a risky portfolio is neither ten business days nor three months;

It could, for instance, be 10 business days with probability 99% and three months with probability 1%.

This is a very simple assumption but it may have already interesting consequences.

Indeed, given the FSA requirement to justify liquidity horizon assumptions for the Incremental Risk Charge modelling, a simple example with the two-points liquidity horizon distribution that we develop below could be interpreted as *a mixture of the distribution under normal conditions and of the distribution under stressed and rare conditions.*

VaR and ES with Stochastic Holding Period III

To make the general idea more precise, it is necessary to distinguish between the two processes:

- the daily P&L of the risky portfolio;
- the P&L of disinvesting and reinvesting in the risky portfolio.

In the following we will assume no transaction costs, in order to fully represent the liquidity risk through the holding period variability.

Therefore, even if the cumulative P&L is the same for the two processes above on the long term, the latter has more variability than the former, due to variable liquidity conditions in the market.

VaR and ES with Stochastic Holding Period IV

If we introduce a third process, describing the dynamics of such liquidity conditions, for instance

- the process of time horizons over which the risky portfolio can be fully bought or liquidated

then the P&L is better defined by the returns calculated over such stochastic time horizons instead of a daily basis. We will use the "stochastic holding period" (SHP) acronym for that process, which belongs to the class of **positive processes** largely used in mathematical finance.

VaR and ES with Stochastic Holding Period V

Liquidity-adjusted VaR or Expected Shortfall (ES)

We define the liquidity-adjusted VaR or Expexted Shortfall (ES) of a risky portfolio as the VaR or ES of portfolio returns calculated over the horizon defined by the SHP process, which is the ‘operational time’ along which the portfolio manager must operate, in contrast to the ‘calendar time’ over which the risk manager usually measures VaR.

None of the previous works on extensions of risk measures to liquity focuses specifically on our setup with random holding period, which represents a simple but powerful idea to include liquidity in traditional risk measures such as Value at Risk or Expected Shortfall.

VaR and ES with Stochastic Holding Period VI

We start with the univariate case.

Let us suppose we have to calculate VaR of a market portfolio whose value at time t is V_t. We call $X_t = \ln V_t$, so that the log-return on the portfolio value at time t over a period h is

$$X_{t+h} - X_t = \ln(V_{t+h}/V_t) \approx \frac{V_{t+h} - V_t}{V_t}.$$

In order to include liquidity risk, the risk manager decides that a realistic, simplified statistics of the holding period in the future will be

Table: Simplified discrete SHP

Holding Period	Probability
10 days	0.99
75 days	0.01

VaR and ES with Stochastic Holding Period VII

To estimate liquidity-adjusted VaR say at time 0, the risk manager will perform a number of simulations of $V_{0+H_0} - V_0$ with H_0 randomly chosen by the statistics above, and finally will calculate the desired risk measure from the resulting distribution.

If the log-return $X_T - X_0$ is normally distributed with zero mean and variance T for deterministic T (e.g. a Brownian motion, i.e. a Random walk), then the risk manager could simplify the simulation using $X_{0+H_0} - X_0|_{H_0} \stackrel{d}{\sim} \sqrt{H_0}\,(X_1 - X_0)$ where $|_{H_0}$ denotes "conditional on H_0". With this practical exercise in mind, let us generalize this example to a generic t.

VaR and ES with Stochastic Holding Period VIII

A process for the risk horizon at time t, i.e. $t \mapsto H_t$, is a positive stochastic process modeling the risk horizon over time. Our risk measure at time t is to be taken on the log-return

$$X_{t+H_t} - X_t.$$

For example, if one uses a 99% Value at Risk (VaR) measure, this will be the

$$\text{1st percentile of } \; X_{t+H_t} - X_t.$$

The request that H_t be just positive means that the horizon at future times can both **increase and decrease**, meaning that liquidity can vary in both directions.

VaR and ES with Stochastic Holding Period IX

There is a large number of choices for positive processes: one can take

- lognormal processes with or without mean reversion,
- Mean reverting square root processes,
- Squared gaussian processes,
- all with or without jumps.

Other examples are possible, such as Variance Gamma or mixture processes, or Levy processes.

VaR and ES with Stochastic Holding Period X

Going back to the previous example, let us suppose that
Assumption. The increments $X_{t+1y} - X_t$ are logarithmic returns of an equity index, normally distributed with annual mean and standard deviation respectively $\mu_{1y} = -1.5\%$ and $\sigma_{1y} = 30\%$. We suppose an exposure of 100 in domestic currency.

The portfolio log-returns under SHP at $t = 0$ are

$$P[X_{H_0} - X_0 < x] = \int_0^\infty P[X_h - X_0 < x] dF_{H,t}(h)$$

i.e. as a mixture of Gaussian returns, weighted by the holding period distribution. Here $F_{H,t}$ denotes the cumulative distribution function of the holding period at time t, i.e. of H_t.

VaR and ES with Stochastic Holding Period XI

Mixtures for heavy-tailed and skewed distributions.

Mixtures of distributions have been used for a long time in statistics and may lead to heavy tails, allowing for modeling of skewed distributions and of extreme events. Given the fact that mixtures lead, in the distributions space, to linear (convex) combinations of possibly simple and well understood distributions, they are tractable and easy to interpret.

Going back to our notation, $\text{VaR}_{t,h,c}$ and $\text{ES}_{t,h,c}$ are the value at risk and expected shortfall, respectively, for an horizon h at confidence level c at time t, namely

$$\mathbb{P}\{X_{t+h} - X_t > -\text{VaR}_{t,h,c}\} = c,$$

$$\text{ES}_{t,h,c} = -\mathbb{E}[X_{t+h} - X_t | X_{t+h} - X_t \leq -\text{VaR}_{t,h,c}].$$

VaR and ES with Stochastic Holding Period XII

In the gaussian log-returns case where

$X_{t+h} - X_t$ is normal with mean $\mu_{t,h}$ and standard deviation $\sigma_{t,h}$

we get

$$\mathrm{VaR}_{t,h,c} = -\mu_{t,h} + \Phi^{-1}(c)\sigma_{t,h}, \quad \mathrm{ES}_{t,h,c} = -\mu_{t,h} + \sigma_{t,h} p(\Phi^{-1}(c))/(1-c)$$

where p is the standard normal pdf and Φ the related cdf.

VaR and ES with Stochastic Holding Period XIII

We calculate VaR and ES for a

- confidence level of 99.96%,
- calculated over the fixed time horizons of 10 and 75 days,
- or under SHP process with statistics at time 0 given by

Holding Period h	Probability $P[H_0 = h]$
10 business days	0.99
75 business days	0.01

We use Monte Carlo simulations. Each year has 250 (working) days. Recall also our

Assumption. The increments $X_{t+1y} - X_t$ are logarithmic returns of an equity index, normally distributed with annual mean and standard deviation respectively $\mu_{1y} = -1.5\%$ and $\sigma_{1y} = 30\%$. We suppose an exposure of 100 in domestic currency.

VaR and ES with Stochastic Holding Period XIV

Table: SHP distributions and Market Risk

Holding Period	VaR 99.96%	(analytic)	ES 99.96%
constant 10 b.d.	20.1	(20.18)	21.7
constant 75 b.d.	55.7	(55.54)	60.0
SHP (Bernoulli 10/75, p_{10}=0.99)	29.6	(29.23)	36.1

More generally, we may derive the VaR and ES formulas for the case where H_t is distributed according to a general distribution

$$\mathbb{P}(H_t \leq x) = F_{H,t}(x), \quad x \geq 0$$

and

$$\mathbb{P}(X_{t+h} - X_t \leq x) = F_{X,t,h}(x).$$

VaR and ES with Stochastic Holding Period XV

We define VaR and ES under a random horizon H_t at time t as

$$\mathbb{P}\{X_{t+H_t} - X_t > -\mathrm{VaR}_{H,t,c}\} = c,$$

$$\mathrm{ES}_{H,t,c} = -\mathbb{E}[X_{t+H_t} - X_t | X_{t+H_t} - X_t \leq -\mathrm{VaR}_{H,t,c}].$$

Using the tower property of conditional expectation it is immediate to prove that in such a case $\mathrm{VaR}_{H,t,c}$ obeys the following equation:

$$\int_0^\infty (1 - F_{X,t,h}(-\mathrm{VaR}_{H,t,c}))dF_{H,t}(h) = c$$

whereas $\mathrm{ES}_{H,t,c}$ is given by

$$\mathrm{ES}_{H,t,c} = -\frac{1}{1-c}\int_0^\infty \mathbb{E}[X_{t+h} - X_t | X_{t+h} - X_t \leq -\mathrm{VaR}_{H,t,c}] \cdot$$

$$\cdot \mathrm{Prob}(X_{t+h} - X_t \leq -\mathrm{VaR}_{H,t,c})dF_{H,t}(h)$$

VaR and ES with Stochastic Holding Period XVI

For the specific Gaussian case above we have

$$\int_0^\infty \Phi\left(\frac{\mu_{t,h} + \mathrm{VaR}_{H,t,c}}{\sigma_{t,h}}\right) dF_{H,t}(h) = c$$

$$\mathrm{ES}_{H,t,c} = \frac{1}{1-c}\int_0^\infty \left[-\mu_{t,h}\Phi\left(\frac{-\mu_{t,h} - \mathrm{VaR}_{H,t,c}}{\sigma_{t,h}}\right) + \sigma_{t,h} p\left(\frac{-\mu_{t,h} - \mathrm{VaR}_{H,t,c}}{\sigma_{t,h}}\right)\right] dF_{H,t}(h)$$

Notice that in general one can try and obtain the quantile $\mathrm{VaR}_{H,t,c}$ for the random horizon case by using a root search, and subsequently compute also the expected shortfall. Careful numerical integration is needed to apply these formulas for general distributions of H_t. The case of the above Table is trivial, since in the case where H_0 is a bernoulli rv integrals reduce to summations of two terms.

VaR and ES with Stochastic Holding Period XVII

We note also that the maximum difference, both in relative and absolute terms, between ES and VaR is reached by the model under random holding period H_0.

Under this model the change in portfolio value shows heavier tails than under a single deterministic holding period.

VaR and ES with Stochastic Holding Period XVIII

In order to explore the impact of SHP's distribution tails on the liquidity-adjusted risk, in the following we will simulate SHP models with H_0 distributed as

(i) an Exponential,

(ii) an Inverse Gamma distribution (obtained by rescaling a distribution $\mathrm{IG}\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$ with $\nu = 3$. Before rescaling, setting $\alpha = \nu/2$, the inverse gamma density is $f(x) = (1/\Gamma(\alpha))(\alpha)^{\alpha} x^{-\alpha-1} e^{-\alpha/x}$, $x > 0$, $\alpha > 0$, with expected value $\alpha/(\alpha - 1)$. We rescale this distribution by $k = 8.66/(\alpha/(\alpha - 1))$ and take for H_0 the random variable with density $f(x/k)/k$)

(iii) and a Generalized Pareto distribution (with scale parameter $k = 9$ and shape parameter $\alpha = 2.0651$, with cdf $F(x) = 1 - \left(\frac{k}{k+x}\right)^{\alpha}$, $x \geq 0$, this distribution has moments up to order α. So the smaller α, the fatter the tails. The mean is, if $\alpha > 1$, $\mathbb{E}[H_0] = k/(\alpha - 1)$),

VaR and ES with Stochastic Holding Period XIX

The three distributions have parameters calibrated in order to obtain a sample with the same 99%-quantile of 75 business days. Outputs are:

Distribution	Mean	Median	99%-q	VaR 99.96% simulation	VaR 99.96% root search
Exponential	16.3	11.3	75.0	39.0	39.2
Pareto (fat)	8.45	3.7	75	41.9	41.9
Inv Gamma (fat+)	8.6	3.7	75.0	46.0	46.7

Distribution	ES 99.96% simulation	ES 99.96% root search	ES/VaR-1
Exponential	44.7	44.7	14 %
Pareto (fat)	57.1	56.9	36%
Inv Gamma (fat+)	73.5	73.0	55 %

VaR and ES with Stochastic Holding Period XX

The SHP process changes the statistical nature of the P&L process: the heavier the tails of the SHP distribution, the heavier the tails of P&L distribution.

- Notice that our Pareto distribution has tails going to 0 at infinity with exponent around 3, as one can see immediately by differentiation of the cumulative distribution function,
- whereas our inverse gamma has tails going to 0 at infinity with exponent about 2.5.
- In this example we have that the tails of the inverse gamma are heavier, and indeed for that distribution VaR and ES are larger and differ from each other more.
- This can change of course if we take different parameters in the two distributions.

VaR and ES with SHP: Multivariate case I

Within multivariate modelling, using a common SHP for many normally distributed risks leads to dynamical versions of the so-called *normal mixtures* and *normal mean-variance mixtures*.

Let log-returns ($X_t^i = \ln V_t^i$, with V_t^i the value at t of the i-th asset)

$$X_{t+h}^1 - X_t^1, \ldots, X_{t+h}^m - X_t^m$$

be normals, means $\mu_{t,h}^1, \ldots, \mu_{t,h}^m$, covariance matrix $Q_{t,h}$. Then

$$P[X_{t+H_t}^1 - X_t^1 < x_1, \;\; X_{t+H_t}^m - X_t^m < x_m] =$$

$$= \int_0^\infty P[X_{t+h}^1 - X_t^1 < x_1, \;\; X_{t+h}^m - X_t^m < x_m] dF_{H,t}(h)$$

is distributed as a mixture of multivariate normals.

VaR and ES with SHP: Multivariate case II

A portfolio V_t of the assets $1, 2, ..., m$ whose log-returns $X_{t+h} - X_t$ ($X_t = \ln V_t$) are a linear weighted combination $w_1, ..., w_m$ of the single asset log-returns $X^i_{t+h} - X^i_t$ would be distributed as

$$P[X_{t+H_t} - X_t < z] = \int_0^\infty P[w_1(X^1_{t+h} - X^1_t) + \ldots + w_m(X^m_{t+h} - X^m_t) < z] dF_{H,t}(h)$$

In particular, in analogy with the unidimensional case, the mixture may potentially generate skewed and fat-tailed distributions, but when working with more than one asset this has the further implication that VaR is not guaranteed to be subadditive on the portfolio.

Then the risk manager who wants to take into account SHP in such a setting should adopt a coherent measure like Expected Shortfall.

VaR and ES with SHP: Multivariate case III

Can we increase returns dependence by common SHP's?

A natural question at this stage is whether the adoption of a common SHP can add dependence to returns that are jointly Gaussian under deterministic calendar time, perhaps to the point of making extreme scenarios on the joint values of the random variables possible.

VaR and ES with SHP: Multivariate case IV

Before answering this question, one needs to distinguish extreme behaviour in the single variables and in their joint action in a multivariate setting.

Extreme behaviour on the single variables is modeled for example by heavy tails in the marginal distributions of the single variables.

Extreme behaviour in the dependence structure of say two random variables is achieved when the two random variables tend to take extreme values in the same direction together. This is called **tail dependence**, and one can have both upper tail dependence and lower tail dependence. More precisely, but still loosely speaking,

tail dependence expresses the limiting proportion according to which the first variable exceeds a certain level given that the second variable has already exceeded that level.

VaR and ES with SHP: Multivariate case V

Tail dependence is technically defined through a limit, so that it is an asymptotic notion of dependence.

“Finite” dependence, as opposed to tail, between two random variables is best expressed by rank correlation measures such as Kendall’s tau or Spearman’s rho.

In case the returns of the portfolio assets are jointly Gaussian with correlations smaller than one, the **adoption of a common random holding period for all assets does not add tail dependence**, *unless the commonly adopted random holding period has a distribution with power tails*.

VaR and ES with SHP: Multivariate case VI

Hence if we want to rely on one of the random holding period distributions in our examples above to introduce upper and lower tail dependence in a multivariate distribution for the assets returns, we need to adopt a common random holding period for all assets that is Pareto or Inverse Gamma distributed.

Exponentials, Lognormals or discrete Bernoulli distributions would not work.

This can be seen to follow for example from properties of the normal variance-mixture model, see our full paper for references.

More precisely

VaR and ES with SHP: Multivariate case VII

Theorem: A common random holding period with less than power tails does not add tail dependence to jointly Gaussian returns.

Assume the log-returns to be $W_t^i = \ln V_t^i$, with V_t^i the value at time t of the i-th asset, $i = 1, 2$, where

$$W_{t+h}^1 - W_t^1, W_{t+h}^2 - W_t^2$$

are 2 correlated Brownian motions, i.e. normals with zero means, variances h and inst correlation less than 1 in absolute value:

$$d\langle W^1, W^2\rangle_t = dW_t^1 \; dW_t^2 = \rho_{1,2} dt, \;\; |\rho_{1,2}| < 1.$$

Then adding a common non-negative random holding period H_0 independent of W's leads to tail dependence in the returns $W_{H_0}^1, W_{H_0}^2$ if and only if $\sqrt{H_0}$ is regularly varing at ∞ with index $\alpha > 0$.

VaR and ES with SHP: Multivariate case VIII

Summarizing, if we work with power tails, the heavier are the tails of the common holding period process H, the more one may expect *tail dependence* to emerge for the multivariate distribution.

By adopting a common SHP for all risks, dependence could potentially appear in the whole dynamics, in agreement with the fact that **liquidity risk is a systemic risk.**

VaR and ES with SHP: Multivariate case IX

We now turn to **finite dependence**, as opposed to tail dependence.

First we note the well known elementary but important fact that one can have two random variables with very high dependence but without tail dependence. Or one can have two random variables with tail dependence but small finite dependence.

For example, if we take two jointly Gaussian Random variables with correlation 0.999999, they are clearly quite dependent on each other but they will not have tail dependence, even if a rank correlation measure such as Kendall's τ would be 0.999, still very close to 1, characteristic of the co-monotonic case. This is a case with zero tail dependence but very high finite dependence.

VaR and ES with SHP: Multivariate case X

On the other hand, take a bivariate student t distribution with few degrees of freedom and correlation parameter $\rho = 0.1$. In this case the two random variables have positive tail dependence and it is known that Kendall's tau for the two random variables is

$$\tau = \frac{2}{\pi} \arcsin(\rho) \approx 0.1$$

which is the same tau one would get for two standard jointly Gaussian random variables with correlation ρ. This tau is quite low, showing that one can have positive tail dependence while having very small finite dependence.

Distribution $\rightarrow$ $\downarrow$ Type of Depend	bivariate gaussian with $\rho = 0.999999$	bivariate t with few degrees of freedom and $\rho = 0.1$
Upper tail depend	No	Yes
High finite depend	Yes	No

VaR and ES with SHP: Multivariate case XI

The above examples point out that one has to be careful in distinguishing large finite dependence and tail dependence.

A further point of interest in the above examples comes from the fact that the multivariate student *t* distribution can be obtained by the multivariate Gaussian distribution when adopting a random holding period given by an inverse gamma distribution (power tails). We deduce the important fact that in this case

a common random holding period with power tails adds positive tail dependence but not finite dependence.

In fact, one can prove a more general result easily by resorting to the tower property of conditional expectation and from the definition of tau based on independent copies of the bivariate random vector whose dependence is being measured.

VaR and ES with SHP: Multivariate case XII

Theorem: A common random holding period does not alter Kendall's tau for jointly Gaussian returns.

Assumptions as in the previous theorem. Then adding a common non-negative random holding period H_0 independent of W's leads to the same Kendall's tau for

$$W^1_{H_0}, W^2_{H_0}$$

as for the two returns

$$W^1_t, W^2_t$$

for a given deterministic time horizon t.

VaR and ES with SHP: Multivariate case XIII

Summing up, this result points out that adding further finite dependence through common SHP's, at least as measured by Kendall's tau, can be impossible if we start from Gaussian returns.

Case of two jointly gaussian returns under deterministic calendar time

type of depend $\rightarrow$ $\downarrow$ Holding Period	upper tail dependence	increased finite dependence
deterministic	No	No
Less than power tails	No	No
Power tails	Yes	No

More generally, at least from a theoretical point of view, it could be interesting to model other kinds of dependence than the one stemming purely from a common holding period (with power tails).

VaR and ES with SHP: Multivariate case XIV

One could have two different holding periods that are themselves dependent on each other in a less simplistic way, rather than being just identical. HOLDING PERIODS WITH FACTOR STRUCTURE?

We will investigate this aspect in further research, but increasing dependence may require, besides the adoption of power tail laws for the random holding periods, abandoning the Gaussian distribution for the basic assets under deterministic calendar time, and possibly using other measures of dependence such as Spearman's rho.

VaR and ES under SHP: Calibrating SHP I

We are aware that multivariate SHP modelling is a purely theoretical exercise and that we just hinted at possible initial developments above.

Nonetheless, a lot of financial data is being collected by regulators, providers and rating agencies, together with a consistent effort on theoretical and statistical studies. This will possibly result in available synthetic indices of liquidity risk grouped by region, market, instrument type, etc.

For instance, Fitch already calculates market liquidity indices on CDS markets worldwide, on the basis of a scoring proprietary model (more on this in the second part).

VaR and ES under SHP: Calibrating SHP II

It could be an interesting exercise to calibrate the dependence structure (e.g. copula function) between a liquidity index (like the Fitch's one), a credit index (like iTRAXX) and a market index (for instance Eurostoxx50) in order to measure the possible (non linear) dependence between the three.

The risk manager of a bank could use the resulting dependence structure within the context of risk integration, in order to simulate a joint dynamics as a first step, to estimate later on the whole liquidity-adjusted VaR/ES by assuming co-monotonicity between the variations of the liquidity index and of the SHP processes.

VaR and ES under SHP: Calibrating SHP III

A lot of information on SHP ‘extreme’ statistics of a OTC derivatives portfolio could be collected from the statistics, across Lehman’s counterparties, of the time lags between the Lehman’s Default Event Date and the trade dates of any replacement transaction. The data could give information on the marginal distribution of the SHP of a portfolio, in a stressed scenario, by assuming a statistical equivalence between data collected ‘through the space’ (across Lehman’s counterparties) and ‘through the time’ under i.i.d. hypothesis (a similar approach is adopted in operational risks, see the full paper for references).

The risk manager of a bank could examine a more specific and non-distressed dataset by collecting information on the ordinary operations of the business units.

SHP: Conclusions I

Within the context of risk integration, in order to include liquidity risk in the whole portfolio risk measures, a stochastic holding period (SHP) model can be useful, being versatile, easy to simulate, and easy to understand in its inputs and outputs.

In a single-portfolio framework, as a consequence of introducing a SHP model, the statistical distribution of P&L moves to possibly heavier tailed and skewed mixture distributions.

In a multivariate setting, the dependence among the SHP processes to which marginal P&L are subordinated, may lead to dependence on the latter under drastic choices of the SHP distribution, and in general to heavier tails on the total P&L distribution.

SHP: Conclusions II

At present, lack of synthetic and consensually representative data forces to a qualitative top-down approach, but it is straightforward to assume that this limit will be overcome in the near future.

We are currently working with the Bank of England on an implementation of this methodology.

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