# Default Correlation, Cluster Dynamics and Single Names: the GPCL Dynamical loss model

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See also the papers
http://www.damianobrigo.it/gpl.pdf (Risk Magazine, June 2007)
http://www.damianobrigo.it/gpclweb.pdf
(International Journal of Theoretical and Applied Finance, June 2007)

Joint Work with Andrea Pallavicini, Roberto Torresetti

Credit Conference, University of Chicago, Stevanovich Center, Oct. 19-20, 2007

# Talk outline

- Information content in CDO market quotes;
- Bottom up and Top down approaches to Loss modeling;
- Common Poisson Shock model: excellent but... repeated defaults
- Address the problem at pool level only: GPL model
- Address the problem also at single name and cluster level: GPCL
- Calibration examples across capital strucure and maturity
- A model instead of an inconsistent Gaussian copula (correlation skew)
- Extensions, Pricing and Further Research

# Index CDO's (iTraxx, CDX...)

Given a pool of names $1, 2, \ldots, M$, typically $M = 125$, each with initial notional $1/M$, the index default leg pays to the protection buyer the loss increment occurring each time one or more names default, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

We denote with $\bar{L}_t$ the portfolio cumulated loss and with $\bar{C}_t$ the number of defaulted names up to time $t$, re-scaled by $M$ (and thus in the interval $[0, 1]$).

Therefore

$$\bar{C}_t = \frac{\text{Number of Defaults by } t}{M}, \quad 0 \le \bar{L}_t \le \bar{C}_t \le 1$$

# Information contained in CDO quotes

Recall the market quoted spreads for indices and tranches:

$$S_0 = \frac{\mathbb{E}_0\Big[\int_0^T D(0,u) d\bar{L}_u\Big]}{\mathbb{E}_0\Big[\sum_{i=1}^b \delta_i D(0,T_i)(1-\bar{C}_{T_i})\Big]}$$

$$S_0^{A,B} = \frac{\mathbb{E}_0\Big[\int_0^T D(0,t) d\bar{L}_u^{A,B}\Big] - U_0^{A,B}}{\mathbb{E}_0\Big[\sum_{i=1}^b \delta_i D(0,T_i)(1-\bar{L}_{T_i}^{A,B})\Big]}$$

where $\bar{L}_{T_i}^{A,B}$ is the tranched loss at points $A, B$ divided by the tranche thickness $B-A$. If $S_0$ and $S_0^{A,B}$ are the only data on default correlation in the market, **we see that the only information are "expected losses", "expected tranche losses" and "expected number of defaults".**

# Loss models: The "BOTTOM UP" and "TOP DOWN" approaches

Index and tranches contain information only on expected losses, expected tranche losses and expected number of defaults.

Modeling loss and default number? 2 approaches: BOTTOM UP and TOP DOWN.

**BOTTOM UP: Model single defaults, correlate them and build the loss from these through recovery assumptions on single names.** Mostly copula models. Static.

**TOP (DOWN?): Model the loss and number of defaults directly as the fundamental objects, and possibly achieve consistency with single names a posteriori.** Dynamics. Market feel.

# The Common Poisson Shocks Framework

We begin with the common Poisson shock framework (CPS).

The occurrence of a default can be originated by different events or "factors", either idiosyncratic or systematic.

Occurrence of event/factor number $e$, with $e = 1 \ldots m$, is modelled as a jump of a Poisson process $N^{(e)}$. Each event can be triggered many times $r = 1, 2, \ldots$ as jumps go on.

Poisson processes driving different factors are independent.

The CPS setup assumes unrealistically that a defaulted name $k$ may default again. We address this limitation.

For now we assume the $r$-th jump of $N^{(e)}$ to trigger a default event for name $k$ with probability $p^{(e)}_{r,k}$,

| | Name $k$ =1 | | |
|---|---|---|---|
| | Factor $e$: 1 | 2 | 3 ... |
| Repetition $r$ : | | | |
| 1 | $p^1_{1,1}$ | $p^2_{1,1}$ | $p^3_{1,1}$ |
| 2 | $p^1_{2,1}$ | $p^2_{2,1}$ | $p^3_{2,1}$ |
| ⋮ | | | |

| | Name $k$ =2 | | |
|---|---|---|---|
| | Factor $e$: 1 | 2 | 3 ... |
| Repetition $r$ : | | | |
| 1 | $p^1_{1,2}$ | $p^2_{1,2}$ | $p^3_{1,2}$ |
| 2 | $p^1_{2,2}$ | $p^2_{2,2}$ | $p^3_{2,2}$ |
| ⋮ | | | |

# The Common Poisson Shocks Framework

We have the dynamics for the single name default process $N_k$, jumping each time name $k$ defaults:

$$N_k(t) := \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{r,k}^{(e)}$$

where $I_{r,k}^{(e)}$ is a Bernoulli variable with probability $\mathbb{Q}\{I_{r,k}^{(e)} = 1\} = p_{r,k}^{(e)}$. $N_k$ turns out to be itself Poisson.

Notice however that $N_k$ and $N_h$ are not independent since their dynamics is explained by the same driving events.

# The Common Poisson Shocks Framework

A key result consists in mapping the single name defaults $N_k$ into a multi-name dynamics explained in terms of *independent* Poisson processes $\widetilde{N}_s$, where $s$ is a subset (or "cluster") of names of the pool:

$$\widetilde{N}_s(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \sum_{s' \supseteq s} (-1)^{|s'|-|s|} \prod_{k' \in s'} I^{(e)}_{r,k'}$$

where $|s|$ is the number of names in the cluster $s$.

In a summation, $s \ni k$ means "across all clusters $s$ containing $k$",
$k \in s$ means "across all elements $k$ of cluster $s$",
$|s| = j$ means "across all clusters of size $j$" and, finally,
$s' \supseteq s$ means "across all clusters $s'$ containing cluster $s$ as a subset".

# GPCL: The Common Poisson Shocks Framework

The non-trivial proof of the independence of $\widetilde{N}_s$ for different subsets $s$ can be found in Lindskog and McNeil (2003).

Notice that a jump in a $\widetilde{N}_s$ process means that all the names in the subset $s$, *and only those names*, have defaulted at the jump time.

We denote by $\widetilde{\lambda}_s$ the intensity of $\widetilde{N}_s(t)$, and we assume it to be deterministic; we present extensions later.

$\widetilde{N}_s(t)$ **will be our new building blocks**: for example single names are

$$dN_k(t) = \sum_{s \ni k} d\widetilde{N}_s(t), \qquad (1)$$

**The first jump times survival copula across names for $N_1, ..., N_k, ..$ is a Marshal Olkin copula. Top Down!**

# The Common Poisson Shocks Framework

$$Z_j(t) := \sum_{|s|=j} \widetilde{N}_s(t). \qquad (2)$$

$Z_j(t)$ describes the simultaneous default of any $j$ names whenever it jumps (by one); being the sum of independent Poisson, is itself Poisson.

Further, since the clusters corresponding to the different $Z_1, Z_2, \ldots, Z_M$ never overlap, the $Z_j(t)$ are independent.

The total number of defaults in the pool by $t$ is

$$Z_t := \sum_{k=1}^{M} N_k(t) = \sum_{j=1}^{M} j Z_j(t) \qquad (3)$$

# Top Down Approach: The GPL Model

**If we accepted repeated defaults, this would be a real top down model for $Z_t/M$ (and then Loss when including recovery modeling).**

How would the aggregate pool default counting process look like exactly?

$$\text{Example}: \quad M = 125, \quad Z_t = 1\ Z_1(t) + 2\ Z_2(t) + \ldots + 125\ Z_{125}(t).$$

If $Z_1$ jumps there is just one default (idiosyncratic), if $Z_{125}$ jumps there are 125 ones and the whole pool defaults one shot (total systemic risk), otherwise for other $Z_i$'s we have intermediate situations (sectors).

A self-exciting feature, although extreme, is present in that more name may default together (rather than one default increasing the intensity of others)

# Improving CPS: Avoiding repeated defaults

Fundamental problem: repeated jumps (=defaults) of the same Poisson processes, **both at the cluster level, $\widetilde{N}_s$, and at the single name level,** $N_k$. These repetitions would also cause the default counting process $Z_t$ to exceed the pool size $M$.

Lindskog and McNeil (2003) suppose that the default intensities of the names are so small to lead to negligible "second-default" probabilities. **However, in our calibration results of GPL below, intensities are large enough to make repeated defaults unacceptable in practice.**

# Improving CPS: Avoiding repeated defaults

**Strategy GPL (Default-counting adjusted approach)**. Modify the aggregated pool default counting process so that this does not exceed the number of names, by simply capping $Z_t$ to $M$, regardless of cluster structures:

$$C_t := \min(Z_t, M)$$

**Strategy GPCL (Cluster adjusted approach)**. Force clusters to jump only once and deduce single names defaults consistently.

The first choice is ok at top level but it does not really go down. We attack this first.

The second choice is a real top down model, which we attack later.

# Top (Down?) Approach: The GPL Model

Consider a number $n$ of independent Poissons $Z_1, \ldots, Z_n$ with intensities $\lambda_1^0, \ldots, \lambda_n^0$. Define the stochastic process (Generalized Poisson Loss, GPL)

$$Z_t = \sum_{j=1}^{n} \alpha_j Z_j(t), \quad C_t := \min(Z_t, M)$$

for increasing integers $\alpha_1, \ldots, \alpha_n$

The density of $Z_t$ (and thus of $C_t$) can be obtained as the inverse Fourier transform of the known characteristic function of $Z_t$

The pool intensity (compensator) can be computed in closed form.

Multiple default intensities can be generalized to Gamma, CIR...

# Calibration

The GPL model is calibrated to the market quotes observed on March 1 and 6, 2006. Deterministic discount rates are listed in Brigo, Pallavicini and Torresetti (2006). Tranche data and DJi-TRAXX fixings, along with bid-ask spreads, are

| | **Att-Det** | **March, 1 2006** | | **March, 6 2006** | | |
|---|---|---|---|---|---|---|
| | | 5y | 7y | 3y | 5y | 7y |
| **Index** | | 35(1) | 48(1) | 20(1) | 35(1) | 48(1) |
| **Tranche** | 0-3 | 2600(50) | 4788(50) | 500(20) | 2655(25) | 4825(25) |
| | 3-6 | 71.00(2.00) | 210.00(5.00) | 7.50(2.50) | 67.50(1.00) | 225.50(2.50) |
| | 6-9 | 22.00(2.00) | 49.00(2.00) | 1.25(0.75) | 22.00(1.00) | 51.00(1.00) |
| | 9-12 | 10.00(2.00) | 29.00(2.00) | 0.50(0.25) | 10.50(1.00) | 28.50(1.00) |
| | 12-22 | 4.25(1.00) | 11.00(1.00) | 0.15(0.05) | 4.50(0.50) | 10.25(0.50) |
| **Tranchlet** | 0-1 | 6100(200) | 7400(300) | | | |
| | 1-2 | 1085(70) | 5025(300) | | | |
| | 2-3 | 393(45) | 850(60) | | | |

# Calibration

The cumulated intensities $\Lambda_i^0(T)$ are real non-decreasing piecewise linear functions in the tranche maturity.

The optimal values for the amplitudes $\alpha$ are selected as follows:

1. set $\alpha_1 = 1$ and all other $\alpha$'s to zero. Calibrate $\Lambda_1^0$;

2. find the best integer value for $\alpha_2$ by calibrating the cumulated intensities $\Lambda_1^0$ and $\Lambda_2^0$ for each value of $\alpha_2$ in the range $[1, 125]$,starting from the previous $\Lambda_1^0$ as a guess;

3. repeat the previous step for $\alpha_i$ with $i = 3$ and so on, by calibrating the cumulated intensities $\Lambda_1^0, \ldots, \Lambda_i^0$, starting from the previously found $\Lambda_1^0, \ldots, \Lambda_{i-1}^0$ as initial guess, until the calibration error is under a pre-fixed threshold or until the intensity $\Lambda_i^0$ can be considered negligible.

# Calibration

The objective function $f$ to be minimized in the calibration is the squared sum of the errors shown by the model to recover the tranche and index market quotes weighted by market bid-ask spreads:

$$f(\alpha, \Lambda^0; \beta, \gamma) = \sum_i \epsilon_i^2, \quad \epsilon_i = \frac{x_i(\alpha, \Lambda^0; \beta, \gamma) - x_i^{\mathrm{Mid}}}{x_i^{\mathrm{Bid}} - x_i^{\mathrm{Ask}}}$$

where the $x_i$, with $i$ running over the market quote set, are the index values $S_0$ for DJi-TRAXX index quotes, and either the index periodic premium rates $S_0^{A,B}$ or the upfront premium rates $U_0^{A,B}$ for the DJi-TRAXX tranche quotes.

# Calibration: All standard tranches up to seven years

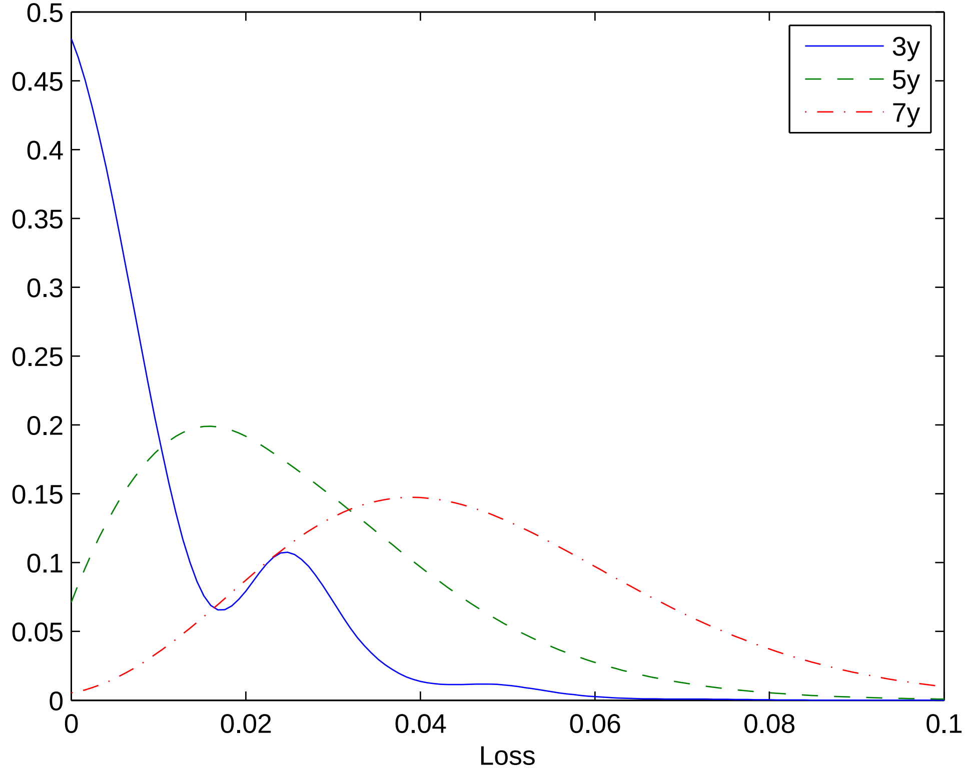
As a first calibration example we consider standard DJi-TRAXX tranches up to a maturity of 7y with constant recovery rate of $40\%$.

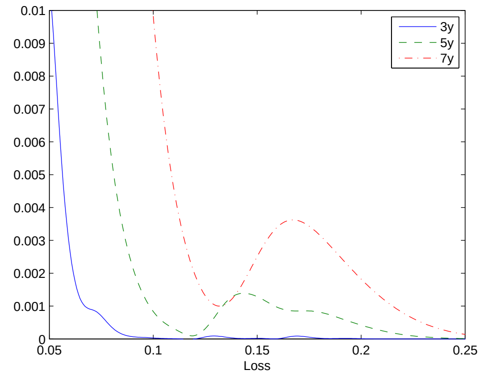
The calibration procedure selects five Poisson processes. The $18$ market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread (we show the ratio calibration error / bid ask spread).

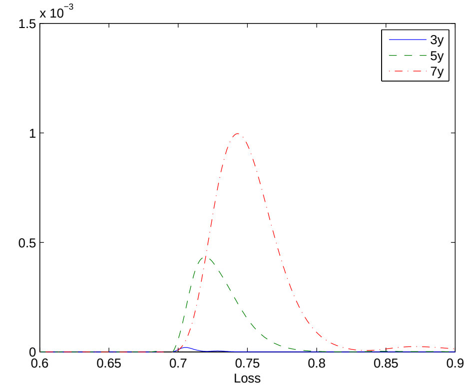
| | **Att-Det** | **Maturities** | | |
|---|---|---|---|---|
| | | 3y | 5y | 7y |
| **Index** | | -0.4 | -0.2 | -0.9 |
| **Tranche** | 0-3 | 0.1 | 0.0 | -0.7 |
| | 3-6 | 0.0 | 0.0 | 0.7 |
| | 6-9 | 0.0 | 0.0 | -0.2 |
| | 9-12 | 0.0 | 0.0 | 0.0 |
| | 12-22 | 0.0 | 0.0 | 0.2 |

| $\boldsymbol{\alpha}$ | $\boldsymbol{\Lambda(T)}$ | | |
|---|---|---|---|
| | 3y | 5y | 7y |
| 1 | 0.535 | 2.366 | 4.930 |
| 3 | 0.197 | 0.266 | 0.267 |
| 16 | 0.000 | 0.007 | 0.024 |
| 21 | 0.000 | 0.003 | 0.003 |
| 88 | 0.000 | 0.002 | 0.007 |

# Calibration: All standard tranches up to seven years



# Calibration: All standard tranches up to seven years

Notice in particular the large portion of mass concentrated near the origin, the subsequent modes (default clusters) when moving along the loss distribution for increasing values, and the bumps in the far tail.

In particular, the components with higher $\alpha$'s are giving rise to the little bumps in the far tail of the loss distribution and help with senior tranches.

These features are common to other approaches, such as the static implied copula by Hull and White (as reimplemented in Torresetti, Brigo and Pallavicini (2006)).

# Calibration: Tranchlets

The market quotes also non-standard tranches, which are quoted over the counter. An interesting case is given by the so called "tranchlets", namely DJi-TRAXX tranches with attachment and detachment points possibly smaller than $3\%$. On the first of march 2006 we obtain market quotes for a set of tranchlets with maturity of five and seven years (see earlier table).

We calibrate the market data with constant recovery rate of $40\%$. The calibration procedure selects five Poisson processes. The $18$ market quotes used by the calibration procedure are recovered, but within an error that is occasionally larger than the bid-ask spread.

# Calibration: Tranchlets

| | **Att-Det** | **Maturities** | |
|---|---|---|---|
| | | 5y | 7y |
| **Index** | | -0.8 | -2.1 |
| **Tranchlet** | 0-1 | 1.1 | -1.4 |
| | 1-2 | 1.7 | -0.6 |
| | 2-3 | -0.1 | -0.4 |
| **Tranche** | 0-3 | 0.1 | 0.4 |
| | 3-6 | -1.9 | 0.2 |
| | 6-9 | 0.4 | 0.6 |
| | 9-12 | 2.8 | 0.9 |
| | 12-22 | -0.4 | -1.5 |

| $\boldsymbol{\alpha}$ | $\boldsymbol{\Lambda(T)}$ | |
|---|---|---|
| | 5y | 7y |
| 1 | 0.834 | 3.336 |
| 2 | 1.070 | 1.070 |
| 13 | 0.008 | 0.015 |
| 21 | 0.004 | 0.013 |
| 104 | 0.002 | 0.007 |

Table 1: Left side: calibration error with respect to the bid-ask spread for tranches quoted by the market. Right side: cumulated intensities of the basic GPL model. Each row corresponds to a different Poisson component with jump amplitude $\alpha$. Recovery rate is $40\%$.

# Pricing and Further research

**Pricing products based on the loss distribution such as tranche options, forward start tranches, cancelabe tranche etc, even path dependent ones, with the calibrated model is simple through simulation, given knowledge of the marginal and transition distributions for the constituent Poisson processes.**

This is maintained also under random (Gamma or scenario or CIR) intensities.

Alternatively, if only the terminal losses are relevant, we may decide to use the inverse Fourier transform of the known characteristic function of the terminal distribution to obtain the loss density and then integrate numerically the payoff against this density.

# Problem

This model, after the capping

$$C_t := \min(Z_t, M)$$

loses the rigorous top down interpretation, in that the default rate is no longer associated with single name models whose defaults are connected through a Marshal Olkin Copula.

So we cannot zoom on defaults of single names not even in principle, nor can we know how losses of subpools are related (CDO$^2$).

**We attack this by indtroducing the second way to avoid repeated defaults in the CPS framewor: The GPLC model.**

# GPCL model: Cluster-adjusted approach

The key to *consistently* avoid repeated cluster defaults (and subsequently single names) is to track, when a cluster jumps, which single-name defaults are triggered, and then force all the clusters containing such names not to jump any longer. Define

$$J_s(t) := \prod_{k \in s} \prod_{s' \ni k} 1_{\{\widetilde{N}_{s'}(t)=0\}} = \prod_{s' \colon s' \cap s \neq \emptyset} 1_{\{\widetilde{N}_{s'}(t)=0\}}$$

The process $J_s(t)$ is equal to $1$ at starting time and it jumps to $0$ whenever a cluster containing one element of $s$ jumps. Or one may view the process $J_s$ as being one when none of the names in $s$ have defaulted and $0$ when some names in $s$ have defaulted.

# GPCL model: Cluster-adjusted approach

We now correct the cluster dynamics by avoiding repeated clusters defaults. We define as new cluster dynamics the following:

$$d\widetilde{N}_s^2(t) = J_s(t^-)d\widetilde{N}_s(t). \tag{4}$$

**Interpretation:** *every time a repeated cluster default process $\widetilde{N}_s$ jumps, this is a jump in our "no-repeated-jumps" framework only if no name contained in $s$ has defaulted in the past, i.e. if no cluster intersecting $s$ has defaulted in the past.*

# GPCL model: Cluster-adjusted approach

Once the clusters defaults are given, single name defaults follow easily. Define the single name dynamics as

$$dN_k^2(t) := \sum_{s \ni k} d\widetilde{N}_s^2 = \sum_{s \ni k} J_s(t^-) d\widetilde{N}_s(t). \tag{5}$$

Now, re-define default counting processes in terms of our new cluster dynamics. We obtain

$$dZ^2{}_j := \sum_{|s|=j} d\widetilde{N}_s^2 = \sum_{|s|=j} J_s(t^-) d\widetilde{N}_s(t). \tag{6}$$

# GPCL model: Cluster-adjusted approach

The pool counting process reads

$$dZ^2 = \sum_{j=1}^{M} j \sum_{|s|=j} d\widetilde{N}_s^2 = \sum_{j=1}^{M} j \sum_{|s|=j} J_s(t^-) d\widetilde{N}_s(t). \tag{7}$$

If not for the cluster-related indicators $J_s(t^-)$, $Z^2$ would be a generalized Poisson process. That is why we term the model $N_k^2, \widetilde{N}_s^2, {Z^2}_j$ the Generalized Poisson Cluster-adjusted Loss model (GPCL).

# Beyond GPL: The GPCL model calibration

The recovery rate is considered as a deterministic constant and set equal to $R = 40\%$. Thus, the underlying driving model definition is

$$C_t := Z^2(t) = \sum_{j=1}^{M} j\, Z_j^2(t), \text{ where } dZ_j^2(t) \sim \text{Poisson}\left(\binom{M - Z_{t^-}^2}{j}\widetilde{\lambda}_j(t)dt\right)$$

while the pool counting and loss processes are defined as

$$\begin{aligned} d\bar{C}_t &:= dZ_t^2/M \\ d\bar{L}_t &:= (1-R)\, dZ_t^2/M \end{aligned}$$

# Beyond GPL: The GPCL model calibration

Given our recovery assumption, the prices of the products to be calibrated, depend only on knowledge of the probability distribution of the pool counting process $C_t$. Thus, our main issue is to calculate this law as fast as possible.

With the GPCL model, the dependence of the intensity of the pool counting process on the process itself

$$dZ_j^2(t) \sim \text{Poisson}\left(\binom{M - Z_{t^-}^2}{j}\widetilde{\lambda}_j(t)dt\right)$$

prevents us either to calculate the relevant characteristic function in closed form (as for GPL instead) or to use the Panjer method.

# Beyond GPL: The GPCL model calibration

Calculate the forward Kolmogorov equation for $p_{Z_t^2}(x) = \mathbb{Q}\{Z_t^2 = x\}$:

$$\frac{d}{dt} p_{Z_t^2}(x) = \sum_{y=0}^{M} A_t(x,y) p_{Z_t^2}(y), \text{ with trans rate matrix } A_t = (A_t(x,y))_{x,y=0,\dots,M}$$

$$A_t(x,y) := \lim_{\Delta t \to 0} \frac{\mathbb{Q}\{Z_{t+\Delta t}^2 = x | Z_t^2 = y\}}{\Delta t} = \binom{M-y}{x-y} \widetilde{\lambda}_{x-y}(t)$$

for $x > y$,

$$A_t(y,y) := \lim_{\Delta t \to 0} \frac{\mathbb{Q}\{Z_{t+\Delta t}^2 = y | Z_t^2 = y\} - 1}{\Delta t} = -\sum_{j=1}^{M-y} \binom{M-y}{j} \widetilde{\lambda}_j(t).$$

for $x = y$, and zero for $x < y$.

# Beyond GPL: The GPCL model calibration

In matrix form we write

$$\frac{d}{dt}\widehat{\pi}_t = A_t \widehat{\pi}_t, \qquad \widehat{\pi}_t := \left[ \begin{array}{ccccc} p_{Z_t^2}(0) & p_{Z_t^2}(1) & p_{Z_t^2}(2) & \ldots & p_{Z_t^2}(M) \end{array} \right]'$$

whose solution is obtained through the exponential matrix,

$$\widehat{\pi}_t = \exp\left(\int_0^t A_u du\right)\widehat{\pi}_0, \quad \widehat{\pi}_0 = [1\ 0\ 0...\ 0]'.$$

Matrix exponentiation can be quickly computed with the Padé approximation (see Golub and Van Loan (1983)), leading to a closed form solution for the probability distribution $p_{C_t} = \widehat{\pi}_t$ of the pool counting process $C_t$. This distribution can then be used in the calibration procedure.

# Beyond GPL: The GPCL model calibration

If we define the cumulated cluster intensities as $\widetilde{\Lambda}_j(t) = \int_0^t \widetilde{\lambda}_j(u)\, du$, then the entries of the matrix undergoing exponentiation in determining the default counting distribution are given by

$$\text{for } x > y: \quad \int_0^t A_u(x,y)du = \binom{M-y}{x-y} \widetilde{\Lambda}_{x-y}(t)$$

$$\text{for } x = y: \quad \int_0^t A_u(y,y)du = -\sum_{j=1}^{M-y} \binom{M-y}{j} \widetilde{\Lambda}_j(t).$$

We assume the $\widetilde{\Lambda}_j$ to be piecewise linear in time, changing their values at payoff maturity dates. We use $\widetilde{\Lambda}_j$ as calibration parameters. We have $bM$ free calibration parameters, if we consider $b$ maturities.

# Beyond GPL: The GPCL model calibration

Many $\widetilde{\Lambda}_j(t)$ will be zero for all maturities, and we can ignore their $Z_j^2(t)$.

Call $\alpha_1 < \alpha_2 < ... < \alpha_n$ the jump sizes with nonzero intensity. Then one renumbers progressively the intensities according to the nonzero increasing $\alpha$: $Z_j^2$ becomes the jump of a cluster of size $\alpha_j$.

The calibration procedure for GPCL is implemented using the $\alpha_j$ in the same way as for GPL. We also calibrate GPL, for comparison.

In the tables we display $\binom{M}{\alpha_j}\widetilde{\Lambda}_j$, i.e. we multiply a cluster cumulated intensity for a given cluster size for the number of clusters with that size at time $0$.

The calibration data set is the DJi-TRAXX main series on the run on October, 2 2006.

# Beyond GPL: The GPCL model calibration

When we calibrate the GPL and GPCL models, and we obtain the calibration parameters presented in Table 2

This is a joint calibration across tranche seniority and maturity, since we are calibrating all and every tranche and index quote with a single model specification.

Both our models perform very well on maturities of 3 years, 5 years and 7 years, for which the calibration error is within the bid-ask spread.

Both models are close to the 10y market values, as we see from the left panel of Table 4. Notice, however, that the GPCL model has a lower calibration error ($10\% - 20\%$ better).

| $\alpha_j$ | $\Lambda_j^0(T)$ | | | |
|---|---|---|---|---|
| | 3y | 5y | 7y | 10y |
| 1 | 0.778 | 1.318 | 3.320 | 4.261 |
| 3 | 0.128 | 0.536 | 0.581 | 1.566 |
| 15 | 0.000 | 0.004 | 0.024 | 0.024 |
| 19 | 0.000 | 0.007 | 0.011 | 0.028 |
| 32 | 0.000 | 0.000 | 0.000 | 0.007 |
| 79 | 0.000 | 0.000 | 0.003 | 0.003 |
| 120 | 0.000 | 0.002 | 0.003 | 0.008 |

| $\alpha_j$ | $\binom{M}{\alpha_j}\widetilde{\Lambda}_j(T)$ | | | |
|---|---|---|---|---|
| | 3y | 5y | 7y | 10y |
| 1 | 0.882 | 1.234 | 3.223 | 3.661 |
| 3 | 0.128 | 0.615 | 0.682 | 1.963 |
| 15 | 0.001 | 0.002 | 0.023 | 0.023 |
| 19 | 0.000 | 0.009 | 0.016 | 0.043 |
| 57 | 0.000 | 0.000 | 0.002 | 0.007 |
| 80 | 0.000 | 0.000 | 0.000 | 0.010 |
| 125 | 0.001 | 0.005 | 0.042 | 0.042 |

Table 2: DJi-TRAXX pool. Left side: cumulated intensities, integrated up to tranche maturities, of the basic GPL model. Each row $j$ corresponds to a different Poisson component with jump amplitude $\alpha_j$. Right side: cumulated cluster intensities, integrated up to tranche maturities, and multiplied by the number of clusters of the same size at time $0$. Each row $j$ corresponds to a different cluster size $\alpha_j$. The amplitudes/cluster-sizes not listed have an intensity below $10^{-7}$. The recovery rate is $40\%$. All calibration errors within one bid-ask

# Beyond GPL: The GPCL model calibration

We also apply the GPL and GPCL methods to the CDX index and tranches, following the same procedure used for the DJi-TRAXX above.

We find better results, that are summarized in Table 3 and in the right panel of Table 4.

| $\alpha_j$ | $\Lambda_j^0(T)$ | | | |
|---|---|---|---|---|
| | 3y | 5y | 7y | 10y |
| 1 | 1.132 | 3.043 | 4.247 | 7.166 |
| 2 | 0.189 | 0.189 | 0.812 | 1.625 |
| 6 | 0.011 | 0.091 | 0.091 | 0.091 |
| 18 | 0.000 | 0.006 | 0.028 | 0.028 |
| 23 | 0.000 | 0.004 | 0.005 | 0.032 |
| 32 | 0.000 | 0.000 | 0.000 | 0.009 |
| 124 | 0.000 | 0.003 | 0.005 | 0.010 |

| $\alpha_j$ | $\binom{M}{\alpha_j}\widetilde{\Lambda}_j(T)$ | | | |
|---|---|---|---|---|
| | 3y | 5y | 7y | 10y |
| 1 | 0.063 | 0.552 | 3.100 | 6.661 |
| 2 | 0.804 | 1.531 | 1.531 | 2.076 |
| 3 | 0.020 | 0.195 | 0.195 | 0.195 |
| 17 | 0.000 | 0.010 | 0.037 | 0.087 |
| 32 | 0.000 | 0.003 | 0.009 | 0.032 |
| 110 | 0.000 | 0.000 | 0.000 | 0.010 |
| 125 | 0.000 | 0.011 | 0.054 | 0.054 |

Table 3: CDX pool. Left side: cumulated intensities, integrated up to tranche maturities, of the basic GPL model. Each row $j$ corresponds to a different Poisson component with jump amplitude $\alpha_j$. Right side: cumulated cluster intensities, integrated up to tranche maturities, and multiplied by the number of clusters of the same size at time $0$. Each row $j$ corresponds to a different cluster size $\alpha_j$. The amplitudes/cluster-sizes not listed have an intensity below $10^{-7}$. The recovery rate is $40\%$.

| | **Att-Det** | **DJi-TRAXX 10y** | |
|---|---|---|---|
| | | **GPL** | **GPCL** |
| **Idx** | | 0.00 | 0.00 |
| **Trn** | 0-3 | 0.76 | 0.62 |
| | 3-6 | -2.35 | -1.93 |
| | 6-9 | 1.21 | 1.04 |
| | 9-12 | -0.40 | -0.36 |
| | 12-22 | 0.02 | 0.02 |
| | 22-100 | 0.00 | 0.00 |

| | **Att-Det** | **CDX 10y** | |
|---|---|---|---|
| | | **GPL** | **GPCL** |
| **Idx** | | 0.00 | -0.06 |
| **Trn** | 0-3 | 1.43 | 1.60 |
| | 3-7 | -0.45 | -0.22 |
| | 7-10 | 0.22 | 0.25 |
| | 10-15 | -0.08 | -0.12 |
| | 15-30 | 0.01 | 0.07 |

Table 4: Calibration errors calculated with the GPL and GPCL models with respect to the bid-ask spread (i.e. $\epsilon_i$) for tranches quoted by the market for the ten year maturity. The left panel refers to DJi-TRAXX market quotes, while the right panel refers to CDX market quotes. Calibration errors for the other maturities are within the bid-ask spread and therefore they are not reported. The recovery rate is $40\%$ .

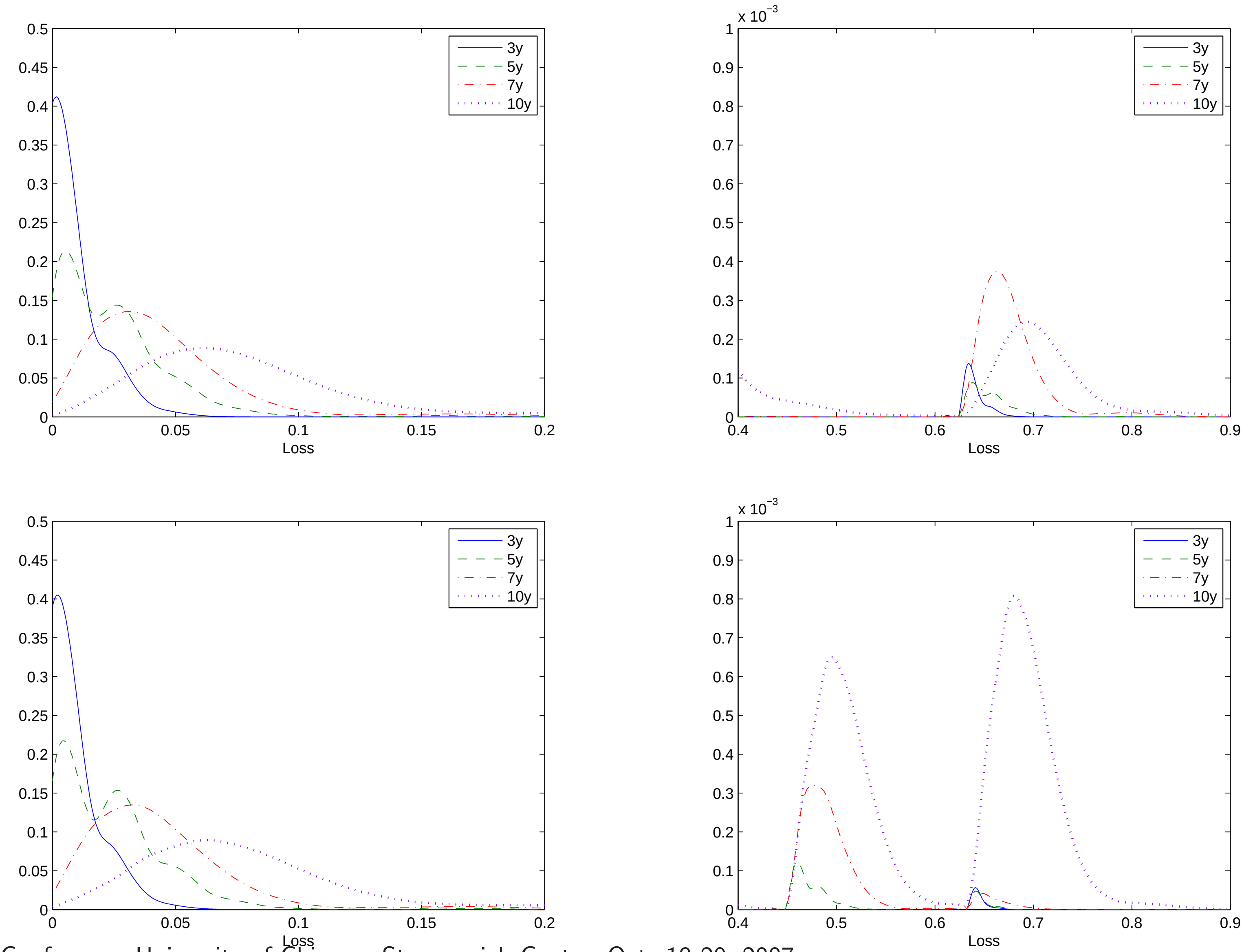





Figure 1:

# Beyond GPL: The GPCL model calibration

Loss distribution evolution of the GPL model (upper panel) and of the GPCL model (lower panel) at all the quoted maturities up to ten years, drawn as a continuous line.

The probability distributions implied by the two dynamical models are similar at gross-grain view, as one can see in Figure 1, but they differ if we observe the fine structure. Indeed, the tails of the two distributions show different bumps. The GPCL model shows a more complex pattern, and, as one can see from Table 2, its highest mode is the maximum portfolio loss, while the GPL model has a less clear tail configuration.

# Model Extensions: Spread dynamics

The valuation of credit index forward contracts or options maturing at time $T = T_a$ requires the calculation of the index spread at those future times, which in turn depends on the default intensity evolution.

The dynamics can be enriched by explicitly adding stochasticity to the Poisson intensities $\widetilde{\lambda}_j(t)$, e.g. resorting to the Gamma, scenario or CIR extensions, similarly to what has been suggested for the GPL model.

# Recovery dynamics

$$d\bar{L}_t = (1 - R_t)d\bar{C}_t \quad \text{(or, more precisely} \quad \bar{L}_t = \int_0^t (1 - R_u)d\bar{C}_u). \tag{8}$$

In general, for ease of computation, we assume recovery at default $R_t$ to be a $\mathcal{G}_t$-adapted and left-continuous (and hence predictable) process taking values in the interval $[0, 1]$.

Here $\mathcal{G}_t$ denotes the filtration consisting of default-free market information and of the default-count monitoring up to time $t$. This implies in particular, that the loss $\bar{L}_t$ is $\mathcal{G}_t$-adapted too, as is reasonable.

# Recovery dynamics

The no-arbitrage condition $d\bar{L}_t \leq \ d\bar{C}_t$ is met if $R_t$ takes values in $[0, 1]$. Equation (8) leaves us with the freedom of defining only two processes among $\bar{L}_t$, $\bar{C}_t$ and $R_t$. The more natural approach would be modeling explicitly $(\bar{C}_t, R_t)$, obtaining $\bar{L}_t$, or modeling explicitly $(\bar{L}_t, R_t)$, obtaining $\bar{C}_t$, all of them adapted.

# Recovery dynamics

In some formulations the predictability of the recovery is not possible. It is also a notion not always realistic: whether one or 125 names default in instant $(t-dt,t]$ (i.e. $dC_t = 1$ or $dC_t = 125$, respectively), we would be imposing the recovery $R_t$ to be the same in both cases and, in particular, to depend only on the information up to $t^-$.

# Recovery dynamics

We now examine possible ways to model the loss more realistically, starting from a GPL or GPCL model formulated in terms of default counting process. This amounts to implicitly model the recovery rate, since the number of defaults and the loss are linked by the recovery at default.

A first approach to implicitly model recovery rates consists in defining the cumulated portfolio loss $\bar{L}_t$ process as a deterministic function of the pool counting process $\bar{C}_t$ via a deterministic map.

# Recovery dynamics through Deterministic mapping

Set

$$\bar{L}_t := \psi(\bar{C}_t),$$

where $\psi$ is a non-decreasing deterministic function with $\psi(0) = 0$ and $\psi(1) \leq 1$. What does this imply in terms of recovery dynamics? We can easily write

$$d\bar{L}_t = \sum_{k=1}^{M} \left[ \frac{\psi(\bar{C}_{t^-} + k/M) - \psi(\bar{C}_{t^-})}{k/M} \right] 1_{\{d\bar{C}_t = k/M\}} d\bar{C}_t$$

which shows that the recovery at default in this case would not be predictable, depending explicitly from $dC_t$, except for very special $\psi$'s.

# Recovery dynamics through Deterministic mapping

A generalization based on a random process transformation (rather than a deterministic function) of the counting process leads to a more sophisticated implicit dynamics of the recovery process.

Consider a stochastic process $u \mapsto \Psi_u$ in time $u$, $\mathcal{G}_u$-adapted and taking values in $[0, 1]$, right-continuous with left limit, and independent of the default counting process $\bar{C}_t$, and use it to map the positive non-decreasing pool counting process $\bar{C}_t$ taking values in $[0, 1]$ into the portfolio cumulated loss $\bar{L}_t$, sharing the same characteristics, i.e. define

$$\bar{L}_t := \Psi_{\bar{C}_t}.$$

# Recovery dynamics through Deterministic mapping

Further, assume (no-arbitrage conditions):

$$\Psi_0 = 0, \quad \Psi_1 \leq 1, \quad \text{and} \quad d\Psi_t \geq 0$$

This way the cumulated portfolio loss can be viewed as a stochastic time change of the process $\Psi$. Further, in order to allow for portfolio total loss, we enforce the stronger condition $\Psi_1 = 1$.

# Recovery dynamics through Deterministic mapping

The time change does not spoil the analytical tractability of the model. If we know the probability distribution function of the pool counting process and of $\Psi$, we can simply derive the probability distribution function of the portfolio loss through an iterated expectation, thanks to independence:

$$\mathbb{Q}\{\bar{L}_t \leq x\} = \mathbb{E}\left[\,\mathbb{Q}\{\bar{L}_t \leq x|\bar{C}_t\}\,\right] = \int \mathbb{Q}\{\Psi_y \leq x\} p_{\bar{C}_t}(y) dy$$

# Recovery dynamics through Deterministic mapping

As a relevant example, assume the process $u \mapsto \Psi_u$ is a Gamma process with shape parameter $\mu(u)$ and scale parameter $\nu$. The monotonicity of the resulting loss process can be easily checked, while the probability distribution of the process can be calculated explicitly. Indeed, as a direct calculation can show, for any times $s < t < T$, the conditional distribution of $\Psi_t$, given $\Psi_s$ and $\Psi_T$ is known in terms of the Beta distribution.

The calculation of the unconditional distribution of the cumulated portfolio loss follows directly.

Exactly as for the previous case based on the deterministic transform $\psi$, here the implicit recovery at default turns out to be not predictable in general.

# GPL and GPCL loss models: conclusions

We **extend the common Poisson shock (CPS) framework** to avoid repeated defaults, leading to the GPL and GPCL dynamical loss models.

GPCL attains **good calibration** as GPL, further allowing for **consistency with single names: one of the few explict top down approaches**.

Consistently accounts for index and tranche market quotes across (i) the whole capital structure **and** (ii) maturity.

Copula models, and in particular the Gaussian copula model leading to implied correlation, cannot achieve (i), let alone (ii).

The model also reads a dynamics for the loss distribution from market data, and we could also use the model to see **Ratings under the risk neutral measure**, although this is not necessarily a good idea.

# GPL and GPCL loss models: conclusions

**Further research** concerns recovery dynamics, calibration and analysis of forward start tranches and tranche options, when liquid quotes will be available, and analysis of calibration stability through history.

Also, basic tranche quotes for **bespoke portfolios** are hardly available, so dependence mapping from the quoted liquid portfolios (itraxx, cdx) to bespoke portfolio needs to be available before we may use GP(C)L for bespoke pools.

A preliminary analysis of **stability** with the GPL model is however presented in Brigo, Pallavicini and Torresetti (2006b), showing good results. This is encouraging and leads to assuming the GPCL stability as well, although a rigorous check is in order in further work.

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