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Consistent Calibration of CDO tranches with the Generalized-Poisson Loss Dynamical model

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See also the paper
<http://www.damianobrigo.it/gpl.pdf>

Joint Work with Andrea Pallavicini, Roberto Torresetti

Talk outline

- Information content in CDO market quotes;
- Expected loss, expected tranche loss and expected number of defaults;
- Bottom up and Top down approaches to Loss modeling;
- The Generalized Poisson Loss model
- Extension
- Calibration examples with the basic model
- Pricing and Further Research

Index CDO's (iTraxx, CDX...)

Given a pool of names $1, 2, \dots, M$, typically $M = 125$, each with initial notional $1/M$, the index default leg pays to the protection buyer the loss increment occurring each time one or more names default, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

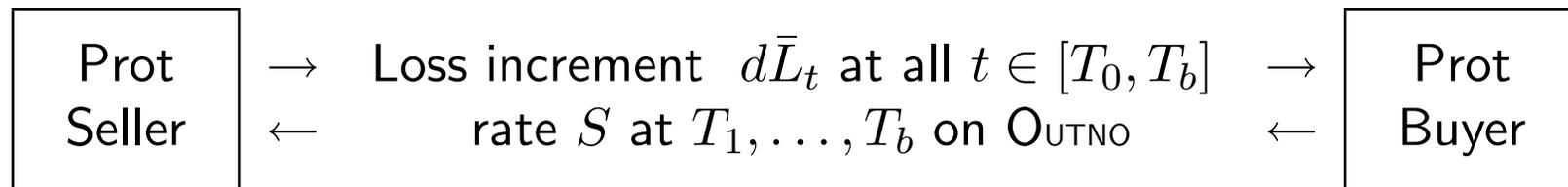
We denote with \bar{L}_t the portfolio cumulated loss and with \bar{C}_t the number of defaulted names up to time t , re-scaled by M (and thus in the interval $[0, 1]$).

Therefore

$$\bar{C}_t = \frac{\text{Number of Defaults by } t}{M}, \quad 0 \leq \bar{L}_t \leq \bar{C}_t \leq 1$$

Index CDO's (iTraxx, CDX...)

In exchange a periodic premium rate (or “spread”) S is paid from the protection buyer to the protection seller, until final maturity T_b . This is computed on a notional (O_{UTNO}) that decreases each time a name in the pool defaults, and decreases of an amount corresponding to the notional of that name, irrespective of the recovery. $O_{\text{UTNO}}(t) = 1 - \bar{C}_t$

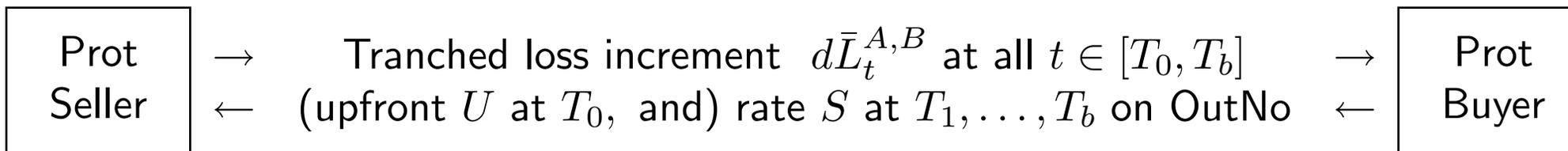


Index CDO's tranches

Synthetic CDO tranches with maturity T are obtained by “tranching” the loss $\bar{L}(t)$. The tranching loss at points A and B in $[0, 1]$ is

$$\bar{L}_t^{A,B} := \frac{1}{B - A} \left[(\bar{L}_t - A) 1_{\{A < \bar{L}_t \leq B\}} + (B - A) 1_{\{\bar{L}_t > B\}} \right].$$

The contract has two legs, the **default leg** and the **premium leg**.



$$\text{OUTNO}(t; A, B) = 1 - \bar{L}_t^{A,B}.$$

Information contained in CDO quotes

Recall the market quoted fair spreads for indices and tranches:

$$S_0 = \frac{\mathbb{E}_0 \left[\int_0^T D(0, u) d\bar{L}_u \right]}{\mathbb{E}_0 \left[\sum_{i=1}^b \delta_i D(0, T_i) (1 - \bar{C}_{T_i}) \right]}$$

$$S_0^{A,B} = \frac{\mathbb{E}_0 \left[\int_0^T D(0, u) d\bar{L}_u^{A,B} \right] - U_0^{A,B}}{\mathbb{E}_0 \left[\sum_{i=1}^b \delta_i D(0, T_i) (1 - \bar{L}_{T_i}^{A,B}) \right]}$$

where $\bar{L}_{T_i}^{A,B}$ is the tranching loss at points A, B divided by the tranche thickness $B - A$. If S_0 and $S_0^{A,B}$ are the only data on default correlation in the market, **we see that the only information are “expected losses”, “expected tranche losses” and “expected number of defaults”.**

Loss models: The “BOTTOM UP” and “TOP DOWN” approaches

Index and tranches contain information only on expected losses, expected tranche losses and expected number of defaults.

Modeling loss and default number? 2 approaches: BOTTOM UP and TOP DOWN.

BOTTOM UP: Model single defaults, correlate them and build the loss from these through recovery assumptions on single names.

TOP DOWN: Model the loss and number of defaults directly as the fundamental objects, and possibly achieve consistency with single names a posteriori.

Loss models: The “BOTTOM UP” approach

BOTTOM UP. In typical reduced form models, transforming the default time τ by its (strictly increasing) cumulated intensity Λ leads to :

$$\Lambda(\tau) = \xi \sim \text{exponential, independent of FX, Interest rates, etc..}$$

If we have names $1, 2, \dots, M$ we may induce “correlation” among the defaults

$$\tau_1 = \Lambda_1^{-1}(\xi_1), \dots, \tau_n = \Lambda_n^{-1}(\xi_n)$$

by **putting dependence among the different ξ through a copula.** If one adds recoveries REC_j , one builds the pool loss from single name losses:

$$\bar{L}_t = \frac{1}{M} \sum_{j=1}^M (1 - \text{REC}_j) 1_{\{\tau_j \leq t\}}, \quad \bar{C}_t = \frac{1}{M} \sum_{j=1}^M 1_{\{\tau_j \leq t\}}$$

Loss models: The “BOTTOM UP” approach

A particular case, with a Gaussian copula collapsing $125 \times 124/2 = 7750$ parameters into 1 is the market “implied correlation” approach.

- **BOTTOM UP:** Easy consistency with single names;
- allows for pricing of CDO squared and other credit payoffs depending on more than the loss of the basic pool; **BUT...**
- The dependence (copula) among single defaults is partly arbitrary;
- Consistent calibration across attachments and maturities is difficult, practically impossible;
- Very difficult to make these models (based on the static notion of copula function) dynamic in order to price forward start CDO or tranche options.

Loss models: The “TOP DOWN” approach

TOP DOWN APPROACH: Model loss-related quantities directly rather than patching single defaults models through a copula.

- a “Market Model” appeal: Focuses on more direct market objects, avoiding arbitrary assumptions on single name default dependencies;
- Possibility to have an authentically dynamic model;
- Calibrate indices and tranches consistently across attachments/maturities;
- Possibility to infer synthetic recovery information on a pool; **BUT...**
- How do losses of different pools “talk” to each other? (CDO squared);
- Consistency with single names: Random Thinning?

Top Down Approach: The GPL Model

The basic Generalized Poisson Loss (GPL) model is an example of the top down approach and can be formulated as follows.

Consider a number n of independent Poisson processes N_1, \dots, N_n with intensities $\lambda_1, \dots, \lambda_n$. Define the stochastic process

$$Z_t = \sum_{j=1}^n \alpha_j N_j(t),$$

for increasing integers $\alpha_1, \dots, \alpha_n$, and model the number of defaults as Z_t .

Top Down Approach: The GPL Model

Example : $M = 125, Z_t = 1 N_1(t) + 2 N_2(t) + \dots + 125 N_{125}(t).$

If N_1 jumps there has been just one default (idiosyncratic default), if N_{125} jumps there are 125 defaults and the whole pool defaults one shot (systemic risk), otherwise for other N_i 's we have intermediate situations.

Some N 's may have zero intensity, which is equivalent to say that the corresponding multiplier is set to zero.

This model explicitly contemplates the possibility of multiple defaults in small time intervals, contrary for example to Schönbucher (2005) and Errais, Giesecke and Goldberg (2006).

Top Down Approach: The GPL Model

A drawback of the model is that the number of defaults in time may increase without limit. If our pool contains M names, we may then consider

$$C_t := \min(Z_t, M) = Z_t 1_{\{Z_t < M\}} + M 1_{\{Z_t \geq M\}}$$

as actual number of defaults. If Z has a known distribution, the distribution of C_t can be easily derived as a byproduct:

$$\mathbb{Q}(C_t \leq x) = 1_{\{x < M\}} \mathbb{Q}(Z_t < x) + 1_{\{x \geq M\}}$$

The GPL Model

The law of Z_t (and thus of C_t) is directly known through its characteristic function. We have easily, thanks to independence of N_i 's,

$$\varphi_{Z_t}(u) = \prod_{j=1}^n \mathbb{E}_0[\exp(-iu\alpha_j N_j(t))] = \prod_{j=1}^n \varphi_{N_j(t)}(\alpha_j u),$$

where $\varphi_{N_j(t)}$ is the characteristic function of the Poisson process N_j . Since we know the Poisson char function, we obtain easily

$$\varphi_{Z_t}(u) = \prod_{j=1}^n \exp \left[\left(\int_0^t \lambda_j(v) dv \right) (e^{i\alpha_j u} - 1) \right] = \exp \left[\sum_{j=1}^n \Lambda_j(t) (e^{i\alpha_j u} - 1) \right]$$

The density of Z_t can be obtained as the inverse Fourier transform

Default intensity

An important feature of loss models is to link default intensities jumps to loss dynamics, so that the default intensity decreases, as long as loss increases, and it is equal to zero when the whole portfolio has defaulted.

Let us consider the compensator A_t of the default-counting point process C_t , namely the nondecreasing predictable process that added to a local martingale gives C_t itself (Doob-Meyer decomposition), satisfying

$$\mathbb{E}_t[C_T - A_T] = C_t - A_t, \quad \mathbb{E}_t[dC_t] = dA_t = h_C(t)dt,$$

If A_t is absolutely continuous its density h is known as the intensity of the process C_t .

Default intensity

The compensator A_t of our default-counting C_t can be computed as

$$A_T = \sum_{j=1}^n \int_0^T \min(\alpha_j, M - Z_s) 1_{\{Z_s < M\}} \lambda_j(s) ds,$$

leading to an intensity h_C for the default-counting process C_t given by

$$h_C(t) = \sum_{j=1}^n \min(\alpha_j, (M - Z_t)^+) \lambda_j(t).$$

The default intensity h_C in the basic GPL model is a stochastic object only through Z_t . It is possible to extend the GPL model by considering the intensities λ_j as stochastic processes, e.g. following a Gamma or CIR process. The default intensity h_C acquires a new source of stochasticity.

The Gamma GPL Model

One interesting extension is the Gamma-intensity Generalized Poisson (GGPL) model.

$$Z_t^G = \sum_{j=1}^n \alpha_j N_j^G(t),$$

where now $N_j^G(t)$ are Cox processes (i.e. Poisson processes with stochastic intensity) whose random cumulated intensities are distributed at any time T as

$$\int_0^T \lambda_j(t) dt =: \Lambda_j(T) \sim \Gamma(k_j(T), \theta_j)$$

where $k > 0$ is the shape parameter and $\theta > 0$ is the scale parameter in the Gamma distribution. We take different $\Lambda_j(T)$ to be independent as j changes.

The Gamma GPL Model

We still compute the characteristic function in closed form as

$$\begin{aligned}\varphi_{Z_T}^G(u) &= \mathbb{E}_0 \left[\mathbb{E}_0 \left[\exp(iuZ_T^G) \mid \Lambda_1(T), \dots, \Lambda_n(T) \right] \right] = \\ &= \prod_{j=1}^n \left[(1 + \theta_j (1 - e^{i\alpha_j u})) \right]^{-k_j(T)}\end{aligned}$$

The Gamma distribution assumption $\Lambda_j(t) \sim \Gamma(k_j(t), \theta_j)$ at every time is consistent with a gamma process assumption for Λ_j , for more details and a piecewise Gamma extension allowing for tractability and a term structure in the parameter θ see Brigo, Pallavicini and Torresetti (2006), where we further discuss the case of stochastic scenario intensities.

The CIR-GPL Model

A different and possibly more interesting extension is the CIR-Generalized Poisson (CIR-GPL) model

$$Z_t^{\text{CIR}} = \sum_{j=1}^n \alpha_j N_j^{\text{CIR}}(t), \quad d\lambda_j = k_j(\theta_j - \lambda_j)dt + \sigma_j \sqrt{\lambda_j} dW_j,$$

with $2k_j\theta_j > \sigma_j^2$, and where the intensities of multiple defaults with different sizes follow different independent CIR processes.

The characteristic function of Z can be computed again in closed form, the calculation being quite similar to the bond price formula for the CIR interest rate model. Alternatively, **jump diffusion** JCIR intensities can be considered, maintaining tractability.

Extended GPL Models: Spread Dynamics

In general the stochastic intensity may help us to model **volatility** when considering for example **forward start CDO's or tranche options**.

For example, consider the index future spread at t :

$$S_t = \frac{\int_t^T D(t, u) \mathbb{E}_t[h_L(u)] du}{\sum_{i=1}^n 1_{\{T_i > t\}} \delta_i D(t, T_i) (1 - \bar{C}_t - \int_t^{T_i} \mathbb{E}_t[h_{\bar{C}}(u)] du)}$$

and recall the compensator,

$$h_C(t) = \sum_{j=1}^n \min(\alpha_j, (M - Z_t)^+) \lambda_j(t)$$

and similarly for h_L . CIR intensities λ_j for the spreads may enrich the dynamics of the index spread.

Recovery assumptions

In order to ensure an arbitrage-free dynamics, the portfolio cumulated loss (\bar{L}_t) and the re-scaled number of defaults (\bar{C}_t) must be non-decreasing processes taking values in the $[0, 1]$ interval, the former with increments always smaller or equal than the increment of the latter.

$$d\bar{L}_t \leq d\bar{C}_t.$$

The portfolio cumulated loss and the number of defaults cannot be independently modelled, since they are coupled by the forward realization of the recovery rate (R_t) at default dates

$$d\bar{L}_t = [1 - R_t]d\bar{C}_t.$$

As a first approach we choose to introduce a constant recovery rate $R = 40\%$

Recovery assumptions

The recovery rate can be expressed also in terms of the intensities of the loss and default rate processes.

If we assume R_t to be adapted (known at time t) then

$$d\bar{L}_t = [1 - R_t]d\bar{C}_t \Rightarrow \mathbb{E}_t [d\bar{L}_t] = [1 - R_t]\mathbb{E}_t [d\bar{C}_t]$$

from which

$$R_t = 1 - \frac{h_L(t)}{h_C(t)}$$

Thus, the recovery rate at default is directly related to the intensities of both the loss and the default rate processes. The choice for the intensity dynamics does induce a dynamics for the recovery rate.

Calibration

The GPL model is calibrated to the market quotes observed on March 1 and 6, 2006. Deterministic discount rates are listed in Brigo, Pallavicini and Torresetti (2006). Tranche data and DJi-TRAXX fixings, along with bid-ask spreads, are

	Att-Det	March, 1 2006		March, 6 2006		
		5y	7y	3y	5y	7y
Index		35(1)	48(1)	20(1)	35(1)	48(1)
Tranche	0-3	2600(50)	4788(50)	500(20)	2655(25)	4825(25)
	3-6	71.00(2.00)	210.00(5.00)	7.50(2.50)	67.50(1.00)	225.50(2.50)
	6-9	22.00(2.00)	49.00(2.00)	1.25(0.75)	22.00(1.00)	51.00(1.00)
	9-12	10.00(2.00)	29.00(2.00)	0.50(0.25)	10.50(1.00)	28.50(1.00)
	12-22	4.25(1.00)	11.00(1.00)	0.15(0.05)	4.50(0.50)	10.25(0.50)
Tranchlet	0-1	6100(200)	7400(300)			
	1-2	1085(70)	5025(300)			
	2-3	393(45)	850(60)			

Calibration

The cumulated intensities $\Lambda_i(T)$ are real non-decreasing piecewise linear functions in the tranche maturity.

The optimal values for the amplitudes α are selected as follows:

1. set $\alpha_1 = 1$ and all other α 's to zero. Calibrate Λ_1 ;
2. find the best integer value for α_2 by calibrating the cumulated intensities Λ_1 and Λ_2 for each value of α_2 in the range $[1, 125]$, starting from the previous Λ_1 as a guess;
3. repeat the previous step for α_i with $i = 3$ and so on, by calibrating the cumulated intensities $\Lambda_1, \dots, \Lambda_i$, starting from the previously found $\Lambda_1, \dots, \Lambda_{i-1}$ as initial guess, until the calibration error is under a pre-fixed threshold or until the intensity Λ_i can be considered negligible.

Calibration

The objective function f to be minimized in the calibration is the squared sum of the errors shown by the model to recover the tranche and index market quotes weighted by market bid-ask spreads:

$$f(\alpha, \Lambda) = \sum_i \epsilon_i^2, \quad \epsilon_i = \frac{x_i(\alpha, \Lambda) - x_i^{\text{Mid}}}{x_i^{\text{Bid}} - x_i^{\text{Ask}}}$$

where the x_i , with i running over the market quote set, are the index values S_0 for DJi-TRAXX index quotes, and either the index periodic premium rates $S_0^{A,B}$ or the upfront premium rates $U_0^{A,B}$ for the DJi-TRAXX tranche quotes.

Calibration: All standard tranches up to seven years

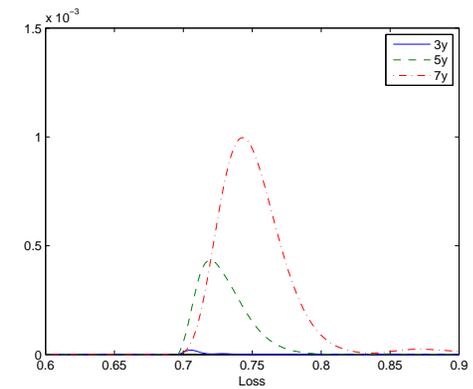
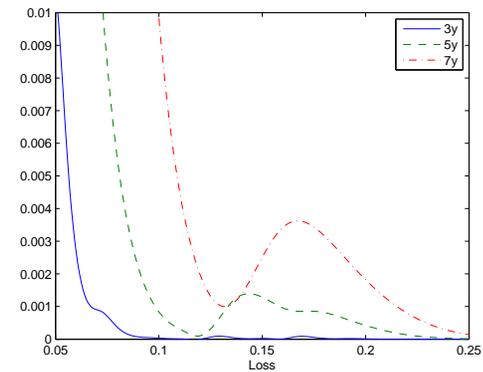
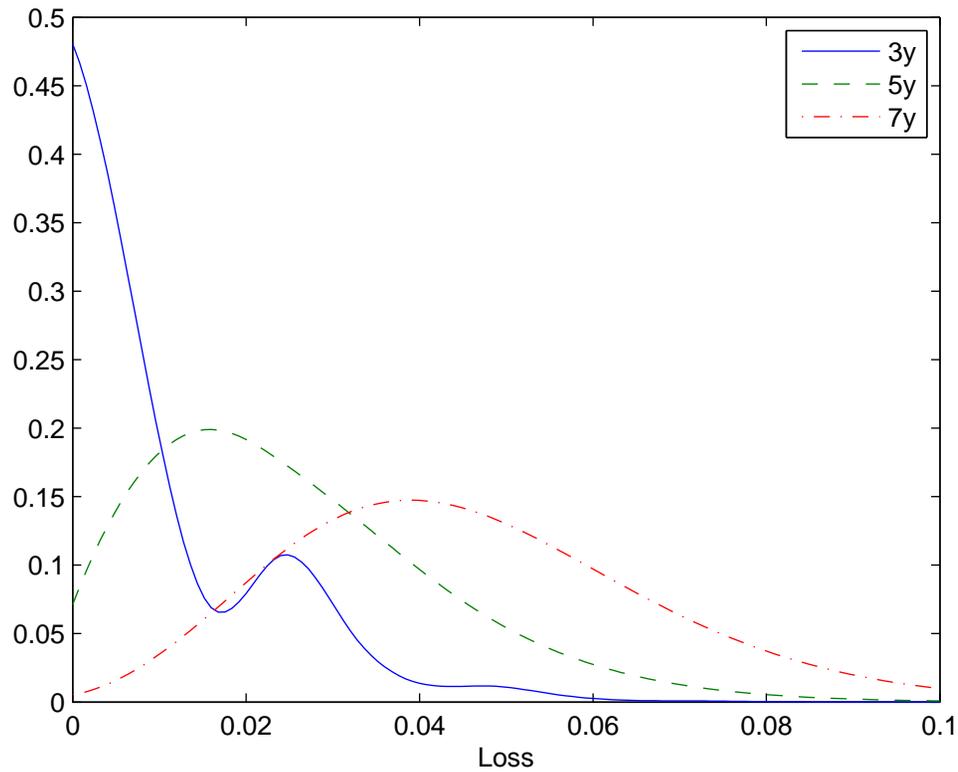
As a first calibration example we consider standard DJi-TRAXX tranches up to a maturity of 7y with constant recovery rate of 40%.

The calibration procedure selects five Poisson processes. The 18 market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread (we show the ratio calibration error / bid ask spread).

	Att-Det	Maturities		
		3y	5y	7y
Index		-0.4	-0.2	-0.9
Tranche	0-3	0.1	0.0	-0.7
	3-6	0.0	0.0	0.7
	6-9	0.0	0.0	-0.2
	9-12	0.0	0.0	0.0
	12-22	0.0	0.0	0.2

α	$\Lambda(T)$		
	3y	5y	7y
1	0.535	2.366	4.930
3	0.197	0.266	0.267
16	0.000	0.007	0.024
21	0.000	0.003	0.003
88	0.000	0.002	0.007

Calibration: All standard tranches up to seven years



Calibration: All standard tranches up to seven years

One possible comparison of our implied loss distribution according to the GPL model is with the implied loss distribution according to Hull and White's (2005) (STATIC!) "perfect copula" approach.

If we compare the implied loss distribution resulting from the calibration of the five year index and tranche quotes with the perfect copula approach as reformulated in Torresetti et al. (2006), we find a qualitative pattern similar to the pattern we have above.

Calibration: All standard tranches up to seven years

Notice in particular the large portion of mass concentrated near the origin, the subsequent modes when moving along the loss distribution for increasing values, and the bumps in the far tail.

These features are common to both approaches. In our GPL models the bumps in the tails of the loss distributions, which seem to be necessary in order to be able to recover the market quotes, are obtained thanks to the multiple jumps components contributing to the loss distribution. In particular, the components with higher α 's are giving rise to the little bumps in the far tail of the loss distribution and help with senior tranches.

Calibration: Tranchlets

The market quotes also non-standard tranches, which are quoted over the counter. An interesting case is given by the so called “tranchlets”, namely DJi-TRAXX tranches with attachment and detachment points possibly smaller than 3%. On the first of march 2006 we obtain market quotes for a set of tranchlets with maturity of five and seven years (see earlier table).

We calibrate the market data with constant recovery rate of 40%. The calibration procedure selects five Poisson processes. The 18 market quotes used by the calibration procedure are recovered, but within an error that is occasionally larger than the bid-ask spread.

Calibration: Tranchlets

	Att-Det	Maturities	
		5y	7y
Index		-0.8	-2.1
Tranchlet	0-1	1.1	-1.4
	1-2	1.7	-0.6
	2-3	-0.1	-0.4
Tranche	0-3	0.1	0.4
	3-6	-1.9	0.2
	6-9	0.4	0.6
	9-12	2.8	0.9
	12-22	-0.4	-1.5

α	$\Lambda(T)$	
	5y	7y
1	0.834	3.336
2	1.070	1.070
13	0.008	0.015
21	0.004	0.013
104	0.002	0.007

Table 1: Left side: calibration error with respect to the bid-ask spread for tranches quoted by the market. Right side: cumulated intensities of the basic GPL model. Each row corresponds to a different Poisson component with jump amplitude α . Recovery rate is 40%.

Pricing and Further research

Pricing products based on the loss distribution such as tranche options, forward start tranches etc with the calibrated model is simple, given knowledge of the marginal and transition distributions for the constituent Poisson processes.

Indeed, if we have a payoff or additive portion of a payoff depending on the loss at one maturity, we simply sample one-shot the independent Poissons N_j at maturity, add them up using the related multiplicity coefficients α , plug the resulting loss in the payoff portion and average over scenarios.

This is maintained also under random (Gamma or scenario) intensities. Alternatively, we may decide to use the inverse Fourier transform of the known characteristic function of the terminal distribution to obtain the loss density and then integrate numerically the payoff against this density.

Pricing and Further research

If a payoff is path dependent on the loss we still may simulate the independent increments of the independent constituent Poisson processes N_j among the relevant instants.

Given independence this can be realized by sampling known independent Poisson laws. Once this has been done, we obtain the constituent processes at every relevant time by adding up their increments, and then we obtain the loss at any time by simply adding the constituent processes times their multiplicity coefficients α .

Then we plug each temporal path of the loss distribution in the payoff and average over scenarios. This procedure is substantially maintained also under possible random (Gamma or scenario) intensities. Simulation is thus easy and based on the ability to sample from a Poisson law.

Pricing and Further research

Further research concerns the possible improvement of the calibration when considering the extended versions of the model.

Also, we plan to analyze the pricing and sensitivities of correlation products with the calibrated model.

A more articulated recovery dynamics and the implications on loss and default counting process intensities are under investigation.

The *future* loss distribution coming from the calibration is to be analyzed

Finally, for products such as CDO squared, we plan to investigate random thinning as a viable technique to “zoom” on single name defaults.

Alternatively, Elouerkhaoui (2006) finds that a model similar to ours is consistent with a bottom up model with Marshall Olkin copula across single names. This also could be use to zoom on single defaults.

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