M3/M4S3 STATISTICAL THEORY II
INTEGRAL WITH RESPECT TO MEASURE :
KEY THEOREMS

The following key theorems describe the behaviour of the Lebesgue-Stieltjes integral. In particular, the theorems specify when it is legitimate to exchange the order of limit and integral operators. In the theorems, we have a general measure space $(\Omega, \mathcal{F}, \nu)$, and measurable set $E \in \mathcal{F}$.

Theorem 1 ***Lebesgue Monotone Convergence Theorem***
If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions, and if

$$\lim_{n\to\infty} f_n = f \qquad \textit{almost everywhere}$$

then

$$\lim_{n\to\infty} \int_E f_n d\nu = \int_E f d\nu$$

Proof. Let the (real) sequence $\{i_n\}$ be defined by

$$i_n = \int_E f_n d\nu.$$

Then, by a previous result

$$i_n = \int_E f_n d\nu \leq \int_E f_{n+1} d\nu = i_{n+1} \qquad \text{as } f_n \leq f_{n+1}$$

so $\{i_n\}$ is increasing. Let L denote the (possibly infinite) limit of $\{i_n\}$. Now, since $f_n \leq f$ almost everywhere for all n, we have (by the same previous result) that

$$\int_E f_n d\nu \leq \int_E f d\nu \Longrightarrow L \leq \int_E f d\nu. \tag{1}$$

Now consider constant c with $0 < c < 1$, and let ψ be any simple function satisfying $0 \leq \psi \leq f$. Let

$$E_n \equiv \{\omega : \omega \in E \text{ and } c\psi(\omega) \leq f_n(\omega)\}$$

and as $E_n \subseteq E$, E_n is measurable, and because $f_n \leq f_{n+1}$, $E_n \subseteq E_{n+1}$ for all n, so $\{E_n\}$ is increasing. Let the limit of the $\{E_n\}$ sequence be denoted

$$F = \bigcup_{i=1}^{\infty} E_n.$$

The set $E \cap F'$ has measure zero, because $\lim\limits_{n\to\infty} f_n = f$ a.e. and $0 \leq c\psi \leq \psi \leq f$. Hence, as $E_n \subseteq E$

$$\int_E f_n d\nu \geq \int_{E_n} f_n d\nu \geq \int_{E_n} c\psi d\nu = c \int_{E_n} \psi d\nu.$$

Taking the limit as $n \to \infty$,

$$L = \lim_{n\to\infty} \int_E f_n d\nu \geq c \lim_{n\to\infty} \int_{E_n} \psi d\nu = c \int_F \psi d\nu = c \int_E \psi d\nu$$

the final step following as $E \cap F'$ has measure zero. Thus, as this holds for all c such that $0 < c < 1$, we must have that

$$L \geq \int_E \psi d\nu$$

whenever $0 \leq \psi \leq f$. Hence L is an upper bound the integral of such a simple function on E. But, by the supremum definition from lectures, the integral of f with respect to ν on E is the **least** upper bound on the integral of such simple functions on E. Hence

$$L \geq \int_E f d\nu. \tag{2}$$

Thus, combining (1) and (2), we have that

$$L = \lim_{n\to\infty} \int_E f_n d\nu = \int_E f d\nu.$$

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Theorem 2 ***Fatou's Lemma*** *(or* ***Lebesgue-Fatou Theorem)***
If $\{f_n\}$ is a sequence of non-negative measurable functions, and if

$$\liminf_{n\to\infty} f_n = f \qquad \textit{almost everywhere}$$

then

$$\int_E f d\nu \leq \liminf_{n\to\infty} \left\{ \int_E f_n d\nu \right\}$$

Proof. The function $\liminf_{n\to\infty} f_n$ is measurable (by the measure theory handout result). For $k = 1, 2, 3, \ldots$ let

$$h_k = \inf \{f_n : n \geq k\}.$$

Then, by definition of infimum, $h_k \leq f_k$ for all k, and thus

$$\int_E h_k d\nu \leq \int_E f_k d\nu \quad \text{for all } k \qquad \Longrightarrow \qquad \liminf_{k\to\infty} \left\{ \int_E h_k d\nu \right\} \leq \liminf_{k\to\infty} \left\{ \int_E f_k d\nu \right\}. \tag{3}$$

Now $\{h_k\}$ is an increasing sequence of non-negative functions, we have in the limit

$$\lim_{k\to\infty} h_k = \liminf_{n\to\infty} f_n = f$$

almost everywhere. Now, by the Monotone Convergence Theorem,

$$\lim_{k\to\infty} \left\{ \int_E h_k d\nu \right\} = \int_E \left\{ \lim_{k\to\infty} h_k \right\} d\nu = \int_E f d\nu$$

Hence, by (3),

$$\int_E f d\nu \leq \liminf_{k\to\infty} \left\{ \int_E f_k d\nu \right\}.$$

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Some corollaries follow immediately from this important theorem

1. If $E_1, E_2, ..., E_n$ are disjoint, with $\bigcup_{i=1}^{n} E_i \equiv E$, and f is non-negative, then

$$\int_E f d\nu = \sum_{i=1}^{n} \left\{ \int_{E_i} f d\nu \right\}$$

Proof: Let $\{\psi_k\}$ be an increasing sequence of simple functions that converge to f, where

$$\psi_k = \sum_{j=1}^{m_k} a_{kj} I_{A_{kj}}$$

say. Then,

$$\begin{aligned} \int_E \psi_k d\nu &= \sum_{j=1}^{m_k} a_{kj} \nu (E \cap A_{kj}) = \sum_{j=1}^{m_k} \sum_{i=1}^{n} a_{kj} \nu (E_i \cap A_{kj}) \qquad \text{as the } E_i \text{ are disjoint} \\ &= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m_k} a_{kj} \nu (E_i \cap A_{kj}) \right\} = \sum_{i=1}^{n} \left\{ \int_{E_i} \psi_k d\nu \right\} \end{aligned}$$

by hence the monotone convergence theorem,

$$\begin{aligned} \int_E f d\nu &= \lim_{k\to\infty} \left\{ \int_E \psi_k d\nu \right\} = \lim_{k\to\infty} \left\{ \sum_{i=1}^{n} \left\{ \int_{E_i} \psi_k d\nu \right\} \right\} = \sum_{i=1}^{n} \left\{ \lim_{k\to\infty} \left\{ \int_{E_i} \psi_k d\nu \right\} \right\} \\ &= \sum_{i=1}^{n} \left\{ \int_{E_i} \left\{ \lim_{k\to\infty} \psi_k \right\} d\nu \right\} = \sum_{i=1}^{n} \left\{ \int_{E_i} f d\nu \right\}. \end{aligned}$$

2. Now consider a **countable** (rather than merely finite) collection $\{E_i\}$ with $\bigcup_{i=1}^{\infty} E_i \equiv E$. Then if f is non-negative

$$\int_E f d\nu = \sum_{i=1}^{\infty} \left\{ \int_{E_i} f d\nu \right\}$$

Proof: For each positive integer n, let $A_n \equiv \bigcup_{i=1}^{n} E_i$, and define $f_n = I_{A_n} f$. Then $\{f_n\}$ is an increasing sequence of non-negative functions, that converges to f (on E). Hence

$$\int_E f d\nu = \lim_{n\to\infty} \left\{ \int_E f_n d\nu \right\} = \lim_{n\to\infty} \left\{ \int_{A_n} f d\nu \right\} = \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \left\{ \int_{E_i} f d\nu \right\} \right\} = \sum_{i=1}^{\infty} \left\{ \int_{E_i} f d\nu \right\}$$

3. Let f be a non-negative function on Ω. Then the function defined on $\mathcal{F}$ by

$$\varphi(E) = \int_E f d\nu$$

is a measure. The only part of the definition of a measure that needs verifying is the countable additivity, by the last result, we have directly that

$$\varphi \left(\bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \varphi(E_i)$$

when the $\{E_i\}$ are disjoint.

For the results above (and the results proved in lectures), we have considered only the integrals of non-negative measurable functions. We now extend them for general measurable functions, using the decomposition into positive and negative part functions $f = f^+ - f^-$ where both f^+ and f^- are measurable and non-negative, and we have

$$\int_E f d\nu = \int_E f^+ d\nu - \int_E f^- d\nu.$$

Recall that we say that f is integrable if both f^+ and f^- are integrable, and now denote the set of all functions integrable on E with respect to ν by $\mathcal{L}_E(\nu)$. From previous arguments we have that

$$f \in \mathcal{L}_E(\nu) \Leftrightarrow f^+ \text{ and } f^- \in \mathcal{L}_E(\nu)$$

Some results can be proved for the functions in this class.

Lemma 1 If $\nu(E) = 0$, then

$$f \in \mathcal{L}_E(\nu) \qquad \text{and} \qquad \int_E f d\nu = 0$$

Proof. We have by definition

$$\int_E f d\nu = \int_E f^+ d\nu - \int_E f^- d\nu = 0 - 0 = 0$$

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Lemma 2 If $f \in \mathcal{L}_{E_2}(\nu)$ and $E_1 \subset E_2$, then $f \in \mathcal{L}_{E_1}(\nu)$.

Proof. By a result from lectures

$$\int_{E_1} f^+ d\nu \le \int_{E_2} f^+ d\nu \qquad \text{and} \qquad \int_{E_1} f^- d\nu \le \int_{E_2} f^- d\nu$$

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Lemma 3 If $\{E_n\}$ is a sequence of disjoint sets with $\bigcup_{n=1}^{\infty} E_n \equiv E$, and $f \in \mathcal{L}_E(\nu)$, then

$$\int_E f d\nu = \sum_{n=1}^{\infty} \left\{ \int_{E_n} f d\nu \right\}$$

Proof. The previous Lemma ensures that $f \in \mathcal{L}_{E_n}(\nu)$ as $E_n \subset E$ for all n. By using the result proved earlier, that if f is non-negative then

$$\int_E f d\nu = \sum_{n=1}^{\infty} \left\{ \int_{E_n} f d\nu \right\}$$

we use the positive and negative part decompositions

$$\begin{aligned}
\int_E f d\nu &= \int_E f^+ d\nu - \int_E f^- d\nu = \sum_{n=1}^{\infty} \left\{ \int_{E_n} f^+ d\nu \right\} - \sum_{n=1}^{\infty} \left\{ \int_{E_n} f^+ d\nu \right\} \\
&= \sum_{n=1}^{\infty} \left[\int_{E_n} f^+ d\nu - \int_{E_n} f^- d\nu \right] \\
&= \sum_{n=1}^{\infty} \left\{ \int_{E_n} \left(f^+ - f^-\right) d\nu \right\} = \sum_{n=1}^{\infty} \left\{ \int_{E_n} f d\nu \right\}
\end{aligned}$$

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Corollary. If $f \in \mathcal{L}_{\Omega}(\nu)$, then the function φ defined on $\mathcal{F}$ by

$$\varphi(E) = \int_E f d\nu$$

is additive.

Proof. As for previous result. ■

Lemma 4 If $f = g$ a.e. on E, and if $g \in \mathcal{L}_E(\nu)$, then $f \in \mathcal{L}_E(\nu)$ and

$$\int_E f d\nu = \int_E g d\nu$$

Proof. Define $A \equiv \{\omega : \omega \in E, f(\omega) = g(\omega)\}$. Then $E \cap A'$ has measure zero, and

$$\int_E f^+ d\nu = \int_A f^+ d\nu = \int_A g^+ d\nu = \int_E g^+ d\nu$$

and

$$\int_E f^- d\nu = \int_A f^- d\nu = \int_A g^- d\nu = \int_E g^- d\nu$$

Adding these equations, we have immediately that $f \in \mathcal{L}_E(\nu)$ and

$$\int_E f d\nu = \int_E g d\nu$$

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Lemma 5 If $f \in \mathcal{L}_E(\nu)$ and c is any real number, then $cf \in \mathcal{L}_E(\nu)$ and

$$\int_E (cf)\, d\nu = c \int_E f d\nu$$

Proof. Consider only the non-trivial case $c \neq 0$. Suppose first $c > 0$, and let g be a non-negative function. For any simple function ψ, say

$$\psi = \sum_{i=1}^{k} a_i I_{A_i}$$

we have

$$\psi \leq g \Leftrightarrow c\psi \leq cg.$$

and

$$\int_E (c\psi)\, d\nu = \sum_{i=1}^{k} (ca_i)\, \nu(E \cap A_i) = c \sum_{i=1}^{k} a_i \nu(E \cap A_i) = c \int_E \psi d\nu$$

Therefore

$$\int_E (cf)\, d\nu = c \int_E f d\nu$$

by the supremum definition, and the result follows for $c > 0$ using this result, and the decomposition $cf = cf^+ - cf^-$. For $c < 0$, write

$$cf = (-c) f^- - (-c) f^+$$

so that the result follows, as $-c > 0$. ■

Lemma 6 If $f, g \in \mathcal{L}_E(\nu)$, then $f + g \in \mathcal{L}_E(\nu)$ and

$$\int_E (f+g)\, d\nu = \int_E f d\nu + \int_E g d\nu$$

Proof. We prove the result two several stages. First suppose that f and g are non-negative, and let $\left\{\psi_n^{(f)}\right\}$ and $\left\{\psi_n^{(g)}\right\}$ be increasing sequences of simple functions with limits f and g respectively. Then $\left\{\psi_n^{(f)} + \psi_n^{(g)}\right\}$ has limit $f + g$, and as

$$\int_E \left(\psi_n^{(f)} + \psi_n^{(g)}\right) d\nu = \int_E \psi_n^{(f)} d\nu + \int_E \psi_n^{(f)} d\nu$$

(see this result by using the measure definition of the integral of a simple function), we have, taking the limit as $n \to \infty$,

$$\int_E (f+g)\, d\nu = \int_E f d\nu + \int_E g d\nu.$$

Now consider the general case; define the following subsets of E

$$\begin{aligned}
E_1 &\equiv \{\omega : f(\omega) \geq 0, g(\omega) \geq 0\} \\
E_2 &\equiv \{\omega : f(\omega) < 0, g(\omega) \geq 0\} \\
E_3 &\equiv \{\omega : f(\omega) \geq 0, g(\omega) < 0, (f+g)(\omega) \geq 0\} \\
E_4 &\equiv \{\omega : f(\omega) < 0, g(\omega) \geq 0, (f+g)(\omega) \geq 0\} \\
E_5 &\equiv \{\omega : f(\omega) \geq 0, g(\omega) < 0, (f+g)(\omega) < 0\} \\
E_6 &\equiv \{\omega : f(\omega) < 0, g(\omega) \geq 0, (f+g)(\omega) \geq 0\}
\end{aligned}$$

Then $E_n, n = 1, 2, ..., 6$ are disjoint, and $\bigcup_{n=1}^{6} E_n \equiv E$. By the Lemma 3, proving that

$$\int_{E_n} (f+g)\, d\nu = \int_{E_n} f d\nu + \int_{E_n} g d\nu$$

for each n is sufficient to prove the result. The proofs for each separate case are very similar; so consider for example set E_3. Then on E, the functions $f, -g$ and $f + g$ are non-negative, and threfore by part one of this proof,

$$\int_{E_3} f d\nu = \int_{E_3} (-g)\, d\nu + \int_{E_3} (f+g)\, d\nu = -\int_{E_3} g d\nu + \int_{E_3} (f+g)\, d\nu$$

and the result follows. ∎

Lemma 7 The function $f \in \mathcal{L}_E(\nu)$ if and only if $|f| \in \mathcal{L}_E(\nu)$. In this instance,

$$\left|\int_E f d\nu\right| \leq \int_E |f|\, d\nu.$$

Proof. We have identified previously that f is integrable if the positive and negative part functions are integrable, and this is the case if and only if the function

$$|f| = f^+ + f^-$$

is integrable. If this is the case, then

$$\left|\int_E f d\nu\right| = \left|\int_E f^+ - f^- d\nu\right| \leq \left|\int_E f^+ d\nu\right| + \left|\int_E f^- d\nu\right| = \int_E |f|\, d\nu$$

∎

Corollary. If $g \in \mathcal{L}_E(\nu)$, and $|f| \leq g$, then $f \in \mathcal{L}_E(\nu)$

Lemma 8 If $f, g \in \mathcal{L}_E(\nu)$, and $f \leq g$ a.e. on E, then

$$\int_E f d\nu \leq \int_E g d\nu$$

that is, the Lebesgue-Stieltjes Integral operator preserves ordering of functions.

Proof. We have $g - f \geq 0$, so the result follows from Integral Result (e) from lectures, and Lemma 6. ∎

Corollary. If $\upsilon(E) < \infty$, and $m \leq f \leq M$ on E, for real values m and M, then by considering simple functions $\psi_m = mI_E$ and $\psi_M = MI_E$, for which $\psi_m \leq f \leq \psi_M$, we have

$$m\upsilon(E) \leq \int_E f d\nu \leq M\upsilon(E)$$

Lemma 9 Suppose $f, g \in \mathcal{L}_E(\nu)$, and that for $A \subset E$,

$$\int_A f d\nu \leq \int_A g d\nu.$$

Then $f \leq g$ a.e. on E.

Proof. Let $F_1 \equiv \{\omega : \omega \in E, f(\omega) \geq g(\omega)\}$, so that $f - g \geq 0$ on F_1. Thus, by the assumption of the Lemma,

$$\int_F (f - g)\, d\nu = 0$$

and hence by $f - g = 0$ or $f = g$ a.e. on F_1, by Integral Result (f) from lectures. ∎

Corollary. If $f, g \in \mathcal{L}_E(\nu)$ and if

$$\int_A f d\nu = \int_A g d\nu.$$

for $A \subset E$, then $f = g$ a.e. on E.

Theorem 3 ***Lebesgue Dominated Convergence Theorem***
If $\{f_n\}$ is a sequence of measurable functions, and if

$$\lim_{n\to\infty} f_n = f \qquad \text{almost everywhere}$$

and $|f_n| \leq g$ for all n, for some $g \in \mathcal{L}_E(\nu)$, then

$$\lim_{n\to\infty} \int_E f_n d\nu = \int_E f d\nu$$

Proof. $\{f_n\}$ and f are measurable functions. By using Fatou's Lemma (Theorem 2) on non-negative sequence $\{g + f_n\}$

$$\int_E (g+f)\, d\nu \leq \liminf_{n\to\infty} \left\{ \int_E (g+f_n)\, d\nu \right\}$$

so that

$$\int_E f d\nu \leq \liminf_{n\to\infty} \left\{ \int_E f_n d\nu \right\}. \tag{4}$$

Similarly, by applying the result to $\{g - f_n\}$, we have that

$$\int_E (g-f)\, d\nu \leq \liminf_{n\to\infty} \left\{ \int_E (g-f_n)\, d\nu \right\} \qquad \therefore \qquad -\int_E f d\nu \leq \liminf_{n\to\infty} \left\{ -\int_E f_n d\nu \right\}$$

Multiplying through by -1, we have by properties of $\limsup$ and $\liminf$ that

$$\int_E f d\nu \geq \limsup_{n\to\infty} \left\{ \int_E f_n d\nu \right\} \tag{5}$$

and hence combining (4) and (5), we have by definition

$$\lim_{n\to\infty} \int_E f_n d\nu = \int_E f d\nu$$

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Corollary. If $\{f_n\}$ is a uniformly bounded sequence (bounded above and below by a pair of real constants) of measurable functions such that

$$\lim_{n\to\infty} f_n = f \qquad \text{almost everywhere}$$

and if $\upsilon(E) < \infty$, then

$$\lim_{n\to\infty} \int_E f_n d\nu = \int_E f d\nu.$$

LEBESGUE-STIELTJES INTEGRALS ON $\mathbb{R}$.

Rather than considering a general sample space Ω, we now consider the specific case when $\Omega \equiv \mathbb{R}$, with corresponding sigma-algebra which is the Borel sigma-algebra. In this case, the measure υ will often be expressed in terms of (or be generated by) an increasing **real** function F on E. Let E be a set in the Borel sigma-algebra. Then for measurable function g, we can express the integral as

$$\int_E g d\nu = \int_E g dF \qquad \text{or} \qquad \int_E g d\nu = \int_E g(x) dF(x)$$

with special cases

$$\int_a^b g\, dF = \int_{(a,b]} g\, dF \qquad \text{and} \qquad \int_{-\infty}^{\infty} g\, dF = \int_{\mathbb{R}} g\, dF$$