M3S3/M4S3
ASSESSED COURSEWORK 2

Deadline: Thursday 16th December 12.00pm
Please hand in to 523 or the General Office

Recall that, in general, the Wald and Rao/Score test statistics derived from a sample of size n, W_n and R_n, for testing

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned}$$

when the log density $\log f_X$ admits a finite second derivative with respect to θ are given by

$$W_n = n\left(\widetilde{\theta}_n - \theta_0\right)^T \widehat{I}_n\left(\widetilde{\theta}_n\right)\left(\widetilde{\theta}_n - \theta_0\right) \tag{1}$$

where $\widetilde{\theta}_n$ is a solution to the likelihood equations,

$$\widehat{I}_n\left(\widetilde{\theta}_n\right) = \begin{cases} I\left(\widetilde{\theta}_n\right) \\ \frac{1}{n}\sum_{i=1}^{n} S\left(X_i, \widetilde{\theta}_n\right) S\left(X_i, \widetilde{\theta}_n\right)^T \\ -\frac{1}{n}\sum_{i=1}^{n} \Psi\left(X_i, \widetilde{\theta}_n\right) \end{cases}$$

is an estimator (with corresponding estimate) of the Fisher Information I derived from the sample, and

$$R_n = Z_n^T \left[I\left(\theta_0\right)\right]^{-1} Z_n \qquad \text{where} \qquad Z_n \equiv Z_n\left(\theta_0\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} S\left(X_i, \theta_0\right) \tag{2}$$

(a) Show that, in the single parameter case, the statistics can be expressed as

$$W_n = -\left(\widetilde{\theta}_n - \theta_0\right)^2 \ddot{l}_n\left(\widetilde{\theta}_n\right) \qquad\qquad R_n = -\left\{\dot{l}_n\left(\theta_0\right)\right\}^2\left\{\ddot{l}_n\left(\theta_0\right)\right\}^{-1}$$ $\longleftarrow$ **TYPO CORRECTED !**

where, $l_n(\theta), \dot{l}_n(\theta), \ddot{l}_n(\theta)$, are the log-likelihood and its first and second derivatives.

[4 MARKS]

(b) Derive the forms of the Wald, Rao/Score and Likelihood Ratio statistics for testing

$$\begin{aligned} H_0 &: \lambda = \lambda_0 \\ H_1 &: \lambda \neq \lambda_0 \end{aligned}$$

if the data follow a Poisson distribution with parameter $\lambda > 0$.

[9 MARKS]

(c) Derive the form of the Wald statistic when the data are presumed normally distributed with parameters $N\left(\mu,\sigma^2\right)$ in a test of

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu \neq 0 \end{aligned}$$

(i) when σ^2 is **unspecified** under the null and the alternative; the MLE for σ^2 under the null is

$$S_0^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

whereas under the alternative, the joint MLE is $\left(\overline{X}, S^2\right)$ where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad\qquad S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2.$$

(ii) when $H_0 : (\mu,\sigma) = \theta_0 = \left(0,\sigma_0^2\right)$ and $H_1 : (\mu,\sigma) \neq \theta_0$.

[7 MARKS]